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Generalised squeezing and information theory approach to quantum entanglement

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Abstract

It is shown that the usual one and two-mode squeezing are based on reducible representations of the SU(1,1) group. Generalised squeezing is introduced with the use of different SU(1,1) rotations on each irreducible sector. Two-mode squeezing entangles the modes and information theory methods are used to study this entanglement. The entanglement of three modes is also studied with the use of the strong subadditivity property of the entropy.

1. Introduction

In a recent paper [1] we have explained that two-mode squeezing is based on reducible representations of the SU(1,1). The various irreducible sectors have been identified and different SU(1,1) rotations have been performed on each of them, generalizing in this way the concept of squeezing. In this paper we extend these ideas. In section 2 we consider one mode squeezing and prove that it is also based on reducible representations of SU(1,1). The two irreducible sectors are identified and different SU(1,1) rotations are applied on each of them, generalising in this way the concept of one-mode squeezing. In section 3 the two-mode case is considered in connection with both the SU(1,1) and SU(2) groups. Some of the results presented in [1] are briefly reviewed here. Each irreducible sector of the SU(1,1) (or SU(2)) group is squeezed independently and the generalised squeezed state is characterised by an infinite number of squeezing parameters. Hamiltonians which will lead to this type of squeezing, are presented.

Two-mode squeezing entangles the two modes. Especially our generalised squeezing entangles them in a very complicated way. One approach to study this entanglement is by using information theory methods. In section 4 we use the subaddivity and strong subadditivity properties of the entropy to define quantities which express the entanglement of two and three quantum systems. Especially interesting are the results for three entangled systems, because they indicate that this case is a non-trivial generalisation of the two system entanglement. The latter case has of course been discussed since the beginning of quantum mechanics; but it is only recently that some preliminary discussion of the former case has appeared [2]. Our results based on information theory methods suggest that the three system entanglement is a very interesting problem that requires further study.

2. Generalised one-mode squeezing

We consider the harmonic oscillator Hilbert space H and express it as

$$H = H_o + H_1 \tag{1}$$

where H_o is the subspace spanned by the even number eigenstates and H_1 the subspace spanned by the odd number eigenstates. We also consider the corresponding projection operators to these subspaces:

$$\pi_o = \sum_{N=0}^{\infty} | 2N > < 2N |$$

$$\pi_1 = \sum_{N=0}^{\infty} | 2N + 1 > < 2N + 1 | \qquad (2)$$

$$\pi_o + \pi_1 = 1$$

The one mode squeezing operators are defined as:

$$S (r, \theta, \lambda) = \exp [- \frac{1}{2} r e^{-i\theta} k_+ - \frac{1}{2} r e^{i\theta} K_-] \exp (i\lambda K_o)$$

$$K_o = \frac{1}{2} a^+ a + \frac{1}{4} \ ; \ K_+ = \frac{1}{2} a^{+2} \ ; \ K_- = \frac{1}{2} a^2$$

$$[K_o, K_\pm] = \pm K_\pm \ ; \ [K_- , K_+] = 2 K_o \qquad (3)$$

$$K^2 = K_o^2 - \frac{1}{2} (K_+ K_- + K_- K_+) = k(k-1) = - \frac{3}{16}$$

They form a reducible representation of SU(1,1). More specifically, they form the k = 1/4 irreducible representation when they act on H_o only; and the k = 3/4 irreducible representation when they act on H_1 only [3]. Related to this is the fact that:

$$[S(r, \theta, \lambda), \pi_o] = [S(r, \theta, \lambda), \pi_1] = 0 \qquad (4)$$

The following unitary operator squeezes independently each irreducible sector:

$$U(r_o \theta_o \lambda_o ; r_1 \theta_1 \lambda_1) = S(r_o, \theta_o, \lambda_o) \pi_o + S(r_1, \theta_1, \lambda_1) \pi_1 \qquad (5)$$

This is more general than the operator of equ.(3). Only in the special case

$$r_o = r_1 \ ; \ \theta_o = \theta_1 \ ; \ \lambda_o = \lambda_1 \qquad (6)$$

the operator (5) reduces to the operator (3). Acting with the operator (5) on a coherent state $| A >$, we get a generalised squeezed state:

$$| A ; r_o \theta_o \lambda_o ; r_1 \theta_1 \lambda_1 > = U(r_o \theta_o \lambda_o ; r_1 \theta_1 \lambda_1) | A >$$

$$= S(r_o, \theta_o, \lambda_o) \pi_o | A > + S(r_1, \theta_1, \lambda_1) \pi_1 | A > \qquad (7)$$

In the special case of equ.(6) this reduces to the usual squeezed states.

In systems described by the Hamiltonian

$$H = \omega' a^{+} a + (\mu_o\, a^{+2} + \mu_o^{*} a^{2})\, \pi_o + (\mu_1\, a^{+2} + \mu_1^{*} a^{2})\, \pi_1 \tag{8}$$

ordinary coherent states will evolve into the generalised squeezed states (7). In the special case $\mu_o = \mu_1$ the Hamiltonian (8) reduces to the Hamiltonian

$$H = \omega\, a^{+} a + \mu\, a^{2} + \mu\, a^{+2} \tag{9}$$

which is associated to the usual squeezed states.

3. Generalised two-mode squeezing

The appropriate group for the study of two-mode quadratic Hamiltonians is Sp(4,R) [4]. In this paper we shall only consider its subgroups SU(1,1) and SU(2) in connection with the Hamiltonians:

$$H_1 = \omega_1\, a_1^{+}\, a_1 + \omega_2\, a_2^{+}\, a_2 + \mu\, a_1\, a_2 + \mu^{*}\, a_1^{+}\, a_2^{+} \tag{10}$$

$$H_2 = \omega_1\, a_1^{+}\, a_1 + \omega_2\, a_2^{+}\, a_2 + \mu\, a_1\, a_2^{+} + \mu^{*}\, a_1^{+}\, a_2 \tag{11}$$

correspondingly. Both of these Hamiltonians have been used extensively in quantum optics problems [5].

Starting with the SU(1,1) group we express the two-mode Hilbert space as

$$H_A \times H_B = \sum_{k=-\infty}^{\infty} H_k \tag{12}$$

where H_k is the subspace spanned by the number eigenstates

$$H_k = (\,|\, N + k,\, N > \;;\; N = \max (o,\, -k),\, \ldots\, \infty\,) \tag{13}$$

We also introduce the corresponding projection operators

$$\pi_k = \sum |\, N + k,\, N > < N + k,\, N \,|$$

$$\sum \pi_k = 1 \tag{14}$$

The two-mode SU(1,1) squeezing operators are defined as

$$S(r,\, \theta,\, \lambda) = \exp \left[-\frac{1}{2} r\, e^{-i\theta} K_{+} + \frac{1}{2} r\, e^{i\theta} K_{-} \right] \exp (i\, \lambda\, K_o)$$

$$K_{+} = a_1^{+}\, a_2^{+} \;\; ; \;\; K_{-} = a_1\, a_2,\; K_o = \frac{1}{2} (a_1^{+}\, a_1 + a_2^{+}\, a_2 + 1)$$

$$K^{2} = \frac{1}{4} (a_1^{+}\, a_1 - a_2^{+}\, a_2)^{2} - \frac{1}{4} \tag{15}$$

They form a reducible representation of SU(1,1). More specifically, when they act on the space H_k only, they form the

$$\ell = \frac{1 + |k|}{2} \tag{16}$$

irreducible representation of SU(1,1) which belongs in the discrete series. Note also that

$$[S(r, \theta, \lambda), \pi_k] = 0 \tag{17}$$

The following unitary operator squeezes independently each irreducible sector:

$$U(\{r_k, \theta_k, \lambda_k\}) = \sum S(r_k, \theta_k, \lambda_k) \pi_k \tag{18}$$

In the special case

$$\begin{aligned} \ldots &= r_{-1} = r_o = r_1 = \ldots \\ \ldots &= \theta_{-1} = \theta_o = \theta_1 = \ldots \\ \ldots &= \lambda_{-1} = \lambda_o = \lambda_1 = \ldots \end{aligned} \tag{19}$$

the operators (18) reduce to the operators (15).

Acting with the operators (18) on two-mode coherent states we get generalised two-mode squeezed states:

$$U(\{r_k \theta_k \lambda_k\}) \mid A_1, A_2 > = \sum_k S(r_k, \theta_k, \lambda_k) \pi_k \mid A_1, A_2 > \tag{20}$$

In the special case of equ.(19) they reduce to the usual two-mode squeezed states. In systems described by the Hamiltonian

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \sum_k (\mu_k a_1 a_2 + \mu_k^* a_1^+ a_2^+)\pi_k \tag{21}$$

ordinary coherent states will evolve into the states of equ.(20). In the special case that all the μ_k are equal to each other, the Hamiltonian (21) reduces to the Hamiltonian (10).

In the case of the SU(2) group we express the two-mode Hilbert space as

$$H_A \times H_B = \sum_j H_{2j+1}$$

$$j = 0, ½, 1, \ldots. \tag{22}$$

where H_{2j+1} is the subspace spanned by the number eigenstates

$$H_{2j+1} = \{ \mid N, 2j-N> \; ; \; N = 0, \ldots. (2j) \} \tag{23}$$

We also introduce the corresponding projection operators

$$\pi_{2j+1} = \sum_{N=0}^{2j} | N, 2j-N> < N, 2j-N|$$

$$\sum \pi_{2j+1} = 1 \tag{24}$$

The SU(2) squeezing operators are defined as:

$$T(r, \theta, \lambda) = \exp \left[-\frac{1}{2} r e^{-i\theta} J_+ + \frac{1}{2} r e^{i\theta} J_- \right] \exp(i\lambda J_o)$$

$$J_+ = a_1^+ a_2 \quad ; \quad J_- = a_1 a_2^+ \quad ; \quad J_o = \frac{1}{2} (a_1^+ a_1 - a_2^+ a_2)$$

$$J^2 = \left[\frac{1}{2} (a_1^+ a_1 + a_2^+ a_2) \right] \left[\frac{1}{2} (a_1^+ a_1 + a_2^+ a_2 + 1) \right] \tag{25}$$

They form a reducible representation of SU(2). When they act on the space H_{2j+1} only, they form the j irreducible representation of SU(2). Note also that:

$$[T(r, \theta, \lambda), \pi_{2j+1}] = 0 \tag{26}$$

The following unitary operator performs SU(2) rotations independently on each irreducible sector:

$$U((r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1})) = \sum T(r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1}) \pi_{2j+1} \tag{27}$$

In the special case:

$$\begin{aligned} r_1 &= r_2 = \ldots \\ \theta_1 &= \theta_2 = \ldots \\ \lambda_1 &= \lambda_2 = \ldots \end{aligned} \tag{28}$$

The operators (27) reduce to the operators (25). Acting with the operators (27) on two-mode coherent states we get the states:

$$U((r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1})) | A_1, A_2 > =$$

$$\sum_j T(r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1}) \pi_{2j+1} | A_1, A_2 > \tag{29}$$

They will be formed during the time evolution of ordinary coherent states in systems described by the Hamiltonian:

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \sum \pi_{2j+1} (\mu_{2j+1} a_1 a_2^+ + \mu_{2j+1}^* a_1^+ a_2) \tag{30}$$

In the special case that all the μ_{2j+1} are equal to each other, the Hamiltonian (30) reduces to the Hamiltonian (11).

The uncertainty properties of the states (20), (29) have been studied in [1]. The results presented there show that both of these states exhibit squeezing.

4. Information theory approach to quantum entanglement

In this section we use quantum information theory methods for the study of two- and three-mode correlated systems. Let ρ be a two-mode density matrix and $<N_1>$, $<N_2>$ the average number of photons in the two modes. As in our previous work [6] we define the information contained in this density matrix as

$$I = S_{max} - S(\rho) = S[\rho_1^{th}(<N_1>) \times \rho_2^{th}(<N_2>)] - S(\rho)$$

$$S(\rho) = -\operatorname{Tr} \rho \ln \rho$$

$$\rho_i^{th}(<N_i>) = \frac{<N_i>^{Ni}}{(1 + <N_i>)^{1+N_i}} \; | N_i > < N_i | \quad ; \quad i = 1, 2 \qquad (31)$$

Following the negentropy ideas of Brillouin we subtract here the entropy of the system from the maximum entropy that the system could have had, with the average number of photons in the two modes been kept fixed. The maximum entropy is equal to the entropy of a thermal system with an average number of photons in the two modes $<N_1>$, $<N_2>$. Taking partial traces, we define:

$$\rho_1 = \operatorname{Tr}_2 \rho \quad ; \quad \rho_2 = \operatorname{Tr}_1 \rho \qquad (32)$$

and express the information (31) as [7, 8]

$$I = I_1 + I_2 + I_{12}$$

$$I_i = S[\rho_i^{th}(<N_i>)] - S(\rho_i)$$

$$I_{12} = S(\rho_1) + S(\rho_2) - S(\rho) \qquad (33)$$

I_i is the information in the mode i; and I_{12} is the information in the correlation between the two modes. The subadditivity property ensures that the I_{12} is non-negative. Numerical evaluation of the quantities I_1, I_2, I_{12} for several examples has been presented in [1].

A non-trivial extension of these ideas occurs in the case of three correlated modes. The information in this case is given by

$$I = S(\rho^{th}) - S(\rho)$$

$$\rho^{th} = \rho_1^{th}(<N_1>) \times \rho_2^{th}(<N_2>) \times \rho_3^{th}(<N_3>) \qquad (34)$$

We define

$$\rho_{ij} = \operatorname{Tr}_k \rho \quad , \quad \rho_i = \operatorname{Tr}_{jk} \rho$$

$$I_i = S[\rho_i^{th}(<N_i>)] - S(\rho_i)$$

$$I_{ij} = S(\rho_i) + S(\rho_j) - S(\rho_{ij}) \qquad (35)$$

I_i is the information in the mode i. I_{ij} is the information in the correlation between the modes (i,j). We then express the total information in the three-mode system as:

$$I = I_1 + I_2 + I_3 + I_{12} + I_{23} + I(12\ ;\ 23) \qquad (36)$$

where

$$I(12\ ;\ 23) = S(\rho_{12}) + S(\rho_{23}) - S(\rho) - S(\rho_2) \quad (37)$$

The strong subadditivity property [9] ensures that the quantity I(12 ; 23) is non-negative. For symmetry reasons, somebody might be tempted to split I(12 ; 23) as:

$$I(12\ ;\ 23) = I_{13} + A \quad (38)$$

so that he can express the information I of equ.(36), as the sum of the three informations in the three modes; the three informations in the correlated pairs of modes; and the quantity A characterising the correlation between all modes. However the quantity A is not necessarily positive and its interpretation as information would be incorrect. Therefore, the information I of a three-mode system is the sum of the three informations in the three modes; the two correlation informations in two of the pairs; and the information I(12 ; 23) of equ.(37) which describes new types of correlations in the three-mode systems. This result can be used as a "guide" of how to study the entanglement of three systems. It is seen that three system entanglement is a non-trivial generalisation of two system entanglement.

5. Discussion

In many cases the concept of squeezing is based on reducible representations of the SU(1,1) (or SU(2)) group. In these cases different SU(1,1) (or SU(2)) rotations on each irreducible sector lead to generalised squeezing. These ideas have been applied to both one-mode and two-mode squeezing.

Two-mode squeezing correlates the two-modes and information theory methods have been used for the study of these correlations. The subadditivity and strong subadditivity properties of the entropy have been used for the study of two and three correlated systems, correspondingly. It has been shown that the entanglement of three systems is a non-trivial generalisation of the entanglement of two systems. Further work is required in this direction.

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