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GEOMETRIC ASPECTS OF UNCERTAINTY AND CORRELATION

Sumiyoshi Abe
Institute for Theoretical Physics III
University of Erlangen-Nürnberg
Staudtstrasse 7
W-8520 Erlangen
The Federal Republic of Germany

Abstract

It is discussed that the metric induced on the quantum evolution submanifold of the projective Hilbert space describes the uncertainties and correlations of the operators generating the quantum-state evolution, and exhibits the inherently-quantized geometry.

1 Introduction

Berry's phase and its extensions [1-6] are the striking phenomena that show how the law of quantum-state evolution is geometric. It is determined by the evolution curve in the projective Hilbert space $\mathcal{P}$, and is independent of a specific choice of the Hamiltonian as long as it gives that projected curve in $\mathcal{P}$.

The phase difference due to the 1-parameter (λ) evolution is seen in the first-order term of $d\lambda$ in the transition amplitude $\langle\psi(\lambda)|\psi(\lambda+d\lambda)\rangle$. On the other hand, geometry of the evolution curve C in $\mathcal{P}$ is characterized by the Fubini-Study metric [7,8] induced on C: $ds^2 = 1 - |\langle\psi(\lambda)|\psi(\lambda+d\lambda)\rangle|^2$. (Here and hereafter, the state vectors are assumed to be normalized.)

Recently, Anandan and Aharonov [9,10] have obtained a remarkable result that if the 1-parameter evolution is generated by a Hermitian operator A, then the relation $ds = \Delta A d\lambda$ holds, where ΔA is the variance $(\Delta A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. This means that the "velocity" of evolution along C is just equal to the uncertainty of the generator of that evolution.

The purpose of this paper is to report briefly the further results recently obtained in the study of geometric aspects of quantum evolution. More detailed discussions will be found in Ref. [11].

2 Geometry of Uncertainty and Correlation

There are a variety of 1-parameter evolutions for a generic quantum state. Each evolution gives each curve in the projective Hilbert space $\mathcal{P}$. It is preferable to consider the multi-dimensional submanifold $\mathcal{N}$ of $\mathcal{P}$, in which various evolution curves are embedded. $\mathcal{N}$ is properly called here the quantum evolution submanifold. If a given state is parametrized by a set of n real numbers $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, then a local coordinate of $\mathcal{N}$ is identified with α. In this case, the metric induced on $\mathcal{N}$ is given by

$$ds^2 = 1 - |\langle\psi(\alpha)|\psi(\alpha + d\alpha)\rangle|^2. \tag{1}$$

If the evolution of the state $|\psi(\alpha)\rangle$ is assumed to be generated by n independent Hermitian operators $\{A_i(\alpha)\}_{i=1,2,\ldots,n}$, that is,

$$-i\partial_i|\psi(\alpha)\rangle = A_i(\alpha)|\psi(\alpha)\rangle \qquad (i = 1, 2, \ldots, n), \tag{2}$$

then Eq.(1) has the form $ds^2 = g_{ij}(\alpha)d\alpha^i d\alpha^j$, where

$$\begin{aligned} g_{ij}(\alpha) = {} & \frac{1}{2}\langle\psi(\alpha)|A_i(\alpha)A_j(\alpha) + A_j(\alpha)A_i(\alpha)|\psi(\alpha)\rangle \\ & -\langle\psi(\alpha)|A_i(\alpha)|\psi(\alpha)\rangle\langle\psi(\alpha)|A_j(\alpha)|\psi(\alpha)\rangle, \end{aligned} \tag{3}$$

provided that $\partial_i \equiv \partial/\partial\alpha^i$ and the summation convention is understood for the repeated upper and lower indices. Thus, one can see that the diagonal g_{ii} and off-diagonal g_{ij} $(i \neq j)$ components are respectively equal to the uncertainties and correlations of the operators generating the evolution [11,12].

The metric (3) defines the Riemannian structure of $\mathcal{N}$. The metric-compatible connection can be expressed as a simple quantum expectation value [11]:

$$\begin{aligned} \Gamma^{\bullet}_{kij} = {} & \frac{1}{4}\langle\psi|[\partial_i B_j + \partial_j B_i - i(B_i B_j + B_j B_i)]B_k \\ & +B_k[\partial_i B_j + \partial_j B_i + i(B_i B_j + B_j B_i)]|\psi\rangle, \end{aligned} \tag{4}$$

where the operators B_i are given by $B_i(\alpha) = A_i(\alpha) - \langle\psi(\alpha)|A_i(\alpha)|\psi(\alpha)\rangle$. With this expression, it is straightforward to ascertain the Riemannian parallelism: $\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^h{}_{ik} g_{hj} - \Gamma^h{}_{jk} g_{ih} = 0$.

Geometric aspects of the uncertainties and correlations can be seen best in the squeezed state example. The single-mode two-photon squeezed state [13] is given by $|z\rangle_\xi = D(z)S(\xi)|0\rangle = \exp(za^\dagger - z^* a)\exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)]|0\rangle$. $a^\dagger$ and a are the usual bosonic creation and annihilation operators. $|0\rangle$ is the vacuum state annihilated by a. $D(z)$ and $S(\xi)$ are called Glauber's displacement operator and the squeeze operator, respectively. The displacement operator gives a correspondence relation between relevant operators and their classical counterparts in the phase space (x, p) with the parametrization $z = (x + ip)/\sqrt{2}$. x and p are respectively equal to the expectation values of the position $X = (a + a^\dagger)/\sqrt{2}$ and momentum $P = (a - a^\dagger)/i\sqrt{2}$ operators in the squeezed state.

Consider the translational evolution: $|z(x,p)\rangle_\xi \longrightarrow |(z + dz)(x + dx, p + dp)\rangle_\xi$, where the squeeze parameter is fixed. From the transition amplitude, the metric $ds^2 = 1 - |_\xi\langle z|z + dz\rangle_\xi|^2$ is directly calculated as

$$ds^2 = \frac{1}{2}(\cosh 2r - \sinh 2r \cos 2\phi)dx^2 + \frac{1}{2}(\cosh 2r + \sinh 2r \cos 2\phi)dp^2 + 2 \times \frac{1}{2}\sinh 2r \sin 2\phi\, dx\, dp, \quad (5)$$

provided the parametrization $\xi = re^{-2i\phi}$ $(0 \leq r,\ 0 \leq \phi < 2\pi)$ has been used. This is the Euclidean metric in a non-Cartesian coordinate. On the other hand, the above translational evolution is generated by the following operators:

$$-i\frac{\partial}{\partial x}|z\rangle_\xi = A_x|z\rangle_\xi, \qquad A_x = -P + \frac{p}{2}, \quad (6a)$$

$$-i\frac{\partial}{\partial p}|z\rangle_\xi = A_p|z\rangle_\xi, \qquad A_p = X - \frac{x}{2}. \quad (6b)$$

The uncertainties and correlations in the squeezed state are the familiar ones:

$$(\Delta A_x)^2 = (\Delta P)^2 = \frac{1}{2}(\cosh 2r - \sinh 2r \cos 2\phi), \quad (7a)$$

$$(\Delta A_p)^2 = (\Delta X)^2 = \frac{1}{2}(\cosh 2r + \sinh 2r \cos 2\phi), \quad (7b)$$

$$C(A_x, A_p) = -C(X, P) = \frac{1}{2}\sinh 2r \sin 2\phi, \quad (7c)$$

where $C(A,B) = C(B,A) = \frac{1}{2}\langle\psi|AB + BA|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle$. These quantities in fact give the components of the metric (5).

The effects of squeezing as the expansion, contraction, and rotation in the phase space has been explored geometrically by the methods of phase-space representations of quantum theory in the literature [14,15]. The metric (5) describes those effects in a peculiar *representation-free* manner.

Since the metric is given in terms of a reference state, it carries some of quantum numbers characterizing that state. Accordingly, $\mathcal{N}$ possesses the quantized structure, in general. In what follows, such examples are given.

The first example is the displaced number state [16]: $|z\rangle_n = D(z)|n\rangle$, where $|n\rangle \equiv (n!)^{-1/2}(a^\dagger)^n|0\rangle$ $(n = 0,1,2,\ldots)$. Consider the translational evolution $|\, z(x,p)\rangle_n \longrightarrow |\, z(x+dx, p+dp)\rangle_n$. The metric is calculated as

$$ds^2 = (n + \frac{1}{2})(dx^2 + dp^2). \tag{8}$$

Therefore, the phase space locally identified with $\mathcal{N}$ associated with the evolution of the displaced number state has a Euclidean metric with a quantized conformal factor.

Another example is the squeezed number state [17]: $|\, \xi\rangle_n = S(\xi)\, |\, n\rangle$ $(n = 0,1,2,\ldots)$. The squeeze parameter is again parametrized as $\xi = re^{-2i\phi}$. Consider the evolution $|\, \xi(r,\phi)\rangle_n \longrightarrow |\, (\xi + d\xi)(r+dr, \phi+d\phi)\rangle_n$. $\mathcal{N}$ is locally labelled by (r,ϕ). The metric is then found to be

$$ds^2 = \frac{1}{2}(n^2 + n + 1)(dr^2 + \sinh^2 2r d\phi^2). \tag{9}$$

This is the metric of the Lobachevsky space [18] with a quantized conformal factor. Its Gaussian curvature [18] is also quantized as $K = -8/(n^2+n+1)$. It is interesting to see that the curvature vanishes in the "classical limit" $n \to \infty$.

3 Conclusions

It has been demonstrated that the Fubini-Study metric induced on the quantum evolution submanifold $\mathcal{N}$ is completely given by the uncertainties and correlations of the operators generating various evolutions, and $\mathcal{N}$ admits the quantized Riemannian structure.

In the above simple examples, only the conformal factors of the metrics are quantized. This may be partially due to the mathematical fact [18] that all two-dimensional spaces are conformally equivalent to the Euclidean space. In general, each component of the metric is individually quantized.

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