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INFORMATION ENTROPY VIA GLAUBER'S Q-REPRESENTATION

C.H. Keitel
Blackett Laboratory, Imperial College, London SW7 2BZ, Great Britain

K. Wódkiewicz
Institute of Theoretical Physics, Warsaw University, Warsaw 00681, Poland

Abstract

We present a convenient way to evaluate the information entropy of a quantum mechanical state via the Glauber Q-representation. As an example we discuss the information entropy of a thermally relaxing squeezed state in terms of its Q-representation and show the validity of the corresponding entropic uncertainty- and Araki-Lieb inequalities.

1 The information entropy

Shannon and Wehrl were the first to describe the information of a quantum mechanical state in terms of its probability distributions [1]. Later, there has also been a substantial amount of work on this topic from the quantum optics point of view [2]. The question of comparability of the information entropy with the Heisenberg uncertainty has been treated as well. The Heisenberg uncertainty has turned out to be of enormous significance because of its experimental measurability. However, it only takes the second moments into account whereas the information entropy is supposed to be an exact measure of the information and thus of the uncertainty or non-information. In comparison to the significant Heisenberg uncertainty inequality, there is a similarly meaningful entropic uncertainty relation. Bialynicki-Birula et al., derived such an inequality more than 15 years ago [3].

In this paper we would like to put forward a possibility to evaluate the information entropy as a function of the Q-representation since this representation is well-known for many interesting quantum mechanical states and completely describes the state. In particular we here would like to investigate the information entropy for the squeezed state which evolves to a thermal state via an appropriate Fokker-Planck equation. Special interest is devoted to the entropic uncertainty relation. As a major result we show that a squeezed state also obeys the minimum entropic uncertainty relation. However, it turns out that the evolution of the squeezed state via the Fokker-Planck equation, does lead to a change of the information entropy and the marginal contributions but surprisingly does not influence the minimality of the uncertainty relation. This even means

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that a thermal state fulfills the entropic uncertainty relation with an equal sign. We moreover investigate the Araki-Lieb inequality [4] for information entropies and find agreement with the well-known results of von Neumann entropies for the thermally relaxing squeezed state.

We start off the paper with some basic facts on entropies and develop an expression for the information entropy in terms of the Glauber Q-representation. The definition of the quantum mechanical entropy is given by:

$$S = -Tr\{\hat{\rho}\ln\hat{\rho}\} \tag{1}$$

with $\hat{\rho}$ being the density matrix operator and assuming the Boltzmann constant to be 1. This often called von Neumann entropy is zero for a pure state and non-zero for a mixed state. It is moreover known to be constant for a closed system which arises from the fact that a unitary time evolution does not change the eigenvalues of $\hat{\rho}$.

Thus, normally, the evolution of the entropy of subsystems of a closed system is of greater interest. Considering two disjunct interacting systems that form together the whole system being described by $\hat{\rho}$, we can introduce the reduced density operators $\hat{\rho}_A = Tr_B\{\hat{\rho}\}$ and $\hat{\rho}_B = Tr_A\{\hat{\rho}\}$, where Tr_A and Tr_B abbreviate the tracing over the variables of the subsystems A and B, respectively. This leads to the definition of the entropy of the subsystem A: $S(\hat{\rho}_A) = -Tr_A\{\hat{\rho}_A\ln\hat{\rho}_A\}$ and to the analogous expression for subsystem B by replacing A by B in the above formula.

These reduced or here called marginal entropies describe information or more directly disorder and uncertainty of A and B and are not necessarily time independent like the entropy of the whole system S of Eq.(1). Information about the interaction of A and B is neither included in $S(\hat{\rho}_A)$ nor in $S(\hat{\rho}_B)$ so that we expect the sum of $S(\hat{\rho}_A)$ and $S(\hat{\rho}_B)$ not to be smaller than S. And, in fact, Araki and Lieb [4] proofed the following triangle inequality:

$$|S(\hat{\rho}_A) - S(\hat{\rho}_B)| \leq S \leq S(\hat{\rho}_A) + S(\hat{\rho}_B). \tag{2}$$

Because of the close relation of entropy and uncertainty and moreover the existence of a lower bound of S, the second inequality can be interpreted as uncertainty relation. The calculation of the above entropies requires the diagonalization of the reduced density operators. Since this is often difficult, the information entropy or Shannon-Wehrl-entropy was introduced according to:

$$\tilde{S}\,(\hat{\rho};\hat{O}) = -\sum_e \langle e|\hat{\rho}|e\rangle \ln\langle e|\hat{\rho}|e\rangle, \tag{3}$$

with

$$\hat{O}|e\rangle = e|e\rangle. \tag{4}$$

The corresponding expressions for the subsystems can be obtained by exchanging $\hat{\rho}$ by $\hat{\rho}_A$ or $\hat{\rho}_B$, where the so far arbitrary operator $\hat{O}$ may be chosen differently. If we are dealing with operators that can be expressed in terms of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ of a boson field, it is reasonable to consider the information entropy

$$\begin{aligned} \tilde{S}\,(\hat{\rho},\hat{a}) &= -\int d^2\alpha \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \ln\left(\frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle\right) && (5)\\ &= -\int d^2\alpha Q(\alpha)\ln Q(\alpha), && (6) \end{aligned}$$

where

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{7}$$

is the well-known Glauber representation of a state $\hat{\rho}$, and where $|\alpha\rangle$ is the boson coherent state with the decomposition of unity $\int d^2\alpha|\alpha\rangle\langle\alpha| = \pi$.

According to following calculations, the information entropy of a squeezed state is not zero as opposed to the von Neumann entropy which is always zero for a pure state. This obviously makes the information entropy more interesting than the von Neumann entropy. The form of Eq.(5) as well as Shannon's early work [1] suggest to define an information entropy for any phase space distribution. The Q-representation, however, has turned out to be appropriate for realitic measurements as shown in the analysis in terms of phase propensities [5] and heterodyne measurements. [6] For the investigation of the entropic uncertainty principle for information entropies, the marginal entropies are evaluated by inserting the marginal Q-representation in the above expressions, instead. Letting α_1 and α_2 be arbitrary coordinates in the complex plane of α, we thus define

$$Q_i(\alpha_i) = \int d\alpha_j Q(\alpha_i, \alpha_j) \tag{8}$$

and

$$S_i = -\int d\alpha_i Q_i(\alpha_i) \ln Q_i(\alpha_i) \tag{9}$$

for $i, j \epsilon \{1, 2\}, i$ unequal j. This leads to the entropic uncertainty relation for information entropies, which, as a major result of this study, will turn out to hold for the squeezed state and its thermally relaxing state.

In the following, we put forward the time independent information entropy of a squeezed state and its evolution to a thermal state via the Fokker-Planck equation and evaluate the corresponding information entropies.

2 Information entropy and entropic uncertainty relations of a squeezed state

2.1 Statics

In this section the squeezed state is described by the time independent Q-representation

$$Q(\alpha, \alpha^*) = \frac{1}{\pi \mathrm{ch}(s)} \exp\left[-|\alpha - \alpha_0|^2 + \frac{\mathrm{th}(s)}{2}\{(\alpha - \alpha_0)^2 + (\alpha^* - \alpha_0^*)^2\}\right], \tag{10}$$

with the corresponding information entropy:

$$S = -\int d^2\alpha Q(\alpha, \alpha^*) \ln Q(\alpha, \alpha^*) = 1 + \ln\frac{\pi}{2} + \ln(e^s + e^{-s}). \tag{11}$$

The letter s here denotes the squeezing parameters and α_0 describes the coherent state which has been squeezed. We now want to compare S with the information entropies obtained out of the marginal Q-representations. Those marginal information entropies are obviously dependent on the choice of coordinates, where our considerations in the following will concentraate on the most interesting Cartesian coordinates.

The Q-representation in Cartesian coordinates ($\alpha_x = \mathrm{Re}\alpha, \alpha_y = \mathrm{Im}\alpha$) has the form

$$Q(\alpha_x, \alpha_y) = \frac{1}{\pi \mathrm{ch}(s)} \exp\left[-\frac{2}{1+e^{2s}}(\alpha_x - (\alpha_0)_x)^2\right] \times \exp\left[-\frac{2}{1+e^{-2s}}(\alpha_y - (\alpha_0)_y)^2\right], \tag{12}$$

leading to the marginal quasi-probability distributions,e.g.:

$$Q_x(\alpha_x) = \int_{-\infty}^{+\infty} d\alpha_y Q(\alpha_x, \alpha_y) = \frac{1}{\pi \mathrm{ch}(s)} \left(\frac{\pi(1+e^{-2s})}{2}\right)^{1/2} \exp\left[-\frac{2}{1+e^{2s}}(\alpha_x - (\alpha_0)_x)^2\right], \tag{13}$$

and thus to the marginal information entropies, e.g.:

$$S_x = -\int Q_x(\alpha_x) \mathrm{ln} Q_x(\alpha_x) d\alpha_x = \frac{1}{2} + \frac{1}{2}\mathrm{ln}\frac{\pi}{2} + \frac{1}{2}\mathrm{ln}(1+e^{2s}), \tag{14}$$

and correspondingly $S_y = \frac{1}{2} + \frac{1}{2}\mathrm{ln}\frac{\pi}{2} + \frac{1}{2}\mathrm{ln}(1+e^{-2s})$. Considering above equations, it is now easy to see that squeezed states fulfill minimum entropic uncertainty

$$S = S_x + S_y, \tag{15}$$

and that the Araki-Lieb inequality is valid as well: $|S_x - S_y| < S$.

A similar consideration can be done for polar coordinates with $\alpha = re^{i\phi}$ and $\alpha_0 = r_0 e^{i\phi_0}$. The integrals here are not as straight forward as in the Cartesian case. For special cases as the weakly squeezed vacuum, however, it was possible to show the validity of the uncertainty and Araki-Lieb relation [7].

2.2 Dynamics

Our interest now turns to the time evolution of the information entropy, its marginal information entropies and its influence on the inequalities investigated in the preceeding section. The time evolution of the Q-representation is governed by the Fokker-Planck equation

$$\partial_t Q = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) + \eta\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right] Q. \tag{16}$$

This equation follows from the well-known Fokker-Planck equation for the P-representation with $Q(\alpha,t) = \int \frac{d^2\beta}{\pi} \exp[-|\alpha-\beta|^2] P(\beta,t)$, $\bar{n}$ is the mean number of photons and η turns out to be $\gamma(\bar{n}+1)$.

We now move on with the solution of the Fokker-Planck equation for Q, assuming the squeezed state to be the initial state at time t equal to 0. Following reference [7] this turns out to be:

$$
\begin{aligned}
Q(\alpha,t) &= \int d^2\beta \frac{1}{\pi n(t)} \exp\left(-\frac{|\alpha-\beta e^{-(\gamma/2)t}|^2}{n(t)}\right) &\quad (17)\\
&\times \frac{1}{\pi \mathrm{ch}(s)} \exp\left[-|\beta-\alpha_0|^2 + \frac{1}{2}\mathrm{th}(s)\{(\beta-\alpha_0)^2 + (\beta^* - \alpha_0^*)^2\}\right] &\quad (18)\\
&= \exp[a(t)|\alpha|^2 + b(t)(\alpha^2+\alpha^{*2}) + c(t)\alpha + c(t)^*\alpha^* + N(t)], &\quad (19)
\end{aligned}
$$

with $n(t) = \frac{\eta}{\gamma}(1-e^{-\gamma t})$. For the rather long analytic expressions of $a(t), b(t), c(t)$ and $N(t)$ we refer to reference [7]. At this point it will only be of interest that the Fokker-Planck equation has preserved the Gaussian character of the initial state.

In the following we would like to point out that also the time dependent Q representation due to the last equation leads to minimum uncertainty. Even more it turns out that every normalized Gaussian function of the following form has this property:

$$Q(\alpha,t) = \exp[a(t)|\alpha|^2 + b(t)(\alpha^2+\alpha^{*2}) + c(t)\alpha + c^*(t)\alpha^* + N(t)], \quad (20)$$

where the real coefficients $a(t), b(t)$ and the complex $c(t)$ are now arbitrary with the only restrictions to fulfill: $a(t) < 0$, $2|b(t)| < |a(t)|$ and $N(t)$ is determined such that $Q(\alpha,t)$ is normalized. Using the same notation as in the static case, some algebra gives rise to the corresponding information entropy and marginal entropies:

$$
\begin{aligned}
S(t) &= 1 + \ln\pi - \frac{1}{2}\ln(a^2(t) - 4b^2(t)), &\quad (21)\\
S_x(t) &= \frac{1}{2} + \frac{1}{2}\ln\pi - \frac{1}{2}\ln(-a(t) - 2b(t)), &\quad (22)\\
S_y(t) &= \frac{1}{2} + \frac{1}{2}\ln\pi - \frac{1}{2}\ln(-a(t) + 2b(t)), &\quad (23)
\end{aligned}
$$

yielding immediately for all times t

$$S(t) = S_x(t) + S_y(t). \quad (24)$$

At $t = 0$ this is in agreement with the minimum Heisenberg uncertainty relation of a Gaussian wavepacket of the above form because the product of uncertainties in space and momentum is exactly one. Since the Fokker-Planck equation does conserve the Gaussian character of the wave function and does moreover give not rise to any phase factor, the Cartesian entropic uncertainty relation is even fulfilled with the equal sign for all times. Thus we have also a minimum uncertainty relation for the thermal state, what is not expected from the Heisenberg uncertainty inequality.

Moreover one finds for any Gaussian distribution that the Araki-Lieb inequality is equivalent to:

$$\frac{1}{2}\left|\ln\frac{-a-2b}{-a+2b}\right| \le -\frac{1}{2}\ln(a^2-4b^2) + 1 + \ln\pi \quad (25)$$

In the case of a thermally squeezed with its particular values for the coefficients $a(t), b(t), c(t)$ and $N(t)$ an even stronger inequalility can be derived:

$$|S_x(t) - S_y(t)| \leq S(t) - (1 + \ln\frac{\pi}{2}). \tag{26}$$

which , however, does not mean that all phase space distributions fulfill the Araki-Lieb or even the improved inequality [7].

In conclusion, we introduced a way to evaluate the information entropy in terms of the Glauber Q-representation. Taking advantage of these entropies, we approached the question of comparability of the Heisenberg uncertainty and the Shannon-Wehrl-entropy like description of information for the example of a thermally relaxing squeezed state. The first just considers second moments and is therefore a - though very important - approximation whereas the other is exact but academic. We find full accordance concerning the validity of the Heisenberg and entropic uncertainty inequalities for the thermally relaxing squeezed state but as expected also observe disagreement in the case when the equal sign holds.

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