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WIGNER FUNCTIONS FOR NONCLASSICAL STATES OF A COLLECTION OF TWO-LEVEL ATOMS

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Abstract

The general theory of atomic angular momentum states is used to derive the Wigner distribution function for atomic angular momentum number states, coherent states, and squeezed states. These Wigner functions $W(\theta,\varphi)$ are represented as a pseudo-probability distribution in spherical coordinates θ and φ on the surface of a sphere of radius $\sqrt{j(j+1)}$ where j is the total angular momentum.

1 Introduction

The phase space description of electromagnetic fields has had great success in leading to an understanding of the relationship between semiclassical and quantum theories of light. It was Sudarshan [1] who proved the optical equivalence theorem, i.e., he derived the relationship between the quantities measured by a photodetector and the mean values of the corresponding operators. He showed that the function appearing in the diagonal coherent state representation, that is calculated from the density matrix, provides a link between the semiclassical and quantum descriptions. This function, now denoted by $P(\alpha)$, is generally singular for nonclassical states [2]. In such cases the Wigner function [3,4] has proved to be especially attractive as an alternative. The Wigner function has also proved to be quite useful in discussing related topics [5] such as the photon number distribution and the phase operator distribution. In these problems, the concept of the area of overlap in phase space has been especially useful.

The nonclassical characteristics of the atomic systems, particularly a collection of two-level atoms, has been a subject of much investigation [6,7]. Much of the work has concentrated on the direct calculation of the variances in the atomic operators such as J_x, J^+, and J^-. Very little has been done on the relationship between the nonclassical

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aspects and the phase space distributions for atomic operators. For general angular momentum systems, Arecchi, et al. [8] introduced the analog of the diagonal coherent state representation

$$\rho = \int P(\alpha,\beta) \,|\alpha,\beta\rangle\langle\alpha,\beta|\, \sin\alpha \, d\alpha \, d\beta , \tag{1}$$

where $|\alpha,\beta\rangle$ represents the atomic coherent state

$$|\alpha,\beta\rangle = \sum_{m=-j}^{+j} \binom{2j}{m+j}^{1/2} \left(\sin\frac{\alpha}{2}\right)^{j+m} \left(\cos\frac{\alpha}{2}\right)^{j-m} e^{-i(j+m)\beta} \,|jm\rangle , \tag{2}$$

and where $|jm\rangle$ is the eigenstate of J^2 and J_z. The parameters α and β correspond to θ and φ except that α is measured off the south pole. The coherent state obtains the minimum of the angular momentum uncertainty relation $\langle \Delta J_{x'}^2 \rangle \langle \Delta J_{y'}^2 \rangle \geq |\langle J_{z'} \rangle|^2/4$, where x', y', and z' form an orthogonal coordinate system with z' in the $\hat{r}$ direction with angular coordinates (α,β). The coherent state is just a rotation of the ground Fock state $|j,-j\rangle$ away from the south pole. Arecchi and co-workers discussed the utility of the function $P(\alpha,\beta)$ in atomic problems, and Scully and co-workers have discussed the Wigner function for spin-$\frac{1}{2}$ particles [9]. Using the general theory of multipole operators [10], Agarwal [11] introduced the Wigner function for systems of arbitrary angular momentum. To arrive at this distribution, we first expand the atomic angular momentum operators as

$$G = \sum_{K=0}^{2j} \sum_{Q=-K}^{+K} G_{KQ} T_{KQ} , \tag{3}$$

where T_{KQ} is the multipole operator defined by

$$T_{KQ} = \sum_{m=-j}^{j} \sum_{m'=-j}^{+j} (-1)^{j-m} \sqrt{2k+1} \begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix} |jm\rangle\langle jm'| , \tag{4}$$

where $\begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix}$ is the usual Wigner $3j$ symbol. The expansion coefficients in Eq. (3) are obtained from the orthogonality of the multipole operators, namely

$$G_{KQ} = \mathrm{Tr}\left(G T_{KQ}^{\dagger}\right) . \tag{5}$$

The Wigner function associated with G is then defined by [11]

$$W(\theta,\varphi) = \sum_{K=0}^{2j} \sum_{Q=-K}^{+K} Y_{KQ}(\theta,\varphi) G_{KQ} , \tag{6}$$

where Y_{KQ} are the usual spherical harmonics. Note that

$$\mathrm{Tr}\, G = \sqrt{\frac{2j+1}{4\pi}} \int W(\theta,\varphi)\, \sin\theta\, d\theta\, d\varphi = 1 \ , \tag{7}$$

a general property desired of any distribution function. Note further that if two operators $G^{(1)}$ and $G^{(2)}$ are represented respectively by the Wigner functions $W^{(1)}$ and $W^{(2)}$, then

$$\mathrm{Tr}\,(G_1 G_2) = \int W^{(1)}(\theta,\varphi)\, W^{(2)}(\theta,\varphi)\, \sin\theta\, d\theta\, d\varphi \ , \tag{8}$$

a defining property of the Wigner distribution. In fact these two features, Eqs. (7) and (8), can be used to derive the form, Eq. (6), of the Wigner function. Thus, unlike the *P* function, all expectation values can be obtained in terms of the Wigner functions alone.

In this paper we shall consider the structure of the Wigner function associated with the important states like (i) Fock states $|j,m\rangle$, (ii) coherent states $|\alpha,\beta\rangle$, and (iii) squeezed states $|\zeta,m\rangle$ associated with a collection of two-level atoms interacting with a squeezed photon bath. We examine how the quantum character of the state is reflected in the properties of the Wigner function.

2 Angular Momentum States $|jm\rangle$

We first obtain the Wigner function for the state $|jm\rangle$. The density matrix can be written in the form

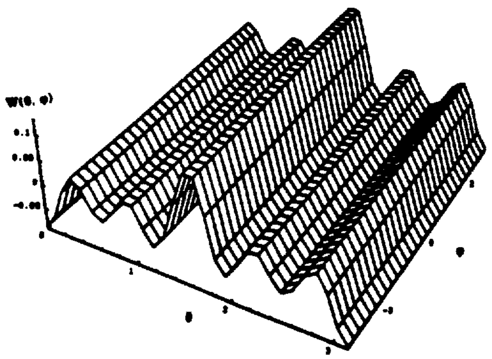
$$\rho = |jm\rangle\langle jm| \ . \tag{9}$$

Upon using Eqs. (4) through (6), that are used in defining the Wigner function *W*, we find that

$$W_{jm}(\theta,\varphi) = \sum_{K=0}^{2j} Y_{K0}(\theta,\varphi)\,(-1)^{j-m}\, \sqrt{2K+1} \begin{pmatrix} j & K & j \\ -m & 0 & m \end{pmatrix}^{*} \tag{10}$$

As expected W_{jm} is independent of φ.

This function is plotted in Fig. 1 as a function of $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$ for $j=5$ and $m=0, -1, \ldots -5$. We plot the distribution both as planar and spherical surfaces. If we suppose that $|jm\rangle$ is an orbital angular momentum state, then quantum-mechanically we would expect the angular momentum vector of length $\sqrt{j(j+1)}\,\hbar$ to be oriented inside a sphere of radius $\sqrt{j(j+1)}\,\hbar$ such that its *z* component is $m\hbar$ where $m = -5, \ldots, 5$. This situation is depicted in Fig. 2 [12]. The Wigner function $W(\theta,\varphi)$, when integrated over the domain of spherical angle, $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$, contributes the most positive probability at precisely these locations in θ. At these θ values there is always one peak on the "wavy sea" that is not cancelled by a trough and so contributes a large amount of probability. In Fig. 1 we plot the function $W(\theta,\varphi)$ as a two-dimensional surface, and also the normalized function $\tilde{W} = W/\sqrt{j(j+1)}$ in spherical coordinates so that the oscillations can be viewed as variations in the surface of a sphere of radius one.



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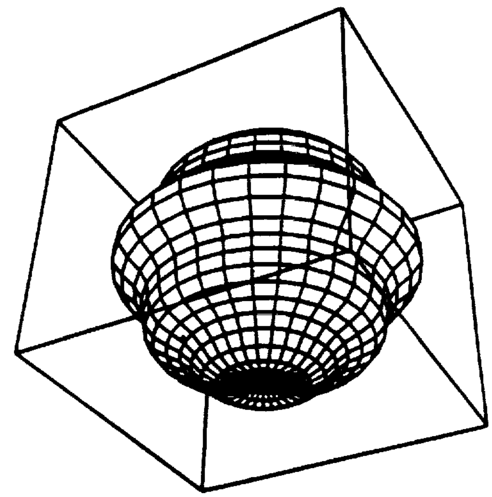

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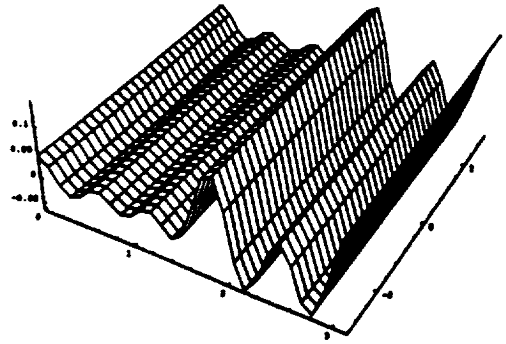

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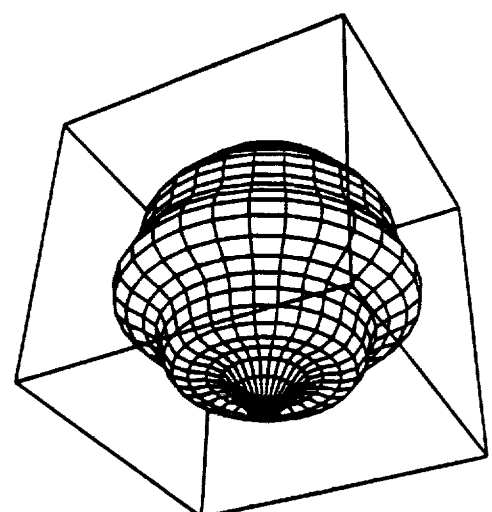

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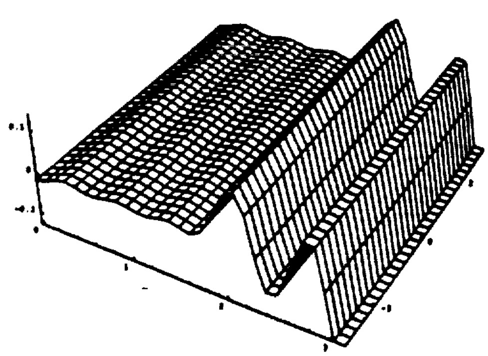

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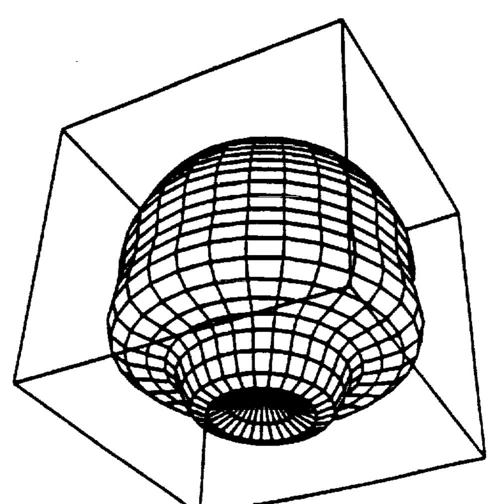

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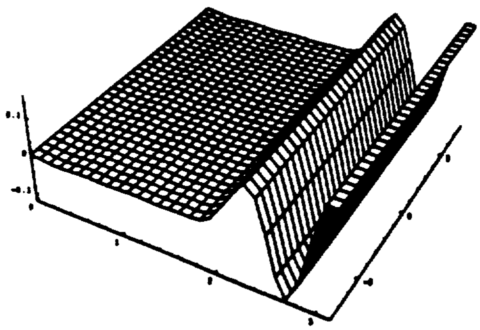

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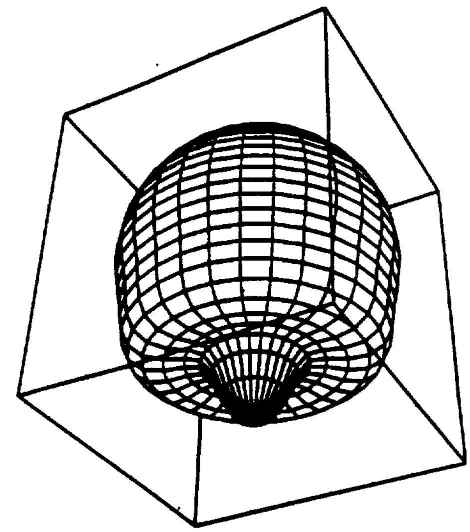

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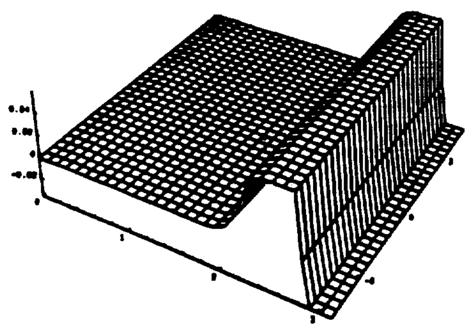

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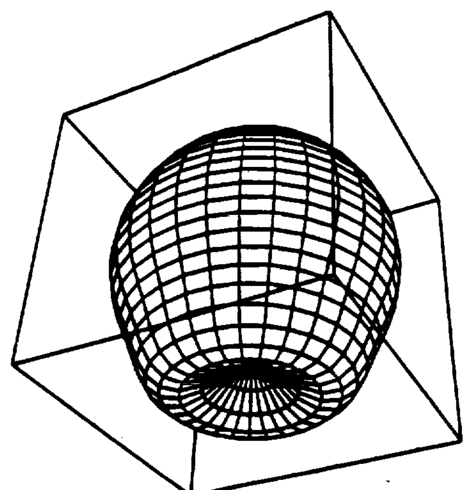

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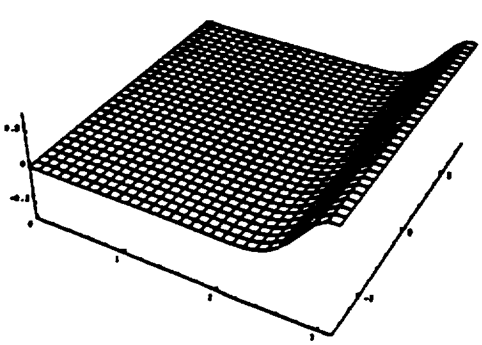

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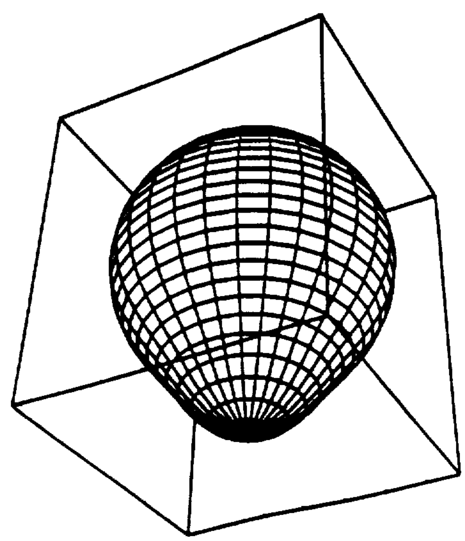


(l)

FIG. 1. Here we plot for $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$, the normalized Wigner function $\tilde{W}^{\rm Fock} = W^{\rm Fock}/\sqrt{j(j+1)}$, where $W^{\rm Fock}(\theta,\varphi)$ is given by Eq. (10). The angular momentum Fock states represented here are $|jm\rangle = |5,m\rangle$ where $m = 0, -1, -2, -3, -4, -5$. When integrated over θ and φ , the Wigner function contributes the most positive probability precisely at the locations where the angular momentum vector for $|jm\rangle$ of length $\sqrt{j(j+1)}\,\hbar$ has z component $m\hbar$ (see Fig. 2). These contributions occur where the dominant positive crest of the Wigner function — the peak that is not cancelled by any troughs — contributes. To bring out all the features of $W(\theta,\varphi)$ we plot it first as a two-dimensional surface function of θ and φ in (a), (c), (e), (g), (i), and (k). This method of presentation brings out the scale of the local positive and negative variations of W with respect to the plane $f(\theta,\varphi) = 0$. Then in (b), (d), (f), (h), (j), and (l), we take a global view by plotting $\tilde{W}(\theta,\varphi) = W(\theta,\varphi)/\sqrt{j(j+1)}$ on a sphere of radius one.

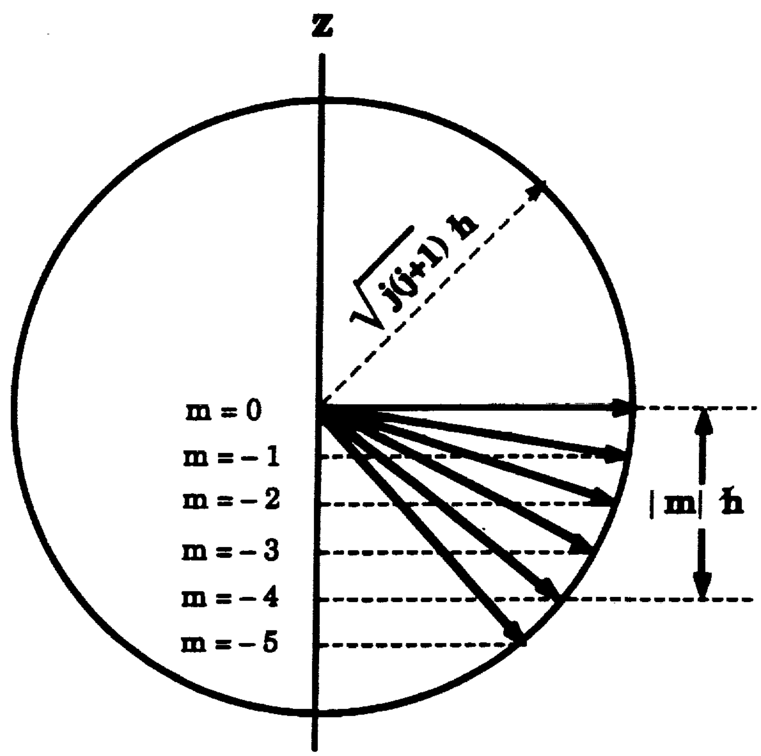


FIG. 2. Here we show a schematic diagram of the angular momentum vector for the Fock states inside a sphere of radius $\sqrt{j(j+1)}\,\hbar$. The vectors all have length $\sqrt{j(j+1)}\,\hbar$ but z component $m\hbar$. These vector locations correspond to the maximal contributions from the Wigner functions shown in Fig. 1. In particular, the Wigner function always has an uncancelled dominant peak at precisely these locations in the angle θ .

3 Atomic Coherent State $|\alpha,\beta\rangle$

We next consider the Wigner function for the atomic coherent state, Eq. (2),

$$\rho = |\alpha,\beta\rangle\langle\alpha,\beta| \ . \tag{11}$$

Using Eqs. (2), (4), and (5), the coefficients of the operator G for the density matrix, Eq. (11), are found to be

$$G_{KQ}^{\text{coherent}} = e^{-iQ\beta}(\tan\alpha/2)^{Q}\sum_{m=-j}^{j}\binom{2j}{j+m}^{1/2}\binom{2j}{j+m+j}^{1/2}(\sin\alpha/2)^{2j+2m}$$

$$\times\ (\cos\alpha/2)^{2j-2m}(-1)^{j-m-Q}(2K+1)^{1/2}\begin{pmatrix} j & j & K \\ m & -m-Q & Q \end{pmatrix} . \tag{12}$$

The Wigner function $W^{\text{coherent}}(\theta,\varphi)$ is then given by Eq. (6) and is plotted in Fig. 3 for $\alpha = \beta = \pi/4$, recalling that α is measured at the south pole. (Again we have normalized $\bar{W} = W/\sqrt{j(j+1)}$.) The coherent state appears as a positive perturbation on the surface of a unit sphere. It is a Gaussian-like distribution located on the sphere's surface at $\theta = 3\pi/4$, $\varphi = \pi/4$; the "Wigner toothache" state. It is just a rotation of the ground Fock state Wigner function from section 2. The Gaussian shape is analogous to that found for the Wigner distribution for coherent states of the single mode radiation field.

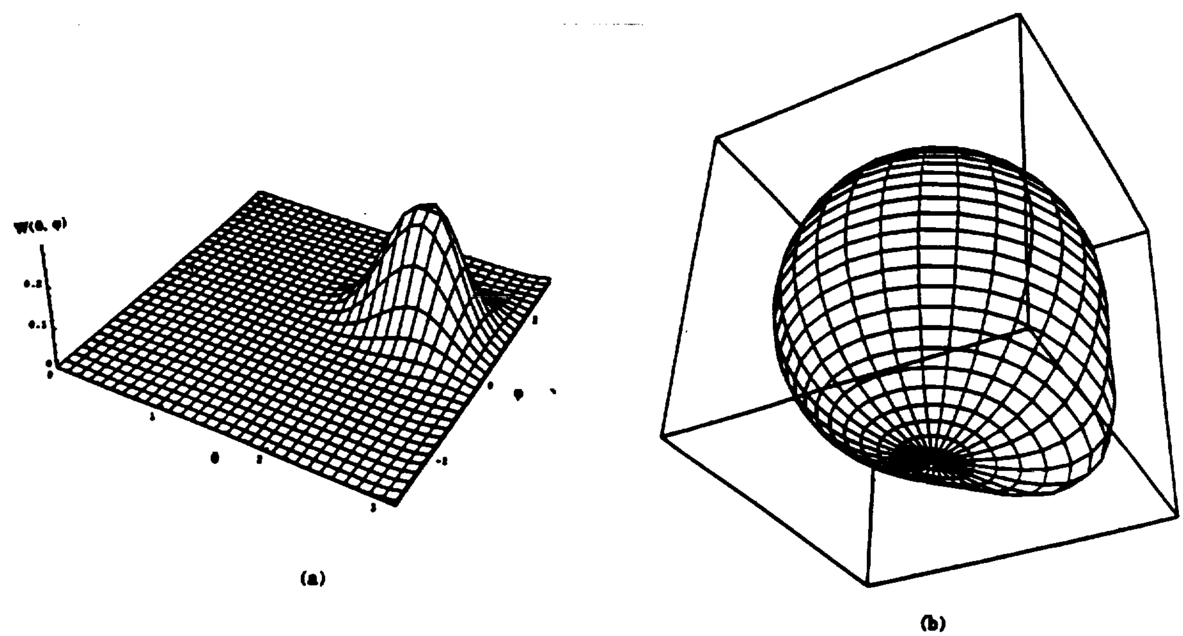


FIG. 3. Here we plot the Wigner distribution $W^{\text{coherent}}/\sqrt{j(j+1)}$ for the coherent state $|\alpha,\beta\rangle$, Eq. (2). We choose the parameters $\alpha=\beta=\pi/4$ that correspond to a Gaussian distribution localized at $\theta = 3\pi/4$, $\varphi = \pi/4$. This distribution is qualitatively similar to that of the coherent state for photons. Again we present a two-dimensional surface view (a) and a spherical coordinate perspective (b).

4 Atomic Squeezed State $|\zeta, m\rangle$

We finally consider the state [12,13] of the angular momentum system defined by

$$|\zeta, m\rangle = \mathcal{A}_m \exp\left(\Theta J_z\right) \exp\left(-i\pi J_y/2\right) |jm\rangle, \tag{13}$$

where $\mathcal{A}_m$ is the normalization constant. For $m = -5$, this state — generated by a non-Hermitian operator — describes the behavior of a collection of two-level atoms interacting with a squeezed coherent photon state. In the state, Eq. (13), the x quadrature, i.e., J_x is squeezed as per Eq. (13),

$$\langle (\Delta J_x)^2 \rangle - \frac{1}{2} \left| \langle J_z \rangle \right| = -\frac{1}{2} \left| \langle J_z \rangle \left(1 - e^{-|\zeta|} \right) \right| , \tag{14}$$

where ζ is defined by

$$e^{2\Theta} = \tanh\,(2|\zeta|) . \tag{15}$$

This relation implies that $\langle (\Delta J_x)^2 \rangle < |\langle J_z \rangle| / 2$ that shows a suppression of x noise, ΔJ_x, in uncertainty relation $\langle (\Delta J_x)^2 \rangle \langle (\Delta J_y)^2 \rangle \geq |\langle J_z \rangle|^2 / 4$ at the expense of the y fluctuations, ΔJ_y. Thus the states of Eq. (13) can be considered as suitable candidates for squeezed states of the general angular momentum system, besides, Agarwal and Puri [13] have shown that the states, Eq. (13), are the eigenstates of the operator $(J^- \cosh |\zeta| + J^+ \sinh |\zeta|)/\sqrt{2 \sinh 2|\zeta|}$ with the eigenvalue m, and that these states are the analog of the two photon coherent states [2] for photons. Note further that Eq. (13) can be written in terms of the elements of the rotation operator coefficients $d^j_{mm'}(\pi/2)$ via the relationship

$$\langle jm | \zeta p \rangle = A_p\, e^{\Theta m}\, d^j_{mp}(\pi/2), \tag{16}$$

where we define

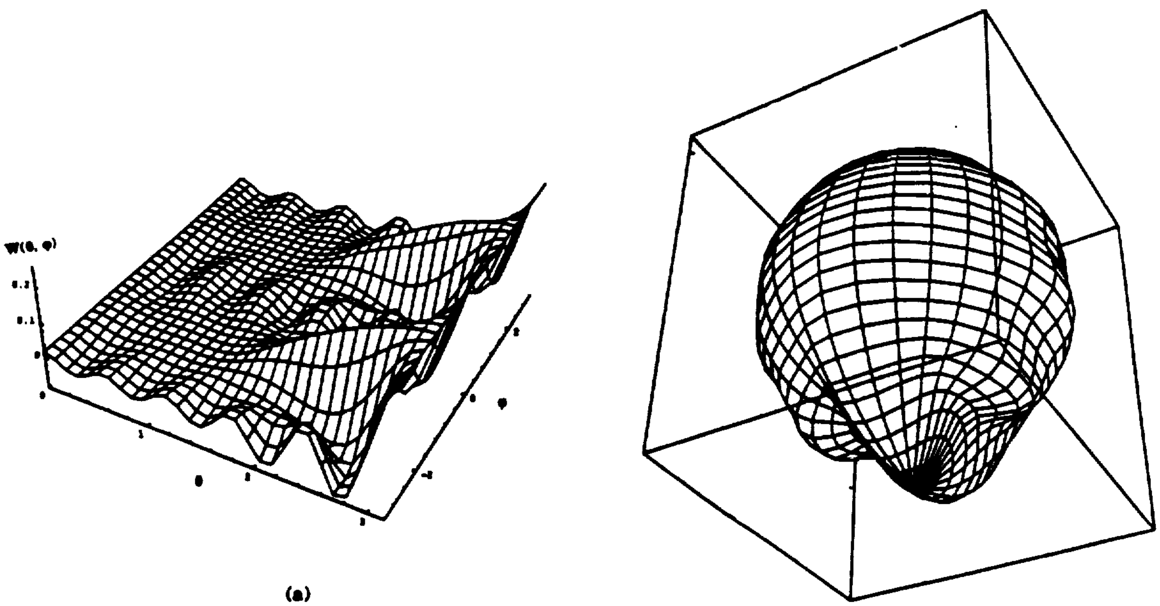
$$d^j_{mp}(\pi/2) = \frac{\left((j+m)!\,(j-m)!\,(j+p)!\,(j-p)! \right)^{1/2}}{2^j} \sum_{q=-j}^{+j} \frac{(-1)^q}{(j-p-q)!\; q!\; (q+p-m)!\; (j+m-q)!} . \tag{17}$$

Upon using Eqs. (13), (16), and (17) for the squeezed state, and Eqs. (4) and (5) for the definition of the Wigner function, we find the coefficients of the squeezed density operator G to be

$$G^{\text{squeezed}}_{KQ} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} (-1)^{j-m} (2K+1)^{1/2} \begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix} \left[\frac{e^{(m+m')\Theta}\, d^j_{mp}\, d^j_{m'p}}{\sum_{m''} \left| d^j_{m''p} \right|^2 e^{2m\Theta}} \right] , \tag{18}$$

where we have also introduced the value of the normalization constant. The Wigner distribution $W^{\text{squeezed}}(\theta, \varphi)$ obtained from Eq. (6), using Eqs. (17) and (18), is plotted in

Fig. 4 for $j = 5$ and $p = -5$. We take the squeezing parameter Θ equal to -2.13×10^{-5} which, in the Agarwal and Puri system of two-level atoms interacting with a squeezed photon bath, corresponds to a mean photon occupation number of $\bar{n} = \sinh^2\left(\frac{1}{2}\operatorname{arctanh}(e^{2\Theta})\right) = 50$ corresponding to $\zeta = 2.65$. The plot is again normalized so that the elongated Gaussian of the squeezed state appears as a "Wigner banana" draped across the surface of sphere of radius one at the south pole. (To see this, one must take the surface in Fig. 4a and mentally map it onto a sphere of radius one, as in Fig. 4b.) Notice that the localization of the state is squeezed in the x direction at the expense of knowledge about the y location. Agarwal and Puri [13] have shown how the atomic states Eq. (13) can be produced if the collection of two-level atoms interacts with a broad band squeezed bath and if one concentrates only on the steady-state solution for the collective system. The parameter ζ characterizes the squeezed bath with average photon number equal to $\sinh^2\zeta$.



(a) (b)

FIG. 4. Here we plot the Wigner function for a squeezed angular momentum state $|\zeta, -5\rangle$ defined by Eq. (13). The function $W^{\text{squeezed}}(\theta,\varphi)$ is computed using Eqs. (6), (17), and (18) for a squeezing parameter of $\Theta = -2.13\times10^{-5}$ corresponding to a mean occupation number of $\bar{n} = 50$. In (a) we plot the function as a surface $W(\theta,\varphi)$ as before. We have normalized the variation in the surface in spherical coordinates to a sphere of radius $\sqrt{j(j+1)}$ in (b) so that the elongated Gaussian appears here as a "Wigner banana" draped across the surface of the sphere of radius one at the south pole. Notice that the squeezed state is more localized in the x' direction than the coherent state, Fig. 3, at the expense of decreased localization or increased noise in the y' direction.

5 Summary and Conclusions

In summary the Wigner distribution for a general angular momentum state has been derived and given explicitly for a Fock state, a coherent state, and a squeezed state. Represented as a pseudo-probability distribution on the sphere of radius one, the Wigner function is plotted for these three situations. These plots enable us to understand the nonclassical nature of the states of a collection of identical two-level atoms since the collection is described by the addition of the spin operators for each atom.

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