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WAVE AND PSEUDO-DIFFUSION EQUATIONS FROM SQUEEZED STATES

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Abstract

We show that the probability distributions $P_n(q,p;y) := |\langle n|p,q;y\rangle|^2$, which are obtained from squeezed states, obey an interesting partial differential equation, to which we give two intuitive interpretations: as a wave equation in one space dimension and also as a pseudo-diffusion equation. We also study the corresponding Wehrl entropies $S_n(y)$, and show that they have minima at zero squeezing, $y = 0$.

1 Introduction

This talk is based mainly on a work which was done in collaboration with Salomon Mizrahi from Brazil.

Squeezed oscillator states are defined in terms of the bosonic creation and annihilation operators, $a^\dagger := \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x})$, and $a := \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x})$, as follows:

$$|z;\xi\rangle = |p,q;\xi\rangle := \mathcal{D}(q,p)\mathcal{S}(\xi)|0\rangle, \quad \text{where} \quad z := (q+ip)/\sqrt{2}\,, \tag{1}$$

and $|0\rangle$ is the ground state of the harmonic oscillator. Both $\mathcal{D}$ and $\mathcal{S}$ are unitary operators. $\mathcal{D}$ creates the coherent state, and is defined by

$$\mathcal{D}(q,p) := \exp[za^\dagger - z^* a] = \exp[ipx - q\frac{\partial}{\partial x}]\,, \tag{2}$$

and $\mathcal{S}(\xi)$ is the **squeezing operator**:

$$\mathcal{S}(\xi) := \exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)]\,, \tag{3}$$

where ξ is a complex variable. For $\xi = 0$, we recover the ordinary (unsqueezed) coherent states. The squeezed states satisfy the completness relation, $\int |p,q;\xi\rangle\langle p,q;\xi|\ \frac{dpdq}{2\pi} = 1$, for every ξ. Therefore,

$$\int P_n(q,p;\xi)\,\frac{dpdq}{2\pi} = 1\,, \quad \text{where} \quad P_n(q,p;\xi) := |\langle p,q;\xi|n\rangle|^2\,, \tag{4}$$

where $|n\rangle$ is the number state. If we interpret the real parameters q and p as the position and momentum variables, then (4) allows us to interpret the non-negative functions P_n as probability distributions in the (q,p)-phase plane, for every n and ξ.

In this talk, I shall consider these P_n for real values of the squeezing parameter ξ, which will be denoted by y. In particular, I shall show that the $P_n(q,p;y)$ satisfy the interesting partial differential equations (9) and (12), to which two intuitive interpretations can be given. Finally, I shall show that the Wehrl entropy $S_n(y)$ (14) of the P_n must have their minima at zero squeezing, $y = 0$.

2 Explicit Form of the Distributions P_n

The distribution $P_n(q,p;\xi) := |\langle n|p,q;\xi\rangle|^2$ gives the probability of finiding n bosons (photons) in the squeezed states $|q,y;\xi\rangle$. It is a physically important quantity, and it has been calculated by different methods. The dependence of $P_n(q,p;\xi)$ on n was studied by Schleich and Wheeler [2]. For $\xi = y$, the P_n is given by the following complicated expression [1,3,7]:

$$P_n(q,p;y) := |\langle p,q;y|n\rangle|^2 = \frac{2\sqrt{\gamma}}{2^n\, n!(\gamma+1)}|\tilde{H}_n(2,\eta;w)|^2 \exp\left[-\frac{q^2+\gamma p^2}{1+\gamma}\right], \quad n \geq 0, \tag{5}$$

where

$$\gamma := e^{2y}, \quad \eta := \frac{1-\gamma}{1+\gamma}, \quad \text{and} \quad w := \frac{q+i\gamma p}{\gamma+1}, \tag{6}$$

and where $\tilde{H}_n(2,\eta;w)$ are the generalized Hermite polynomials ($\mathcal{GHP}$), which are defined in terms of the raising operatores $R(\alpha,\beta;x) = \alpha x - \beta\frac{\partial}{\partial x}$, as follows [1]:

$$\tilde{H}_n(\alpha,\beta;x) = R^n(\alpha,\beta;x)\cdot 1 = \sum_{s=0}^{[n/2]} \frac{n!}{(n-2s)!s!}\left(-\frac{\alpha\beta}{2}\right)^s (\alpha x)^{n-2s}. \tag{7}$$

These polynomials are equal to the standard Hermite polynomials for $\alpha = 2$ and $\beta = 1$. In the limit, $\beta \to 0$, these $\tilde{H}_n(x)$ becomes simple powers of x: $\tilde{H}_n(\alpha,0,x) = \alpha^n x^n$. Therefore, in the limit of zero squeezing, $\gamma \to 1$, we have $\eta \to 0$, so that the above $\mathcal{GHP}$'s become simple powers of w. Thus, for $y \to 0$, equation (5) gives the following well-known Poisson distribution of the unsqueezed coherent states:

$$P_n(q,p;0) = \frac{\rho^{2n}}{2^n n!}\exp\left[-\frac{\rho^2}{2}\right], \quad n \geq 0, \quad \text{where} \quad \rho^2 := q^2+p^2, \tag{8}$$

When discussing probability distributions, it is useful to think of the regions that are surrounded by the **equipotential curves**, $P_n(q,p;y) = const.$; I shall call these regions **potential regions**. Thus, the potential regions of the above Poisson distribution $P_n(q,p;0)$ are concentric circles in the (q,p)-plane. But for $y \neq 0$, these regions will have approximately elliptical shapes, whose the major axes lie along the p-axis for $y < 0$ and along the q-axis for $y > 0$. These regions become more elongated in one direction and narrower in the other, as $|y|$ increases.

3 The Partial Differential Equation for the P_n

Since the integral (4) of the distributions $P_n(q,p;y)$ over the whole (q,p)-space remains constant under squeezing, it is useful to think of the change of $P_n(q,p;y)$ as functions of y as a

redistribution of probability densities in phase space, which maintains the positivity condition $P_n(q,p;y) \geq 0$ for all y. This redistribution of the $P_n(q,p;y)$ is governed by the following interesting and amazingly simple partial differential equation:

$$\frac{\partial}{\partial\gamma}P_n(q,p;y(\gamma)) = \frac{1}{4}\left(\frac{\partial^2}{\partial q^2} - \frac{1}{\gamma^2}\frac{\partial^2}{\partial p^2}\right) P_n(q,p;y(\gamma)) , \quad \text{where} \quad \gamma := e^{2y} . \tag{9}$$

This equation was originally obtained [1] by straightforward but lengthy differentiation of the expression (5) , and by using the following property of the $\mathcal{GHP}$ [1]:

$$\frac{\partial}{\partial\eta}\tilde{H}_n(\alpha,\eta,w) = -\frac{1}{4}\frac{\partial^2}{\partial w^2}\tilde{H}_n(\alpha,\eta,w) . \tag{10}$$

However, we can now derive it by two other more general methods [5], as reported in the summary section.

4 Interpretation as Wave and Pseudo-Diffusion Equations

I shall now present two possible intuitive interpretations of the above differential equation:

(I) **D'Alembert or Wave Equation:** The following is a new interpretation, which was not discussed in [1]: For a *fixed squeezing parameter* y, equation (9) looks like the wave equation for one space dimension q, if we think of the p variable in (9) as the time variable t:

$$\left(\frac{\partial^2}{\partial q^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \Phi(q,t;y) = -4\pi\rho(q,t;y) , \quad \text{where} \quad \rho(q,t;y) = -\frac{1}{\pi}\frac{\partial}{\partial\gamma}P_n(q,t;y(\gamma)) , \tag{11}$$

In this interpretation, the parameter γ would then play the role of the speed of light $c(n)$ in matter, which depends on the parameter y, similar to the dependence of $c(n)$ on the index of refraction index n. If the P_n are thought of as electromagnetic potentials $\Phi(q,t;y)$, then $4\frac{\partial}{\partial\gamma}P_n(q,p;y(\gamma))$ will play the role of a time-dependent charge distributions $-4\pi\rho(q,t;y)$.

(II) **Pseudo-Diffusion Equation:** By substituting $\frac{\partial}{\partial y} = 2e^{2y}\frac{\partial}{\partial\gamma}$ into (9) , we obtain a more symmetric differential equation for the P_n:

$$\frac{\partial}{\partial y}P_n(q,p;y) = \frac{1}{2}\left(e^{2y}\frac{\partial^2}{\partial q^2} - e^{-2y}\frac{\partial^2}{\partial p^2}\right) P_n(q,p;y) . \tag{12}$$

This equation is also new and permits a more pertinent intuitive understanding of the redistribution process of the P_n, by comparing (12) with the diffusion equation in two dimensions [6]:

$$\frac{\partial}{\partial t}T(q,p;t) = \sigma\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right) T(q,p;t) , \tag{13}$$

where σ is the diffusion coefficient. Equations (12) and (13) are similar, if we interpret the squeezing parameter y as the time variable. However, the two equations differ in two interesting aspects:

(1) The sign in front of $\frac{\partial^2}{\partial p^2}$ in (12) is negative rather than positive. Such a *"negative diffusion coefficient"* leads to *" infusion" rather than diffusion in the p-direction.* Consequently, as y increases, the equi-probability curves, $P_n(q,p;y) = const.$, move towards the origin along the p-axis, but away from the origin along the q-axis. Therefore, we expect the probability regions to be concentric elongated "quasi ellipses" which are extended along the p-axis for $y \to -\infty$. They become more and more circular as y approaches zero, and then stretch outwards along the q-axis, as $y \to \infty$. For the above reasons, we shall call equations (9) and (12) **"pseudo diffusion equation"**.

(2) The "diffusion coefficients" $\exp[2y]/2$ and $-\exp[-2y]/2$ and in front of $\frac{\partial^2}{\partial q^2}$ and $\frac{\partial^2}{\partial p^2}$ in (12) depend on y. For $y \to +\infty$, the term $\frac{1}{2}e^{2y}\frac{\partial^2}{\partial q^2}P_n$ dominates the r.h.s. of (12) , whereas for $y \to -\infty$, the second term dominates. This dependence on y can be given an interesting intuitive explanation: Let us consider the redistribution process when y is very large: In this case the probability densities $P_n(q,p;y)$ are extended in the q-direction and tightly squeezed or compressed in the p-axis, which makes it difficult to compress them further along the p-axis. For this reason the "infusion coefficient" becomes so small, namely $\propto \exp[-2y]$. In contrast, the diffusion along the q-axis must become faster and faster, in order to diffuse all the incoming density flux from the other orthogonal p-direction, which is entering the cigar-shaped potential regions through their lengthy boundaries.

5 The Wehrl Entropy for the P_n

A useful measure for the information content of the probability distributions $P_n(q,p;y)$ is the Gibbs or Wehrl entropy [7], which is defined by

$$S_n(y) := -\int P_n(q,p;y) \ln P_n(q,p;y)\, \frac{dpdq}{2\pi}\,. \tag{14}$$

Because of the symmetry $P_n(q,p;-y) = P_n(p,q;y)$, the entropy (14) is even in y: $S_n(-y) = S_n(y)$. Therefore, at $y=0$ each $S_n(y)$ must have either a maximum or a minimum. We shall now argue that $S_n(0)$ should correspond to a minumum: We assume that $S_n(y)$ does not oscillate as a function of y. Therefore, it is enough to argue that $S_n(y)$ grows with $|y|$ for large values of $|y|$. For large positive y, equation (12) behaves essentially like a one-dimensional diffusion equation in the q-variable. But it is well-known that the solutions of diffusion equations lead to entropies which increase with time [6]. Therefore, the $S_n(y)$ must increase as $y \to \infty$. But since the $S_n(y)$ are even in y, they must also grow as $y \to -\infty$. Hence, the $S_n(0)$ must lie at the bottom of the curves $S(y)$ vs. y.

Finally, we note that the von Neumann entropy $S_{vN}(\rho) := -\mathrm{Tr}\,(\rho ln \rho)$ for the pure states $\rho := |n\rangle\langle n|$ must vanisch. In contrast, explicit calculations of the Wehrl entropies of the Poisson distributions (8) shows that $S_n(0) \geq 1$ for all n, in accordance with a conjecture by Wehrl [7], which was proved by Lieb [8].

To summarize this section: in contrast to diffusion equations, where the entropies of their solutions always increase with time, the entropies $S_n(y)$ for the solutions of the above pseudo-diffusion equation first decrease monotonically as y grows from $-\infty$ to zero, but then increase monotonically as y grows from zero to $+\infty$.

6 Summary and Outlook

Two equivalent partial differential equations (9) and (12) were presented and then interpreted, as wave and as pseudo-diffusion equations. The probability densities $P_n(q,p;y)$ (5) provide infinite number of their solutions.

By the time of writing the present lecture notes, we succeeded in proving, by two general methods, that the expectation values $\langle q,p;\xi|O|q,p;\xi\rangle$ of an arbitrary operator O, satisfy a generalized version of the above partial differential equations, which also include rotations, i.e for the general squeezing $\xi = re^{i\phi}$. Interesting examples of O are the number operators N and N^2; their expectation values provide the simplest solutions of (9) and (12) . Also the projection operator $|q,p;\xi\rangle\langle q,p;\xi|$, and consequently its Wigner function, satisfy these equations.

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