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ICASE Fluid Mechanics

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THE THREE-DIMENSIONAL FLOW PAST A RAPIDLY ROTATING CIRCULAR CYLINDER

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ABSTRACT

The high Reynolds number (Re) flow past a rapidly rotating circular cylinder is investigated. The rotation rate of the cylinder is allowed to vary (slightly) along the axis of the cylinder, thereby provoking three-dimensional flow disturbances, which are shown to involve relatively massive ($O(\mathrm{Re})$) velocity perturbations to the flow away from the cylinder surface.

Additionally, three integral conditions, analogous to the single condition determined in two dimensions by Batchelor (**1**), are derived, based on the condition of periodicity in the azimuthal direction.

1   Introduction

One of the most important and fundamental theorems of Fluid Mechanics to have been developed in relatively recent times is the closed streamline theory of Batchelor (1). This states that in a two-dimensional flow involving closed streamlines

$$\oint \operatorname{curl} \omega \cdot d\mathbf{s} = 0, \tag{1}$$

where ω is the vorticity and $d\mathbf{s}$ the line element along a streamline. At large Reynolds numbers, when viscous effects can be neglected, this condition, taken with the inviscid (Euler) equations of motion, leads to the result that the vorticity is constant within a region of closed streamlines.

This theorem was generalised by Blennerhassett (2) to the situation which, rather than being one of closed streamlines, involves helical streamlines. Specifically a three-dimensional flow with three velocity components, all of which are independent of the axial direction; this implies constant axial pressure gradient. This analysis, in addition to confirming Batchelor's (1) result, leads to an additional result linking the constant axial pressure gradient and the viscous terms. This result, taken in conjunction with the inviscid flow equations yields the result that the axial velocity must be proportional to the streamfunction for the motion in the plane normal to the axial vector.

Little progress appears to have been made in further extending the ideas of Batchelor (1) into the three-dimensional regime, in spite of the importance of flows of this type. It is the aim of this paper to address this issue.

Two-dimensional flows involving closed streamlines involve additional interesting subtleties. Even though the work of Batchelor (1) involves the inclusion of (small) viscous effects, there is insufficient flow physics in the constant vorticity result to actually determine its value. This must be determined by recourse to boundary-layer regions located on body surfaces. Riley (3) presented an example of two-dimensional flow inside an elliptic container, driven by slippage of the container walls. By Batchelor's (1) theorem the flow in the core of the container must be that of uniform vorticity, and Riley (3) showed how just one particular value of the vorticity produced the appropriate behaviour (exponential decay at the outer edge) inside the wall boundary layers. These ideas have subsequently been extended by other authors to other situations (see below), including situations involving two wall layers, see Duck (4).

One of the classical results of potential-flow theory is that of uniform flow past a circular cylinder, with superimposed circulation. Physically this circulation may be caused by the rotation of the cylinder. A number of authors have studied the relationship between the rate of rotation and this circulation. Glauert (5) considered the large Reynolds number, large rotation rate problem, whilst Moore (6) considered the finite Reynolds number, large rotation rate problem. Loc (7) and Ingham (8) have considered fully numerical solutions (finite Reynolds numbers, finite rotation rates), whilst Nikolayev (9), Negoda and Sychev (10), Sychev (11), Lam (12) have investigated the large Reynolds number (finite rotation rate) problem, partly using the ideas of Riley (3). There appears to be a monotonic relationship between rotation rate and circulation, and the simple model of outer, potential flow together with a unidirectional boundary layer is only appropriate for rotation rates above some critical

1

value. As this critical value is approached, a stagnation point forms off the surface of the cylinder wall, but inside the boundary layer; this is sufficient to disrupt the entire model at lower values of rotation rate.

In this paper we consider a three-dimensional analogue of the above problem. We take a uniform, straight circular cylinder, and a uniform flow far from the cylinder directed perpendicular to the axis of the cylinder, and the cylinder surface is rotating with a large angular velocity, which is dependent on axial location. We further suppose that the Reynolds number (as defined in Section 2) is large. The restriction on large angular velocity renders the problem tractable to analytic techniques, whilst pointing the way in which more general rotation rates may be tackled. This analysis is presented in Section 3, and the corresponding numerical results are in Section 4. In Section 5 we formulate three integral conditions, analogous to the condition of Batchelor (1) described above. In Section 6 we present our conclusions.

2 Formulation

We take a straight circular cylinder of radius a, together with polar coordinates (ar, θ, az), with r measured radially, θ azimuthally in an anti-clockwise direction and z axially. The surface of the cylinder is rotating with angular velocity $\omega + \gamma \cos \lambda z$, where ω, γ and λ are constants. Referred to the coordinate system described above, we suppose that far from the cylinder the fluid velocity takes the form $U_\infty(\cos \theta, -\sin \theta, 0)$. The flow Reynolds number is defined by

$$\mathrm{Re} = \frac{U_\infty a}{\nu}, \tag{2}$$

where ν is the kinematic viscosity of the fluid (assumed constant). In this paper we are primarily interested in the regime $\mathrm{Re} \gg 1$.

Two further non-dimensional parameters may be defined, namely

$$\Omega = \frac{\omega a}{U_\infty}, \tag{3}$$

and

$$\epsilon = \frac{\gamma a}{U_\infty}, \tag{4}$$

describing the rate of rotation and degree of three-dimensionality introduced into the problem.

To make further progress in this paper, we assume that $|\epsilon| \ll 1$ and further that $|\Omega| \gg 1$ (but that $\epsilon = o(\Omega^{-1})$). Thus we focus our attention on small amounts of three-dimensionality and high rotation rates.

We write the velocity vector as $U_\infty \mathbf{u}$, and the pressure as $\rho U_\infty^2 p$, where ρ is the density of the fluid (assumed constant). We then develop the solution in powers of ϵ, namely

$$\mathbf{u} = \mathbf{u}_0 + \epsilon \tilde{\mathbf{u}} + O(\epsilon^2), \tag{5}$$
$$p = p_0 + \epsilon \tilde{p} + O(\epsilon^2). \tag{6}$$

For $r - 1 = O(1)$, when $\mathrm{Re} \gg 1$ we assume the basic flow takes the form

$$\mathbf{u}_0 = \left(\cos \theta \left(1 - \frac{1}{r^2} \right), -\sin \theta \left(1 + \frac{1}{r^2} \right) + \frac{\Omega}{r}, 0 \right), \tag{7}$$

this being merely the potential flow solution. There also exists a boundary layer wherein $r - 1 = O(\mathrm{Re}^{-1/2})$. The problem for $\mathbf{u}_0$ in the boundary layer has been studied in the past by a number of authors, as noted in the previous section, however as $\Omega \to \infty$,

$$\mathbf{u}_0 \to (0, \Omega, 0) + O(1) \tag{8}$$

for $r - 1 = O(Re^{-1/2})$.

The full problem for $\tilde{\mathbf{u}}$ is

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \tag{9}$$

$$\tilde{\mathbf{u}} \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \tilde{\mathbf{u}} = -\nabla \tilde{p} + \frac{1}{\mathrm{Re}} \nabla^2 \tilde{\mathbf{u}}, \tag{10}$$

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with the following boundary conditions

$$\tilde{\mathbf{u}} = (0, \cos \lambda z, 0) \text{ on } r = 1, \tag{11}$$

$$\tilde{\mathbf{u}} \rightarrow \mathbf{0} \text{ as } r \rightarrow \infty. \tag{12}$$

In the following section we consider the solution of (10) as Ω and $\text{Re} \rightarrow \infty$.

3 The limits Re $\to \infty$, $\Omega \to \infty$

Since the perturbation velocity $\tilde{u}$ is triggered by an $O(1)$ amount (see (11)) it is tempting to speculate that this in turn will lead to general perturbation quantities away from the cylinder of $O(1)$. However this is certainly not the case. It turns out that the only plausible solution for $r = O(1)$, i.e. the only asymptotic development that leads to self-consistency involves magnitudes very much larger than $O(1)$.

Considering first the solution for $r - 1 = O(1)$, then on account of the linearity of the system (9), (10), we may write

$$\tilde{u} = (v(r, \theta) \cos \lambda z, u(r, \theta) \cos \lambda z, w(r, \theta) \sin \lambda z), \tag{13}$$

$$\tilde{p} = p(r, \theta) \cos \lambda z. \tag{14}$$

We also have (see (7)) that as Re $\to \infty$,

$$\mathbf{u}_0 \to (V_{01}, \Omega U_{00} + U_{01}, 0), \tag{15}$$

where

$$
\begin{aligned}
U_{00} &= 1/r, \\
V_{01} &= \left(1 - \frac{1}{r^2}\right) \cos \theta, \\
U_{01} &= -\left(1 + \frac{1}{r^2}\right) \sin \theta.
\end{aligned}
\tag{16}
$$

The only meaningful series development of the perturbation velocities turns out to be

$$
\begin{aligned}
v &= \Omega \operatorname{Re} v_0(r) + \cdots + \operatorname{Re} v_1(r, \theta) + \cdots, \\
u &= \operatorname{Re} u_1(r, \theta) + \cdots + u_2(r) + \cdots, \\
w &= \Omega \operatorname{Re} w_0(r) + \cdots + \operatorname{Re} w_1(r, \theta) + \cdots, \\
p &= \Omega \operatorname{Re} p_0(r, \theta) + \cdots + \Omega p_1(r) + \cdots.
\end{aligned}
\tag{17}
$$

These expansions prove to be the key to the solution of our problem, although *a priori*, it is difficult to justify these expansions, and so *a posteriori* verification is therefore necessary. In Section 5, some *prima facie* justification for this form of solution development is given.

Consider now the leading order terms in (17) that are independent of θ. It turns out, perhaps surprisingly, that the flow is not governed by the inviscid Euler equations, but rather is predominantly viscous in nature. Taking $O(\Omega)$ terms in the radial momentum equation yields the following equation

$$-\frac{2u_2}{r^2} = -\frac{dp_1}{dr} + v_{0rr} + \frac{1}{r}v_{0r} - \lambda^2 v_0 - \frac{v_0}{r^2}. \tag{18}$$

The $O(\mathrm{Re}^{-1})$ terms in the azimuthal momentum equation yield

$$u_{2rr} + \frac{1}{r}u_{2r} - \lambda^2 u_2 - \frac{u_2}{r^2} = 0, \tag{19}$$

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whilst the $O(\Omega)$ terms in the axial momentum equation yield

$$w_{0rr} + \frac{1}{r}w_{0r} - \lambda^2 w_0 = -\lambda p_1. \tag{20}$$

The continuity equation, to leading order, may be written

$$v_{0r} + \frac{1}{r}v_0 + \lambda w_0 = 0. \tag{21}$$

With this system there is no difficulty in imposing the full and correct boundary conditions; we therefore have

$$\begin{align*}
v_0(r = 1) &= w_0(r = 1) = 0, \\
u_2(r = 1) &= 1,
\end{align*} \tag{22}$$

with

$$v_0, w_0, u_2 \to 0 \quad \text{as} \quad r \to \infty. \tag{23}$$

The solution for u_2 can be written in terms of Bessel functions, namely

$$u_2 = \frac{K_1(\lambda r)}{K_1(\lambda)}, \tag{24}$$

whilst it is possible to eliminate w_0 and p_1, from (18), (20), (21) to yield

$$v_0'''' + \frac{2}{r}v_0''' - \left(\frac{3}{r^2} + 2\lambda^2\right)v_0''$$
$$+ \left(\frac{3}{r^3} - \frac{2\lambda^2}{r}\right)v_0' + \left(\lambda^4 + \frac{2\lambda^2}{r^2} - \frac{3}{r^4}\right)v_0 = \frac{2u_2\lambda^2}{r^2}, \tag{25}$$

to which the following additional (alternative) boundary condition is appropriate

$$v_0'(r = 1) = 0. \tag{26}$$

Once u_2 is determined, it is quite straightforward (although a numerical task) to determine $v_0(r)$, $w_0(r)$; numerical results will be presented in the following section. It is worth noting that v_0, u_2 and w_0, as described above, also represent an exact solution of the linearised Navier Stokes equations ((9), (10)) in the absence of the uniform flow.

We now turn to consider the leading order terms in (17) that are dependent upon both r and θ and which are, indeed, determined primarily through inviscid equations. Again, the linear nature of (9), (10) greatly simplifies the solution technique, this time allowing us to write

$$\begin{align*}
v_1 &= \overline{v}_1(r)\sin\theta, \\
u_1 &= \overline{u}_1(r)\cos\theta, \\
w_1 &= \overline{w}_1(r)\sin\theta, \\
p_0 &= \overline{p}_0(r)\cos\theta.
\end{align*} \tag{27}$$

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We may also write

$$V_{01} = \overline{V}_{01}\cos\theta,$$
$$U_{01} = \overline{U}_{01}\sin\theta, \tag{28}$$

with

$$\overline{V}_{01} = \left(1 - \frac{1}{r^2}\right),$$
$$\overline{U}_{01} = -\left(1 + \frac{1}{r^2}\right). \tag{29}$$

Taking terms $O(\Omega\,\mathrm{Re})$ in the radial, azimuthal and axial momentum equations respectively, leads to

$$\overline{V}_{01}\frac{dv_0}{dr} + v_0\frac{d\overline{V}_{01}}{dr} + \frac{\overline{v}_1}{r^2} - \frac{2\overline{u}_1}{r^2} = -\frac{d\overline{p}_0}{dr}, \tag{30}$$

$$v_0\frac{d\overline{U}_{01}}{dr} - \frac{1}{r^2}\overline{u}_1 + \frac{\overline{U}_{01}}{r}v_0 = \frac{1}{r}\overline{p}_0 \tag{31}$$

$$\overline{V}_{01}\frac{dw_0}{dr} + \frac{1}{r^2}\overline{w}_1 = \lambda\overline{p}_0, \tag{32}$$

whilst continuity leads to

$$\frac{1}{r}\frac{d}{dr}(r\overline{v}_1) - \frac{\overline{u}_1}{r} + \lambda\overline{w}_1 = 0. \tag{33}$$

On this system we may impose decay of all components of the solution as $r \to \infty$, whilst on $r = 1$, we may have only

$$\overline{v}_1(r = 1) = 0. \tag{34}$$

The no-slip constraints on $\overline{u}_1$ and $\overline{w}_1$ are therefore violated, but these may be rectified by the inclusion of a thin boundary layer of thickness $O(\Omega^{-\frac{1}{2}}\,\mathrm{Re}^{-\frac{1}{2}})$ on $r = 1$.

Specifically, we write

$$Y = (r - 1)\Omega^{\frac{1}{2}}\,\mathrm{Re}^{\frac{1}{2}} = O(1), \tag{35}$$
$$u_1 = \hat{u}_1(Y)e^{i\theta} + c.c., \tag{36}$$
$$w_1 = \hat{w}_1(Y)e^{i\theta} + c.c., \tag{37}$$

and so

$$i\hat{u}_1 = \hat{u}_{1YY} - i\overline{p}_0(1), \tag{38}$$
$$i\hat{w}_1 = \hat{w}_{1YY} + \lambda\overline{p}_0(1), \tag{39}$$

giving

$$\hat{u}_1 = -\overline{p}_0(1)\left[1 - e^{-(1+i)Y/\sqrt{2}}\right], \tag{40}$$

$$\hat{w}_1 = -i\lambda\overline{p}_0(1)\left[1 - e^{-(1+i)Y/\sqrt{2}}\right]. \tag{41}$$

We see therefore, that on the $r - 1 = O(1)$ scale, intriguingly both viscous and inviscid effects are important simultaneously, and to a large degree, independently. In the following section we consider a number of numerical results arising from the results of this section, and go on to consider the limits $\lambda \to \infty$ and $\lambda \to 0$.

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4 Results and large/small λ behaviour

The system (22), (25), (30)-(34) was solved using a conventional fourth order Runge-Kutta method. Results are shown in figure 1 for $\bar{p}_0(r = 1)$ (- - - -), $p_1(r = 1)$ (solidus), $u_{2r}(r = 1)(- \cdot - \cdot -)$, where the dependence of these quantities with λ is shown.

We may make some further analytic progress by considering the limits of large and small λ. Taking first $\lambda \to \infty$, then by (24)

$$u_2 \to e^{-\hat{Y}}, \tag{42}$$

where

$$\hat{Y} = \lambda(r - 1) = O(1), \tag{43}$$

i.e.

$$u_{2r}|_{r=1} \to -\lambda. \tag{44}$$

The above implies the perturbation to the flow is confined to within a thin $r - 1 = O(\lambda^{-1})$ boundary layer (although we must impose the restriction that $\lambda = o(\Omega^{\frac{1}{2}} Re^{\frac{1}{2}})$ in order that this layer remains outside the $Y = O(1)$ layer discussed in the previous section). After some algebra, it is possible to show that

$$v_0 = \frac{1}{4\lambda^2}\hat{Y}^2 e^{-\hat{Y}} + O(\lambda^{-3}), \tag{45}$$

$$w_0 = \frac{e^{-\hat{Y}}}{\lambda^2}\left(\frac{1}{4}\hat{Y}^2 - \frac{1}{2}\hat{Y}\right) + O(\lambda^{-3}), \tag{46}$$

$$p_1 = \frac{-e^{-\hat{Y}}}{\lambda}\left(\frac{3}{2} - \hat{Y}\right) + O(\lambda^{-2}), \tag{47}$$

$$\bar{p}_0 = \frac{e^{-Y}}{\lambda^3}\left(\frac{1}{8}\hat{Y}^4 + \frac{1}{12}\hat{Y}^3 - \frac{5}{8}\hat{Y}^2 - \frac{5}{8}\hat{Y}\right) + O(\lambda^{-4}). \tag{48}$$

Comparison of (47) evaluated on $\hat{Y} = 0$ with the corresponding numerical results of figure 1 shows good agreement as λ increases. Unfortunately it is not possible to compare $\bar{p}_0(r = 1)$ with $\bar{p}_0(\hat{Y} = 0)$ without further substantial algebra since $\bar{p}_0(Y = 0) = O(\lambda^{-4})$.

The alternative limit of $\lambda \to 0$ is slightly more complicated, because two key radial lengthscales emerge. For $r = O(1)$, we have that (24) reduces to

$$u_2 \sim \frac{1}{r}, \tag{49}$$

which interestingly is the azimuthal velocity component corresponding to a line vortex, i.e. effectively the two-dimensional result. There is also a large radial scale, $R = \lambda r = O(1)$. (We do have the restriction that $\lambda = o(\Omega)$ in order that over the lengthscales under consideration, U_{00} remains dominant over U_{01} in (16)).

For $r = O(1)$, the solution develops in the form

$$v_0 = \lambda^2\left[\log \lambda \hat{v}_0(r) + O(1)\right], \tag{50}$$

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with the leading order general solution taking the form

$$\hat{v}_0 = Ar + Br\log r + Cr^3 + \frac{D}{r}. \tag{51}$$

For $R = O(1)$, we must consider the first two leading terms in the solution development, namely

$$v_0 = \lambda\left[\log\lambda\tilde{v}_0(R) + \tilde{v}_1(R) + O((\log\lambda)^{-1})\right]. \tag{52}$$

The complete analytic solution for $\tilde{v}_0(R)$ and $\tilde{v}_1(R)$ does not appear possible, however it suffices to consider the limit $R \to 0$, for which

$$\tilde{v}_0(R) \sim A_0 R + B_0 R\log R + C_0 R^3 + \frac{D_0}{R}, \tag{53}$$

whilst $\tilde{v}_1(R)$ (which is forced directly by the u_2 term on the right-hand-side of (25)) takes the form

$$\begin{aligned}
\tilde{v}_1(R) &\sim A_1 R + B_1 R\log R + C_1 R^3 \\
&+ \frac{D_1}{R} - \frac{1}{4}R(\log R)^2.
\end{aligned} \tag{54}$$

Boundedness constraints demand

$$C = D_0 = D_1 = 0, \tag{55}$$

whilst matching of (49) as $r \to \infty$ with (51) as $R \to 0$ requires

$$B_0 = \frac{1}{4}, \tag{56}$$

$$B = -\frac{1}{4}. \tag{57}$$

Further imposing the two boundary conditions on $r = 1$ yields

$$\hat{v}_0 = \frac{1}{8}r - \frac{1}{4}r\log r - \frac{1}{8r}. \tag{58}$$

This then leads to

$$w_0 = \frac{1}{2}\lambda\log\lambda\log r + O(\lambda), \tag{59}$$

$$p_1 = -1/r^2. \tag{60}$$

Fortunately it is possible to determine the leading-order term of p_1 without substantial algebra (and additional numerical effort), even though p_1 is an order lower in $\log\lambda$ than may first be expected; equation (4.19) yields the result that $p_1(r = 1) \to -1$ as $\lambda \to 0$, a result that agrees with our numerical results. However it is not possible to determine $\bar{p}_0(r = 1)$ without further substantial algebra and numerical effort.

In the following section we go on to consider results for more general classes of two-dimensional flows involving closed streamlines, which are perturbed in some three-dimensional manner.

5 The periodicity requirement and the associated integral conditions

The result of Batchelor (1) may be viewed as arising from a condition of periodicity within the region of closed streamlines of the various physical flow quantities. It is needed because the Euler equations are, themselves, not sufficient to enforce periodicity; an alternative viewpoint is that the Euler equations are unable to capture the mean flow physics, i.e. the flow corresponding to zero wavenumber, a result that is little surprising. This point is equally important in the three-dimensional context, and is now investigated.

We may write (9), (10) in the form

$$\mathbf{u}_0 \wedge \tilde{\omega} + \tilde{\mathbf{u}} \wedge \omega_0 = \nabla \tilde{H} + \mathrm{Re}^{-1} \nabla \wedge \tilde{\omega}, \tag{61}$$

where $\tilde{H}$ is the $O(\epsilon)$ total lead, i.e.

$$\tilde{H} = \tilde{p} + \mathbf{u}_0 \cdot \tilde{\mathbf{u}}, \tag{62}$$

and we have written the vorticity vector

$$\omega = \omega_0 + \epsilon \tilde{\omega} + O(\epsilon^2). \tag{63}$$

Following Batchelor (1) and Blennerhassett (2) the work of this section is most efficiently carried out in terms of a coordinate system based on the two-dimensional , undisturbed streamfunction, specifically in terms of (ψ, ξ, z). Here ψ represents the streamfunction of the undisturbed ($\mathbf{u}_0$) flow, and ξ is orthogonal to ψ and z; z remains the axial coordinate. If the undisturbed flow $\mathbf{u}_0$ is irrotational, then ξ may be taken to be the standard velocity potential ϕ. The infinitesimal line element is $\left(\dfrac{d\psi}{q_0}, h_2 d\xi, dz \right)$. Here h_2 is the ξ coordinate metric, which in the case of an irrotational $\mathbf{u}_0$ is merely $1/q_0$, and $q_0 = |\mathbf{u}_0|$.

Referred to the (ψ, ξ, z) coordinate system, the velocity vector may be written

$$\mathbf{u} = (\epsilon v \cos \lambda z, q_0 + \epsilon u \cos \lambda z, \epsilon w \sin \lambda z) + O(\epsilon^2), \tag{64}$$

and the pressure as

$$p = p_0 + \epsilon p_1 \cos \lambda z + O(\epsilon^2). \tag{65}$$

Referred to the (ψ, ξ, z) coordinate system, we may write the vorticity terms as follows

$$\begin{aligned} \omega_0 &= (0, 0, \omega_0) \\ &= (0, 0, \frac{q_0}{h_2} \frac{\partial}{\partial \psi}(h_2 q_0)), \end{aligned} \tag{66}$$

$$\begin{aligned} \tilde{\omega} &= (\omega_1 \sin \lambda z, \omega_2 \sin \lambda z, \omega_3 \cos \lambda z) \\ &= \left(\frac{\sin \lambda z}{h_2}(w_\xi + \lambda u h_2), \sin \lambda z(-\lambda v - q_0 w_\psi) , \right. \\ &\quad \left. \frac{q_0 \cos \lambda z}{h_2} \left[\frac{\partial}{\partial \psi}(u h_2) - \frac{\partial}{\partial \xi}\left(\frac{v}{q_0} \right) \right] \right). \end{aligned} \tag{67}$$

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We may then write the three components of (61) in the following form

$$
\frac{u\omega_0}{q_0^2} + \frac{uh_{2\psi}}{h_2} - \frac{q_{0\psi}u}{q_0} - \frac{1}{h_2}\frac{\partial}{\partial\xi}\left(\frac{v}{q_0}\right)
$$

$$
= \frac{1}{q_0}\frac{\partial\tilde{p}}{\partial\psi} + \frac{\mathrm{Re}^{-1}}{h_2 q_0^2}\left[\frac{\partial\omega_3}{\partial\xi} - \lambda h_2\omega_2\right], \tag{68}
$$

$$
-v\omega_0 = \frac{1}{h_2}\frac{\partial\tilde{p}}{\partial\xi} + \frac{1}{h_2}\frac{\partial}{\partial\xi}(q_0 u)
$$

$$
+ \ \mathrm{Re}^{-1}\left[\lambda\omega_1 - q_0\frac{\partial\omega_3}{\partial\psi}\right] \tag{69}
$$

$$
-\frac{w_\xi}{h_2} = -\frac{\lambda\tilde{p}}{q_0} + \frac{\mathrm{Re}^{-1}}{h_2}\left[\frac{\partial}{\partial\psi}(h_2\omega_2) - \frac{\partial}{\partial\xi}\left(\frac{\omega_1}{q_0}\right)\right]. \tag{70}
$$

The continuity equation, in terms of these variables is

$$
\frac{\partial}{\partial\psi}(h_2 v) + \frac{\partial}{\partial\xi}\left(\frac{u}{q_0}\right) + \frac{\lambda w h_2}{q_0} = 0. \tag{71}
$$

If we integrate each of (68)–(71) around a complete circuit in ξ (lying entirely within a region of closed streamlines), then periodicity of the flow demands

$$
\oint\left\{h_2\frac{u\omega_0}{q_0^2} + uh_{2\psi} - q_{0\psi}\frac{uh_2}{q_0} - \frac{h_2}{q_0}\frac{\partial\tilde{p}}{\partial\psi}\right.
$$

$$
\left. - \ \frac{\mathrm{Re}^{-1}}{q_0^2}\left[\frac{\partial\omega_3}{\partial\xi} - \lambda h_2\omega_2\right]\right\} d\xi = 0, \tag{72}
$$

$$
\oint\left\{v\omega_0 + \mathrm{Re}^{-1}\left[\lambda\omega_1 - q_0\frac{\partial\omega_3}{\partial\psi}\right]\right\} h_2 d\xi = 0, \tag{73}
$$

$$
\oint\left\{\frac{-\lambda\tilde{p}}{q_0} + \frac{\mathrm{Re}^{-1}}{h_2}\frac{\partial}{\partial\psi}(h_2\omega_2)\right\} h_2 d\xi = 0. \tag{74}
$$

These results are quite general, and indeed exact within the framework of the linearised Navier Stokes equations (9), (10). If we now direct our attention on the problem considered earlier in this paper, then $\mathbf{u_0}$ is irrotational, and so in line with our comments regarding ξ above, we can write

$$
\oint\frac{1}{q_0^2}\left\{2q_{0\psi}u + \frac{\partial\tilde{p}}{\partial\psi} + \mathrm{Re}^{-1}\left[\frac{\partial\omega_3}{\partial\xi} - \frac{\lambda\omega_2}{q_0}\right]\right\} d\xi = 0, \tag{75}
$$

$$
\oint\left[\frac{\lambda\omega_1}{q_0} - \frac{\partial\omega_3}{\partial\psi}\right] d\xi = 0, \tag{76}
$$

$$
\oint\left\{\frac{-\lambda\tilde{p}}{q_0^2} + \mathrm{Re}^{-1}\frac{\partial}{\partial\psi}(h_2\omega_2)\right\} d\xi = 0. \tag{77}
$$

Note that the result of Batchelor (1) is retrieved from (76) by allowing $\lambda \to 0$.

At this stage we can partly confirm the solution development of Section 3. Equation (77) above strongly suggests that $\omega_2 = O$ (Re), which in turn suggests that v and/or w must also be $O(Re)$. This is partly confirmed by (75). Indeed, it can be shown that our solutions given in Section 3 do satisfy the above conditions, which clearly illustrate the importance of viscosity on the $r - 1 = O(1)$ scale.

6 Conclusions

In this paper we have considered the effects of three-dimensionality introduced into the problem of a rapidly rotating circular cylinder in a uniform flow. The effects include a substantial ($O(\text{Re})$) response in the three velocity components, in the bulk of the fluid. This effect is confirmed by the analysis of Section 5 in which three integral conditions were derived (analogous to the solitary integral condition in Batchelor's (1) two-dimensional work). All the indications are that this massive response with three-dimensionality will be a generic feature of similar flows. It is likely that the results of this paper have repercussions for important practical applications, perhaps the most important of which being that of high lift aerofoils.

This paper has deliberately focused on the large rotation rate problem (i.e. $\Omega \to \infty$). However the extremely important regime of $\Omega = O(1)$ remains to be studied. This regime involves a number of additional questions, perhaps the key aspect being the nature of the solution as the closed streamline region of the base (two-dimensional) flow is exited. It could well be that some form of mild discontinuity exists, for which a thin shear layer would be required. This whole problem would be a non-trivial numerical undertaking, but it is to be expected that the $O(\text{Re})$ velocity scales will persist. Equally we have implicitly assumed here that $\Omega \neq O(\text{Re}^{\frac{1}{n}})$, n positive integer, although if this condition is relaxed, it seems likely that the solution will be modified, but in a relatively minor manner.

The $\epsilon = O(1)$ problem, at this stage, would appear to be a formidable task, quite possibly involving a solution of the full Navier Stokes equations. Finally, although our study has concentrated on sinusoidal disturbances, in the axial direction, it is of course straightforward to extend our ideas to non-periodic axial disturbances using standard transform techniques.

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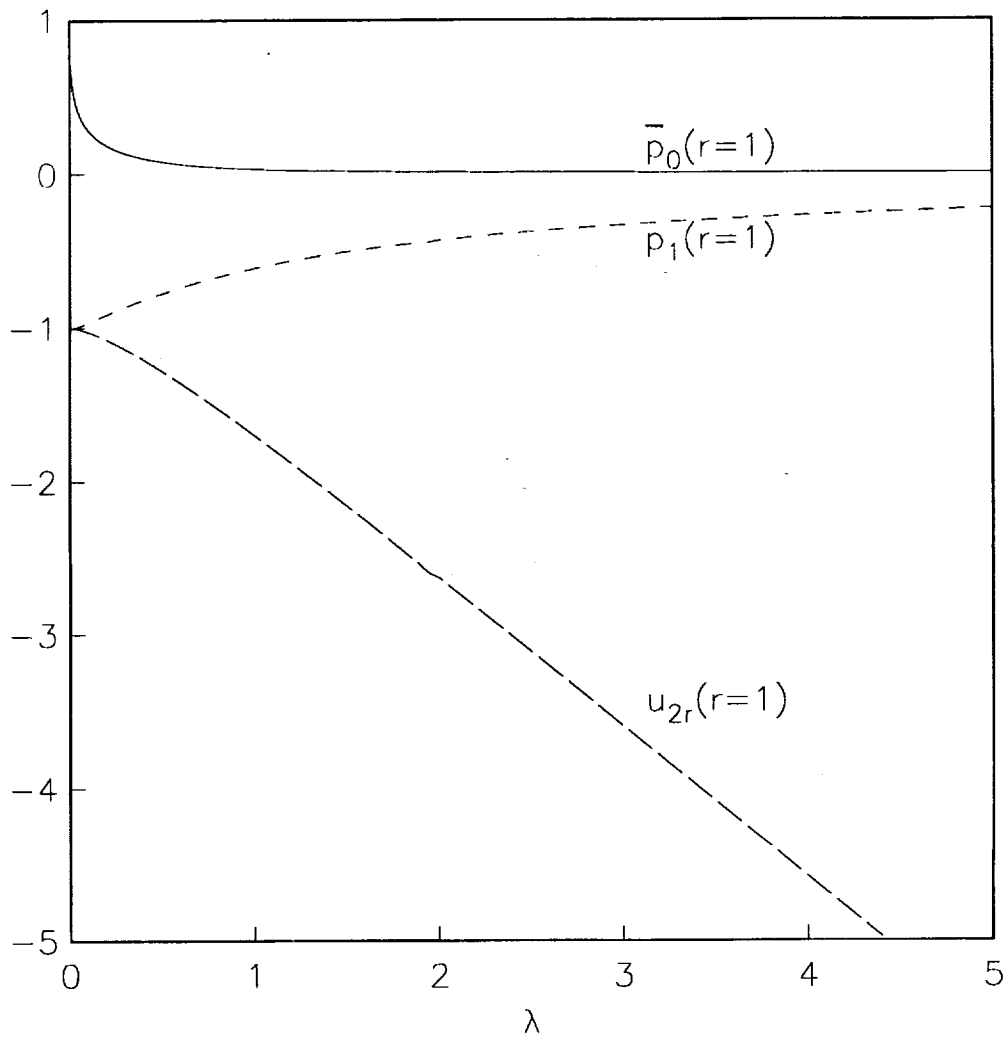**Figure 1** Variation of $\bar{p}_0(r = 1)$, $p_1(r = 1)$ and $u_{2r}(r = 1)$ with λ.

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The high Reynolds number (Re) flow past a rapidly rotating circular cylinder is in-
vestigated. The rotation rate of the cylinder is allowed to vary (slightly) along
the axis of the cylinder, thereby provoking three-dimensional flow disturbances,
which are shown to involve relatively massive (O(Re)) velocity perturbations to the
flow away from the cylinder surface.

Additionally, three integral conditions, analogous to the single condition determined
in two dimensions by Batchelor (1), are derived, based on the condition of period-
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