IN-34
181213
P-27

NASA Technical Memorandum 108993

A Priori Error Estimates for an *hp*-Version of the Discontinuous Galerkin Method for Hyperbolic Conservation Laws

Kim S. Bey
J. Tinsley Oden

(NASA-TM-108993) A PRIORI ERROR ESTIMATES FOR AN hp-VERSION OF THE DISCONTINUOUS GALERKIN METHOD FOR HYPERBOLIC CONSERVATION LAWS (NASA) 27 p

N94-13467

Unclas

G3/34 0181213

June 1993

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-0001

A Priori Error Estimates for an *hp*-Version of the Discontinuous Galerkin Method for Hyperbolic Conservation Laws

K.S. Bey
NASA Langley Research Center
Hampton, VA 23681

J.T. Oden
The Texas Institute for Computational Mechanics
University of Texas at Austin
Austin, TX 78712

June 1993

Abstract

A priori error estimates are derived for *hp*-versions of the finite element method for discontinuous Galerkin approximations of a model class of linear, scalar, first-order hyperbolic conservation laws. These estimates are derived in a mesh-dependent norm in which the coefficients depend upon both the local mesh size h_K and a number p_K which can be identified with the spectral order of the local approximations over each element. The results generalize those of Johnson and Pitkaranta to *hp*-methods.

1 Introduction

The discontinuous Galerkin method has received renewed interest as a higher-order scheme for approximating solutions to hyperbolic conservation laws. The method can be interpreted as a natural higher-order extension of finite volume methods while overcoming some of the difficulties associated with standard Galerkin or spectral methods. Originally studied for linear problems with constant coefficients and fixed-order approximations by Lesaint and Raviart [1] and for linear problems with variable coefficients by Johnson and his collaborators

[2, 3], the method can be regarded as an elementwise application of the standard Galerkin method in which jumps on element boundaries are admitted naturally in the formulation. Since the usual condition of continuity of the solution along inter-element boundaries is not enforced, the element equations are decoupled, thereby eliminating the need for solving large systems of algebraic equations, even as the polynomial degree of the approximation increases. The solution in neighboring elements is mildly coupled through the flux across element boundaries. These element boundary fluxes are approximated using a numerical flux function which incorporates the hyperbolic character of the conservation law in much the same way as the finite volume method. The significant difference between the discontinuous Galerkin method and higher-order finite volume methods is that the coefficients in the polynomial approximation of the solution within an element are obtained by solving the conservation law and not by some post-processing of solution mean values.

As with any higher-order method, special treatment is required to prevent oscillations in solutions which contain steep gradients or discontinuities. In the work of Cockburn, Hou, and Shu [4], the discontinuous Galerkin method was shown to be TVB (Total Variation Bounded) in the solution mean values, provided that the solution satisfied certain conditions. These conditions were used to construct a projection strategy for controlling oscillations in the mean values. Unfortunately, these conditions are not sufficient to eliminate oscillations in the pointwise values of the solution. Extensions of the projection ideas to the entire solution can be found in Bey and Oden [5] and Flaherty et. al. [6]. The numerical experiments of Bey and Oden [5] showed that with the discontinuous formulation, oscillations are confined to elements containing the discontinuity (in contrast to global oscillations resulting from the standard Galerkin or spectral methods) and that the order of accuracy of the method is $p+1$ in smooth regions when using uniform meshes with polynomial approximations of degree p.

The discontinuous Galerkin method is ideally suited to adaptive *hp* strategies and to parallel computing. An *a priori* estimate of the error in the solution provides a basis for an adaptive strategy since one then knows how the error behaves as a function of the mesh size and the degree of the polynomial approximation in an element. In this note, we derive an *a priori* error estimate for an *hp* version of the discontinuous Galerkin method. This estimate extends the previous work of Johnson and Pitkaranta [2] who analyzed an h version of the method with a fixed polynomial degree.

2 Model Problem

For simplicity, we consider a convex polygonal domain Ω. The domain boundary $\partial\Omega$ with an outward unit normal vector $\mathbf{n}(x)$ consists of two parts: an inflow boundary Γ_- to be defined below and an "outflow" boundary $\Gamma_+ = \partial\Omega \setminus \Gamma_-$. We consider the following linear scalar hyperbolic model problem,

$$u_\beta + au = f \quad \text{in } \Omega \subset \mathcal{R}^2 \tag{1}$$

$$\boldsymbol{\beta} \cdot \mathbf{n}\, u = \boldsymbol{\beta} \cdot \mathbf{n}\, g \quad \text{on } \Gamma_- \tag{2}$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_-)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ is a constant unit vector, $u_\beta = \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u$, $a = a(\mathbf{x})$ is a bounded measurable function on Ω such that $0 < a_0 \leq a(\mathbf{x})$, and $\Gamma_- = \{\mathbf{x} \in \partial\Omega | \boldsymbol{\beta} \cdot \mathbf{n}(x) < 0\}$. Note that while solutions to (1) may be discontinuous across characteristic lines $\mathbf{x}(s)$ defined by $\frac{\partial \mathbf{x}}{\partial s} = \boldsymbol{\beta}$, the solution is continuous in the direction parallel to $\boldsymbol{\beta}$.

For $f \in L^2(\Omega)$, the space of admissible functions for solutions to (1) is given by $V(\Omega) = \{v \in L^2(\Omega) | v_\beta \in L^2(\Omega)\}$. We note that the trace of functions in $V(\Omega)$ exist only in the direction $\boldsymbol{\beta}$; therefore, we further restrict Ω so that $\boldsymbol{\beta} \cdot \mathbf{n} \neq 0$ on $\partial\Omega$. If u is a solution to (1) with boundary values satisfying (2), then u also satisfies the following variational equality:

$$\int_\Omega (u_\beta + au) v dx + \int_{\Gamma_-} uv |\boldsymbol{\beta} \cdot \mathbf{n}| ds = \int_\Omega fv + \int_{\Gamma_-} gv |\boldsymbol{\beta} \cdot \mathbf{n}| ds \quad \forall v \in V(\Omega) \tag{3}$$

3 Notation and Preliminaries

For a domain D in $\mathcal{R}^2$, let $(v,w)_D = \int_D vw d\mathbf{x}$ and $||v||_D^2 = (v,v)_D$. Let $||\cdot||_{m,D}$ denote the norm in the usual Sobolev space $H^m(D)$.

The starting point for the discontinuous Galerkin method is (3) defined on a partition of Ω. Let $\mathcal{P}_h$ denote a partitioning of Ω into $N_K = N_K(\mathcal{P}_h)$ subdomains K with boundaries ∂K such that

(i) $N_K(\mathcal{P}_h) < \infty$

(ii) $\bar{\Omega} = \bigcup\{\bar{K} : K \in \mathcal{P}_h\}$

(iii) For any pair of elements $K, L \in \mathcal{P}_h$ such that $K \neq L$, $K \cap L = \emptyset$

(iv) K are Lipschitzian domains with piecewise smooth boundaries

(v) $\partial K_- = \{\mathbf{x} \in \partial K |\ \boldsymbol{\beta} \cdot \mathbf{n}_K < 0\}$ and $\partial K_+ = \partial K \setminus \partial K_-$

(vi) $\Gamma_-^h = \bigcup_{K=1}^{N_K} \partial K \cap \Gamma_-$ coincides with Γ_- for every $h > 0$

(vii) $\Gamma_{KL} = \partial K \cap \partial L$ is an entire edge of both K and L

Let $V(K) = \{v \in L^2(K) |\ v_\beta \in L^2(K)\}$, then $V(\mathcal{P}_h) = \prod_{K=1}^{N_K} V(K)$. Note that a function $v \in V(\mathcal{P}_h)$ need not be continuous across element interfaces. We use the following notations concerning functions $v, w \in V(\mathcal{P}_h)$:

$$\begin{aligned} v^{\text{int } K} &= v|_K(\mathbf{x}), \quad \mathbf{x} \in \partial K \\ v^{\text{ext } K} &= v|_L(\mathbf{x}), \quad \mathbf{x} \in \partial K \cap \partial L \\ v^{\pm} &= \lim_{\epsilon \to 0} v(\mathbf{x} \pm \epsilon \boldsymbol{\beta}) \end{aligned}$$

$$\langle v, w\rangle_\gamma = \int_\gamma vw|\boldsymbol{\beta}\cdot\mathbf{n}_e|\, ds \quad \gamma \subset \partial K$$

$$\langle\langle v\rangle\rangle_\gamma^2 = \langle v, v\rangle_\gamma$$

$$||v||_\Omega^2 = \sum_{K=1}^{N_K} ||v||_K^2 = \sum_{K=1}^{N_K}\int_K v^2\, d\mathbf{x}$$

The problem corresponding to (1)-(2) defined on any partition $\mathcal{P}_h$ is as follows:

Find $u(\mathbf{x}) \in V(\mathcal{P}_h)$ such that for every $K \in \mathcal{P}_h$

$$\left.\begin{aligned} u_\beta + au &= f \quad \text{in } K \\ u^{\text{int } K}\boldsymbol{\beta}\cdot\mathbf{n}_K &= u^{\text{ext } K}\boldsymbol{\beta}\cdot\mathbf{n}_K \quad \forall\mathbf{x}\in\partial K\setminus\partial\Omega \\ u^{\text{int } K}\boldsymbol{\beta}\cdot\mathbf{n}_K &= g\,\boldsymbol{\beta}\cdot\mathbf{n}_K \quad \forall\mathbf{x}\in\partial K\cap\Gamma_- \end{aligned}\right\} \quad \mathbf{P1}$$

Let **P2** denote the following variational boundary value problem for any partition $\mathcal{P}_h$:

$$\left.\begin{aligned} \text{Find } u(\mathbf{x}) &\in V(\mathcal{P}_h) \text{ such that} \\ B(u,v) &= \mathcal{L}(v)\ \forall\, v\in V(\mathcal{P}_h) \end{aligned}\right\} \quad \mathbf{P2}$$

where

$$B(u,v) = \sum_{K=1}^{N_K}\{(u_\beta + au, v)_K + \langle u^+ - u^-, v^+\rangle_{\partial K_-\setminus\Gamma_-} + \langle u, v\rangle_{\partial K_-\cap\Gamma_-}\} \tag{4}$$

$$\mathcal{L}(v) = \sum_{K=1}^{N_K}\{(f,v)_K + \langle g, v\rangle_{\partial K_-\cap\Gamma_-}\} \tag{5}$$

Note that with the definition of $V(\mathcal{P}_h)$ we can apply Green's formula to (4) to obtain

$$B(u,v) = \sum_{K=1}^{N_K}\{(u, -v_\beta + av)_K + \langle u^-, v^- - v^+\rangle_{\partial K_-\setminus\Gamma_-} + \langle u, v\rangle_{\partial K_+\cap\Gamma_+}\} \tag{6}$$

Lemma 1 *Let the bilinear form $B(\cdot,\cdot)$ be defined by (4). Then there exist positive constants α and M such that*

$$B(v,v) \geq \alpha |||v|||_B^2 \quad \forall v \in V(\mathcal{P}_h) \tag{7}$$

$$B(v,w) \leq M|||v|||_\beta \, |||w|||_\beta \quad \forall v,w \in V(\mathcal{P}_h) \tag{8}$$

where

$$|||v|||_B^2 \stackrel{\text{def}}{=} \sum_{K=1}^{N_K} \{||v||_K^2 + \langle\langle v^+ - v^- \rangle\rangle_{\partial K_- \backslash \Gamma_-}^2 + \langle\langle v \rangle\rangle_{\partial K \cap \partial\Omega}^2\} \tag{9}$$

$$\begin{aligned} |||v|||_\beta^2 &\stackrel{\text{def}}{=} \sum_{K=1}^{N_K} \{||v_\beta||_K^2 + ||v||_K^2 + \langle\langle v^+ \rangle\rangle_{\partial K_- \backslash \Gamma_-}^2 \\ &+ \langle\langle v^- \rangle\rangle_{\partial K_- \backslash \Gamma_-}^2 + \langle\langle v \rangle\rangle_{\partial K \cap \partial\Omega}^2\} \end{aligned} \tag{10}$$

Proof: (i) From (4) we have

$$B(v,v) = \sum_{K \in \mathcal{P}_h} \{(v_\beta + av, v)_K + \langle v^+ - v^-, v^+ \rangle_{\partial K_- \backslash \Gamma_-} + \langle v, v \rangle_{\partial K_- \cap \Gamma_-}\}$$

Applying Green's formula to the first term:

$$\begin{aligned} (v_\beta, v)_K &= \frac{1}{2}\int_K (v^2)_\beta d\mathbf{x} = \frac{1}{2}\int_{\partial K} v^2 \boldsymbol{\beta} \cdot \mathbf{n}_K \, ds \\ &= \frac{1}{2}(\langle\langle v^+ \rangle\rangle_{\partial K_-}^2 + \langle\langle v^- \rangle\rangle_{\partial K_+}^2) \end{aligned}$$

and noting that

$$\sum_{K=1}^{N_K} \langle\langle v^- \rangle\rangle_{\partial K_+}^2 = \sum_{K=1}^{N_K} \{\langle\langle v^- \rangle\rangle_{\partial K_- \backslash \Gamma_-}^2 + \langle\langle v^- \rangle\rangle_{\partial K_+ \cap \Gamma_+}^2\}$$

yields

$$B(v,v) = \sum_{K=1}^{N_K} \{(av,v)_K + \frac{1}{2}\langle\langle v^+ - v^- \rangle\rangle_{\partial K_- \backslash \Gamma_-}^2 + \frac{1}{2}\langle\langle v \rangle\rangle_{\partial K_+ \cap \Gamma_+}^2\}$$

from which the first inequality follows.

(ii) Adding the definitions of $B(\cdot,\cdot)$ in (4) and (6) yields

$$
\begin{aligned}
2B(v,w) &= \sum_{K=1}^{N_K}\{(v_\beta,w)_K-(v,w_\beta)_K+2(av,w)_K \\
&+ \langle v^+-v^-,w^+\rangle_{\partial K_-\backslash\Gamma_-}+\langle v^-,w^--w^+\rangle_{\partial K_-\backslash\Gamma_-}+\langle v,w\rangle_{\partial K\cap\partial\Omega}\}
\end{aligned}
$$

Applying the triangle inequality to the jump terms and the Holder's inequality to the integrals and resulting sums yields

$$
\begin{aligned}
B(v,w) &\leq \frac{1}{2}\max(1,2||a||_{\infty,\Omega})\{\sum_{K=1}^{N_K}[||v_\beta||_K^2+2||v||_K^2+\langle\langle v^+\rangle\rangle^2_{\partial K_-\backslash\Gamma_-} \\
&+ 3\langle\langle v^-\rangle\rangle^2_{\partial K_-\backslash\Gamma_-}+\langle\langle v\rangle\rangle^2_{\partial K\cap\partial\Omega}]\}^{\frac{1}{2}} \\
&\times \{\sum_{K=1}^{N_K}[||w_\beta||_K^2+2||w||_K^2+3\langle\langle w^+\rangle\rangle^2_{\partial K_-\backslash\Gamma_-} \\
&+ \langle\langle w^-\rangle\rangle^2_{\partial K_-\backslash\Gamma_-}+\langle\langle w\rangle\rangle^2_{\partial K\cap\partial\Omega}]\}^{\frac{1}{2}}
\end{aligned}
$$

from which the second inequality follows.

Remark: We note that it is sufficient to take the constants in the bounds of the Lemma to be the numbers

$$\alpha = \min(\min_{\mathbf{x}\in\Omega} a(\mathbf{x}),\frac{1}{2}) \tag{11}$$

$$M = \frac{3}{2}\max(1,2||a||_{\infty,\Omega}) \tag{12}$$

∎

4 Discontinuous Galerkin Approximation

Approximate solutions to **P2** are sought in a finite dimensional subspace of $V(\mathcal{P}_h)$ which we denote by $V_p(\mathcal{P}_h)$ and define precisely below. The discontinuous Galerkin approximation is obtained by replacing $u, v \in V(\mathcal{P}_h)$ by $u_h^p, v_h^p \in V_p(\mathcal{P}_h)$ as follows:

$$\text{Find } u_h^p \in V_p(\mathcal{P}_h) \text{ such that}$$

$$B(u_h^p, v_h^p) = \mathcal{L}(v_h^p) \quad \forall v_h^p \in V_p(\mathcal{P}_h) \tag{13}$$

where $B(u_h^p, v_h^p)$ is given in (4) and $\mathcal{L}(v_h^p)$ is given in (5).

4.1 The Finite Dimensional Space $V_p(\mathcal{P}_h)$

Let the elements $K \in \mathcal{P}_h$ be quadrilateral elements which are affine maps of a master element $\hat{K} = [-1,1] \times [-1,1]$, i.e., $K = F_K(\hat{K})$ as illustrated in Fig. 1. Let $h_K = \text{diam}(K)$, S be a sphere contained in K, and $\rho_K = \sup\{\text{diam}(S)\}$. For the analysis we assume that $\mathcal{P}_h$ belongs to a family $\mathcal{F}$ of quasiuniform refinements, that is for every $K \in \mathcal{P}_h$, there exist positive constants σ and τ independent of $h = \max_{K \in \mathcal{P}_h} h_K$ such that

$$\frac{h}{h_K} \leq \tau \quad \text{and} \quad \frac{h_K}{\rho_K} \leq \sigma \tag{14}$$

The finite dimensional space $V_p(\mathcal{P}_h) \subset V(\mathcal{P}_h)$ is defined as follows

$$V_p(\mathcal{P}_h) = \{v \in L^2(\Omega) : v|_K \circ F_K = \hat{v}_K \in Q^{p_K}(\hat{K})\}$$

where $Q^{p_K}(\hat{K})$ is the space of tensor products of complete polynomials of degree $\leq p_K$ defined on the master element $\hat{K}$. We use the notation that $v_K \in Q^{p_K}(K)$ to imply that $\hat{v}_K \in Q^{p_K}(\hat{K})$. The basis for $Q^{p_K}(\hat{K})$ is formed by tensor products of one-dimensional Legendre polynomials. Note that in general the elements of $V_p(\mathcal{P}_h)$ are discontinuous across element interfaces and that the degree of the polynomial approximation may vary from one element to the next.

In proving *a priori* error estimates for solutions of (13), we will need the following basic approximation properties of functions belonging to $V_p(\mathcal{P}_h)$. Since functions in $V_p(\mathcal{P}_h)$ are

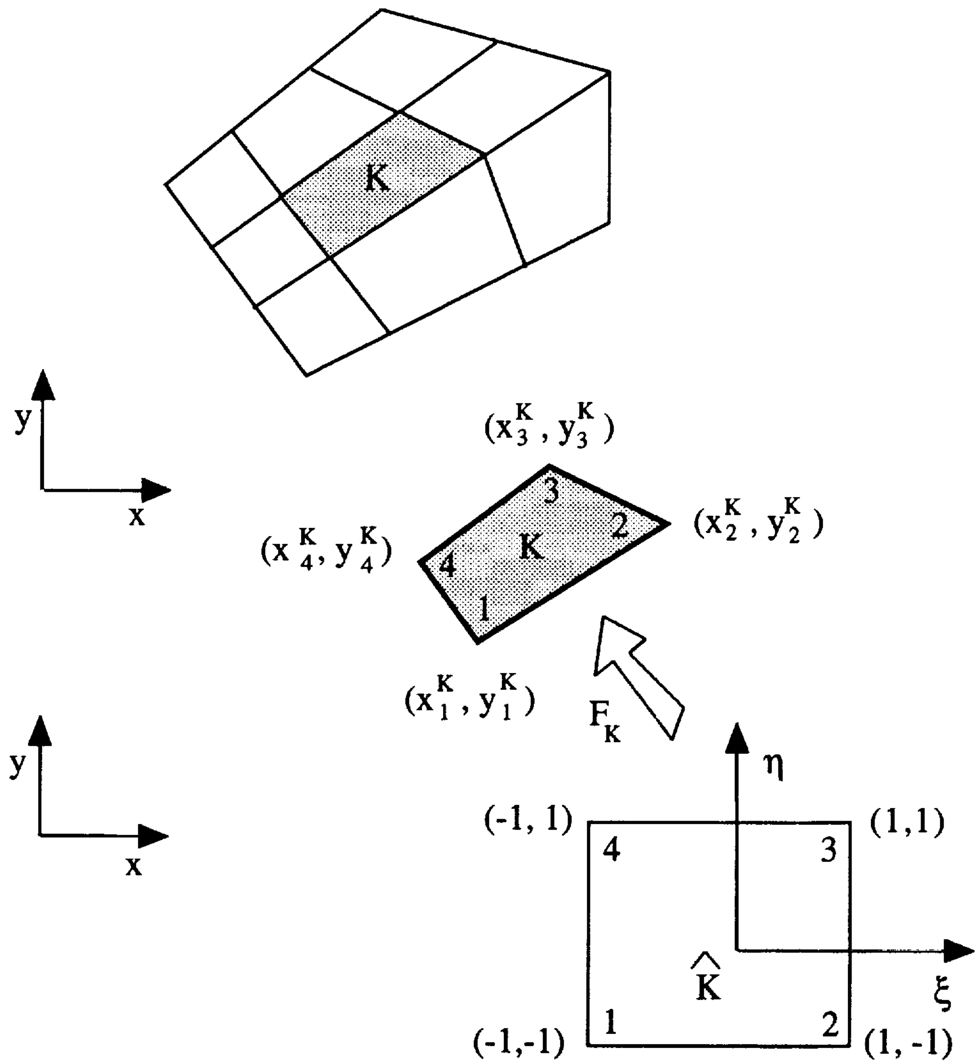

Figure 1: The mapping of the master element $\hat{K}$ onto a typical element $K \in \mathcal{P}_h$.

discontinuous at element interfaces, we are primarily concerned with polynomial approximations on a single element and its boundaries.

Lemma 2 *Let $K \in \mathcal{R}^2$ be an affine map of a master element $\hat{K} = [-1,1] \times [-1,1]$, that is $K = F_K(\hat{K})$. Let γ denote any edge of ∂K which is an affine map of a master edge $\hat{\gamma} = [-1,1]$. Let $\hat{w}_K$ be a polynomial of degree p_K defined on the master element. Let $w_K = \hat{w}_K \circ F_K$ denote the image of $\hat{w}_K$ under the transformation F_K. Then $\boldsymbol{\beta} \cdot \nabla w_K$ satisfies the following:*

$$||\boldsymbol{\beta} \cdot \nabla w_K||_K \leq C\frac{p_K^2}{h_K}||w_K||_K \tag{15}$$

$$\int_\gamma (\boldsymbol{\beta} \cdot \nabla w_K)^2 |\boldsymbol{\beta} \cdot \mathbf{n}_\gamma| ds \leq C\frac{p_K^4}{h_K^2} << w_K >>_\gamma^2 \tag{16}$$

where the constants C are independent of h_K, p_K, and w_K.

Proof: For polynomials of degree p_K on the master element we have that (see Dorr [7])

$$|\hat{w}_K|_{s,\hat{K}} \leq ||\hat{w}_K||_{s,\hat{K}} \leq C p_K^{2s} ||\hat{w}_K||_{\hat{K}} \tag{17}$$

$$|\hat{w}_K|_{s,\hat{\gamma}} \leq ||\hat{w}_K||_{s,\hat{\gamma}} \leq C p_K^{2s} ||\hat{w}_K||_{\hat{\gamma}} \tag{18}$$

where the constants $C > 0$ depends on s, but not on p_K or $\hat{w}_K$.

For affine mappings F_K, a standard scaling argument (see Ciarlet [8]) yields that for $s \geq 0$ an integer, there exist constants $C > 0$ such that

$$|w_K|_{s,K} \leq C h_K^{1-s} |\hat{w}_K|_{s,\hat{K}} \tag{19}$$

$$|w_K|_{s,\gamma} \leq C h_K^{\frac{1}{2}-s} |\hat{w}_K|_{s,\hat{\gamma}} \tag{20}$$

$$|\hat{w}_K|_{s,\hat{K}} \leq C h_K^{s-1} |w_K|_{s,K} \tag{21}$$

$$|\hat{w}_K|_{s,\hat{\gamma}} \leq C h_K^{s-\frac{1}{2}} |w_K|_{s,\gamma} \tag{22}$$

where C depend on s, σ, and τ (see (14)), but not on h_K, p_K, or w_K.

The first estimate (15) follows by combining (9), (7), and (21). The second estimate (6) follows from (20), (18), and (22). ∎

We also have the following result from Babuška and Suri [9] concerning the polynomial approximation of functions on a single element.

Lemma 3 (Babuška and Suri [9]) *Let $K \in \mathcal{P}_h$, γ denote any edge of ∂K, and $u \in H^s(K)$. Then there exists a constant $C = C(s, \tau, \sigma)$ (see (14)) independent of u, p_K, and h_K, and a sequence $z_h^p \in Q^{p_K}(K), p_K = 1, 2, \ldots$ such that for every $0 \le r \le p_K$*

$$||u - z_h^p||_{r,K} \le C\frac{h_K^{\nu-r}}{p_K^{s-r}}||u||_{s,K}, \ s \ge 0 \tag{23}$$

$$||u - z_h^p||_{0,\gamma} \le C\frac{h_K^{\nu-\frac{1}{2}}}{p_K^{s-\frac{1}{2}}}||u||_{s,K}, \ s \ge \frac{1}{2} \tag{24}$$

where $\nu = \min(p_K + 1, s)$.

4.2 *A Priori* Error Estimate

The discontinuous Galerkin method (13) was first analyzed by Lesaint and Raviart [1] for a given fixed value of p_K, i.e. for the case in which $p_K = p$ for every element $K \in \mathcal{P}_h$. The error in a solution u_h to (13) approximating an exact solution $u \in H^s(\Omega)$ to **P2** was shown to be

$$||u - u_h||_{L^2(\Omega)} \le Ch^{s-1}||u||_{s,\Omega}$$

This estimate is not optimal in the sense of interpolation error estimates and was improved by Johnson and Pitkaranta [2]. Using a mesh-dependent norm they showed that

$$|||u - u_h|||_{h,\beta} \le Ch^{s-\frac{1}{2}}||u||_{s,\Omega}$$

where

$$|||e|||_{h,\beta}^2 = |||u - u_h|||_{h,\beta}^2$$

$$\stackrel{\text{def}}{=} \sum_{K\in\mathcal{P}_h} h_K ||e_\beta||^2_{L^2(K)} + ||e||^2_{L^2(\Omega)} + \sum_{K\in\mathcal{P}_h} \langle\langle e^+ - e^-\rangle\rangle^2_{\partial K_-\backslash\Gamma_-} + \langle\langle e\rangle\rangle^2_{\partial\Omega}$$

While this estimate is not optimal in the sense of interpolation error estimates for $||e||_{L^2(\Omega)}$, it is optimal with respect to $\sqrt{h_K}||e_\beta||_{L^2(K)}$ and $\langle\langle e^+ - e^-\rangle\rangle_{\partial K_-\backslash\Gamma_-}$.

We shall derive estimates similar to Johnson and Pitkaranta [2]. Taking into account that p_K is not constant, we shall use the following mesh-dependent norm

$$|||v|||^2_{hp,\beta} \stackrel{\text{def}}{=} \sum_{K\in\mathcal{P}_h} \{\frac{h_K}{p_K^2}||v_\beta||^2_K + ||v||^2_K + \langle\langle v^+ - v^-\rangle\rangle^2_{\partial K_-\backslash\Gamma_-} + \langle\langle v\rangle\rangle^2_{\partial K\cap\partial\Omega}\} \tag{25}$$

The presence of the polynomial degree p_K in this norm deserves comment. For each element $K \in \mathcal{P}_h$ we assign to K a positive integer λ_K which serves as a weighting factor tuned to allow optimality in some sense of the error estimate in a mesh-dependent norm. Later, the choice $\lambda_K = p_K$ will prove to be the proper one since it allows for the development of quasi-optimal *hp*-estimates.

We first prove that the bilinear form in (13) satisfies an inf-sup condition with respect to this norm on the space $V_p(\mathcal{P}_h)$ and therefore that the solution depends continuously on the data.

Lemma 4 *For every $v_h^p \in V_p(\mathcal{P}_h)$, there exists a $w_h^p \in V_p(\mathcal{P}_h)$ such that*

$$B(v_h^p, w_h^p) \geq C_B|||v_h^p|||^2_{hp,\beta} \tag{26}$$

$$|||w_h^p|||_{hp,\beta} \leq C|||v_h^p|||_{hp,\beta} \tag{27}$$

where the positive constants C_B and C are independent of h_K, p_K, and v_h^p. Moreover, the solution u_h^p to (13) satisfies the following global stability estimate:

$$|||u_h^p|||_{hp,\beta} \leq C(||f||_\Omega + << g >>_{\Gamma_-})$$

Proof: Define the restriction of $w_h^p \in V_p(\mathcal{P}_h)$ to an element $K \in \mathcal{P}_h$ as

$$w_h^p|_K = v_h^p|_K + \delta \frac{h_K}{p_K^2} \boldsymbol{\beta} \cdot \nabla v_h^p|_K \tag{28}$$

where $\delta \in (0,1]$ is defined later in the proof. Dropping the h, p, and K scripts for ease in notation, we have

$$\begin{aligned}
B_K(v,w) &= \int_K (v_\beta + av)(v + \delta \frac{h_K}{p_K^2} v_\beta)\, d\mathbf{x} \\
&+ \int_{\partial K_- \setminus \Gamma_-} (v^+ - v^-)(v^+ + \delta \frac{h_K}{p_K^2} v_\beta{}^+)|\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds \\
&+ \int_{\partial K_- \cap \Gamma_-} v^+(v^+ + \delta \frac{h_K}{p_K^2} v_\beta{}^+)|\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds \\
&\geq a_0 ||v||_K^2 + \delta \frac{h_K}{p_K^2} ||v_\beta||_K^2 + \int_K v v_\beta\, d\mathbf{x} \\
&+ \delta \frac{h_K}{p_K^2} \int_K v v_\beta\, d\mathbf{x} + << v^+ >>^2_{\partial K_- \cap \Gamma_-} + << v^+ >>^2_{\partial K_- \setminus \Gamma_-} \\
&- \int_{\partial K_- \setminus \Gamma_-} v^+ v^- |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds + \delta \frac{h_K}{p_K^2} \int_{\partial K_- \setminus \Gamma_-} (v^+ - v^-) v_\beta{}^+ |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds \\
&+ \delta \frac{h_K}{p_K^2} \int_{\partial K_- \cap \Gamma_-} v^+ v_\beta{}^+ |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds
\end{aligned}$$

Noting that

$$\int_K v v_\beta\, d\mathbf{x} = \frac{1}{2} \int_{\partial K_+} (v^-)^2 |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds - \frac{1}{2} \int_{\partial K_-} (v^+)^2 |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds \tag{29}$$

and that from Lemma 2

$$\begin{aligned}
|\int_K v v_\beta\, d\mathbf{x}| &\leq c_1 \frac{p_K^2}{h_K} ||v||_K^2 \\
|\int_\gamma v^+ v_\beta{}^+ |\boldsymbol{\beta} \cdot \mathbf{n}_K|\, ds| &\leq c_2 \frac{p_K^2}{h_K} << v^+ >>^2_\gamma
\end{aligned}$$

we have

$$
\begin{aligned}
B_K(v,w) \geq & (a_0 - c_1\delta)||v||_K^2 + \delta\frac{h_K}{p_K^2}||v_\beta||_K^2 \\
+ & \frac{1}{2} << v^+ >>^2_{\partial K_-\backslash\Gamma_-} + (\frac{1}{2} - c_2\delta) << v >>^2_{\partial K_-\cap\Gamma_-} \\
- & \int_{\partial K_-\backslash\Gamma_-} v^+ v^- |\boldsymbol{\beta}\cdot\mathbf{n}_K|\, ds + \delta\frac{h_K}{p_K^2}\int_{\partial K_-\backslash\Gamma_-}(v^+ - v^-){v_\beta}^+|\boldsymbol{\beta}\cdot\mathbf{n}_K|\, ds
\end{aligned}
$$

Using the Schwarz inequality and the previous inequalities, one can show that

$$
|\delta\frac{h_K}{p_K^2}\int_{\partial K_-\backslash\Gamma_-}(v^+ - v^-){v_\beta}^+|\boldsymbol{\beta}\cdot\mathbf{n}_K|\, ds| \leq \frac{3c_2}{2}\delta(<< v+ >>^2_{\partial K_-\backslash\Gamma_-} + << v^- >>^2_{\partial K_-\backslash\Gamma_-})
$$

Now summing over all the elements $K \in \mathcal{P}_h$ and realizing that

$$
\begin{aligned}
\sum_{K\in\mathcal{P}_h}\{ & \tfrac{1}{2} << v^- >>^2_{\partial K_-\backslash\Gamma_-} - \frac{3c_2}{2}\delta << v^- >>^2_{\partial K_-\backslash\Gamma_-} \\
+ & (\frac{1}{2} - \frac{3c_2}{2}\delta) << v^+ >>^2_{\partial K_-\backslash\Gamma_-} - \int_{\partial K_-\backslash\Gamma_-} v^+ v^- |\boldsymbol{\beta}\cdot\mathbf{n}_K|\, ds\} \\
\geq & \frac{1}{2} << v >>^2_{\Gamma_+} + \min(1, \frac{1}{2} - \frac{3c_2}{2}\delta)\sum_{K\in\mathcal{P}_h} << v^+ - v^- >>^2_{\partial K_-\backslash\Gamma_-}
\end{aligned}
$$

results in

$$
\begin{aligned}
B(v_h^p, w_h^p) \geq & (a_0 - c_1\delta)||v||_\Omega^2 + \delta\sum_{K\in\mathcal{P}_h}\frac{h_K}{p_K^2}||v_\beta||_K^2 \\
+ & (\frac{1}{2} - c_2\delta) << v >>^2_{\Gamma_-} + \frac{1}{2} << v >>^2_{\Gamma_+} \\
+ & \min(1, \frac{1}{2} - \frac{3c_2}{2}\delta)\sum_{K\in\mathcal{P}_h} << v^+ - v^- >>^2_{\partial K_-\backslash\Gamma_-}
\end{aligned}
$$

Choosing $\delta = \min(\frac{1}{4}, \frac{a_0}{2c_1}, \frac{1}{6c_2})$ yields the first inequality.

The second inequality easily follows from the definition of w_h^p and Lemma 2. The third inequality follows by combining (26) with (13) and (5):

$$
C_B|||u_h^p|||^2_{hp,\beta} = B(u_h^p, \hat{v}_h^p) = \mathcal{L}(\hat{v}_h^p)
$$

where $v_h^p|_K = u_h^p|_K + \delta\beta \cdot \nabla u_h^p|_K$ and applying Holder's inequality to $\mathcal{L}(v_h^p)$ defined in (5). ∎

We now have all the preliminary results needed to prove an *a priori* error estimate for an *hp*-version of the discontinuous Galerkin method.

Theorem 1 *Let $u \in H^r(\Omega)$ be a solution to* **P2** *and let u_h^p be a solution to (13). Then there exists a positive constant C independent of h_K, p_K, and u such that the error, $e = u - u_h^p$, satisfies the following estimate*

$$|||e|||_{hp,\beta} \leq C \left\{ \sum_{K \in \mathcal{P}_h} \left[\frac{h_K^{2\nu_K - 1}}{p_K^{2r-2}} \max\left(1, \; \frac{h_K}{p_K^2}, \frac{1}{p_K}\right) ||u||_{r,K}^2 \right] \right\}^{\frac{1}{2}} \tag{30}$$

where $\nu_K = \min(p_K + 1, r)$.

Proof: Let $\Pi_h^p u \in V_p(\mathcal{P}_h)$ be an *hp*-approximation of u that satisfies the estimates in Lemma 3 and write

$$e = u - u_h^p = u - \Pi_h^p u + \Pi_h^p u - u_h^p$$

which implies that

$$|||e|||_{hp,\beta}^2 \leq 2(|||u - \Pi_h^p u|||_{hp,\beta}^2 + |||u_h^p - \Pi_h^p u|||_{hp,\beta}^2) \tag{31}$$

Subtracting (13) from **P2** yields the orthogonality condition that

$$B(e, v_h^p) = B(u - \Pi_h^p u, v_h^p) - B(u_h^p - \Pi_h^p u, v_h^p) = 0 \quad \forall\, v_h^p \in V_p(\mathcal{P}_h)$$

Combining this with Lemma 4 yields that

$$C_B |||u_h^p - \Pi_h^p u|||_{hp,\beta}^2 \leq B(u_h^p - \Pi_h^p u, \hat{v}) = B(u - \Pi_h^p u, \hat{v}) \tag{32}$$

for a particular choice of $\hat{v} \in V_p(\mathcal{P}_h)$. To simplify the notation, let

$$\eta = u - \Pi_h^p u \quad \text{and} \quad w = u_h^p - \Pi_h^p u$$

and recall from Lemma 4 that the particular choice of $\hat{v} \in V_p(\mathcal{P}_h)$ for which (32) holds also satisfies the estimate

$$|||\hat{v}|||_{hp,\beta} \le C|||w|||_{hp,\beta} \tag{33}$$

Next we seek to bound from above $B(\eta, \hat{v})$ on the right hand side of (32). Using the definition in (6), we have

$$\begin{aligned} B(\eta,\hat{v}) &= \sum_{K\in\mathcal{P}_h} \{(\eta, -\hat{v}_\beta + a\hat{v})_K + \langle \eta^-, \hat{v}^- - \hat{v}^+\rangle_{\partial K_-\setminus\Gamma_-} + \langle \eta, \hat{v}\rangle_{\partial K_+\cap\Gamma_+}\} \\ &\le \sum_{K\in\mathcal{P}_h} \frac{p_K}{\sqrt{h_K}}\|\eta\|_K \cdot \frac{\sqrt{h_K}}{p_K}\|\hat{v}_\beta\|_K + \|a\|_{\infty,K}\|\eta\|_K\|\hat{v}\|_K \\ &+ \langle\langle \eta^-\rangle\rangle_{\partial K_-\setminus\Gamma_-}\langle\langle \hat{v}^+ - \hat{v}^-\rangle\rangle_{\partial K_-\setminus\Gamma_-} + \langle\langle\eta\rangle\rangle_{\partial K_+\cap\Gamma_+}\langle\langle\hat{v}\rangle\rangle_{\partial K_+\cap\Gamma_+}\} \end{aligned}$$

Using Holder's inequality for sums results in

$$\begin{aligned} B(\eta,\hat{v}) &\le \max(1, ||a||_{\infty,\Omega})\sqrt{\sum_{K\in\mathcal{P}_h}\frac{p_K^2}{h_K}||\eta||_K^2}\sqrt{\sum_{K\in\mathcal{P}_h}\frac{h_K}{p_K^2}||\hat{v}_\beta||_K^2} + \sqrt{\sum_{K\in\mathcal{P}_h}||\eta||_K^2}\sqrt{\sum_{K\in\mathcal{P}_h}||\hat{v}||_K^2} \\ &+ \sqrt{\sum_{K\in\mathcal{P}_h}\langle\langle\eta^-\rangle\rangle^2_{\partial K_-\setminus\Gamma_-}}\sqrt{\sum_{K\in\mathcal{P}_h}\langle\langle\hat{v}^+ - \hat{v}^-\rangle\rangle^2_{\partial K_-\setminus\Gamma_-}} \\ &+ \sqrt{\sum_{K\in\mathcal{P}_h}\langle\langle\eta\rangle\rangle^2_{\partial K_+\cap\Gamma_+}}\sqrt{\sum_{K\in\mathcal{P}_h}\langle\langle\hat{v}\rangle\rangle^2_{\partial K_+\cap\Gamma_+}} \end{aligned} \tag{34}$$

Applying Holder's inequality for sums again yields

$$B(\eta,\hat{v}) \le C\left\{\sum_K\left[\frac{p_K^2}{h_K}\|\eta\|_K^2 + \|\eta\|_K^2 + \langle\langle\eta-\rangle\rangle^2_{\partial K_+\setminus\Gamma_+} + \langle\langle\eta\rangle\rangle^2_{\partial K_+\cap\Gamma_+}\right]\right\}^{\frac{1}{2}} |||\hat{v}|||_{hp,\beta} \tag{35}$$

Combining (35), (33), and (32) results in

$$|||u_h^p - \Pi_h^p u|||_{hp,\beta} \le C\left\{\sum_K\left[\frac{p_K^2}{h_K}\|\eta\|_K^2 + |\eta\|_K^2 + \langle\langle\eta^-\rangle\rangle^2_{\partial K_+\setminus\Gamma_+} + \langle\langle\eta\rangle\rangle^2_{\partial K_+\cap\Gamma_+}\right]\right\}^{\frac{1}{2}} \tag{36}$$

Using the estimates (23)–(24) in Lemma 3, we have

$$||\eta||_K \le C\frac{h_K^{\nu_K}}{p_K^r}||u||_{r,K}$$

$$||\eta_\beta||_K \le C(\boldsymbol{\beta}\cdot\mathbf{n}_K)||\eta||_{1,K} \;\le\; C\frac{h_K^{\nu_K-1}}{p_K^{r-1}}||u||_{r,K}$$

$$\langle\langle\eta^-\rangle\rangle_{\partial K_+\setminus\Gamma_+} \;\le\; C\frac{h_K^{\nu_K-\frac{1}{2}}}{p_K^{r-\frac{1}{2}}}||u||_{r,K}$$

where $\nu_K = \min(p_K+1, r)$.

Substituting the above estimates involving η into (36) yields

$$|||u_h - \Pi_h^p u|||_{hp,\beta} \;\le\; C\left\{\sum_{K\in\mathcal{P}_h}\left[\left(\frac{h_K^{2\nu_K-1}}{p_K^{2r-2}} + \frac{h_K^{2\nu_K}}{p_K^{2r}} + \frac{h_K^{2\nu_K-1}}{p_K^{2r-1}}\right)||u||_{r,K}^2\right]\right\}^{\frac{1}{2}} \tag{37}$$

Recalling (31), we see that all that remains is to bound $|||u - \Pi_h^p u|||_{hp,\beta}$ which follows from Lemma 3 and (25):

$$|||u - \Pi_h^p u|||_{hp,\beta} \le C\left\{\sum_{K\in\mathcal{P}_h}\left[\left(\frac{h_K^{2\nu_K}}{p_K^{2r}} + \frac{h_K^{2\nu_K-1}}{p_K^{2r}} + \frac{h_K^{2\nu_K-1}}{p_K^{2r-1}}\right)||u||_{r,K}^2\right]\right\}^{\frac{1}{2}} \tag{38}$$

Combining (31), (37), and (38) completes the proof. ∎

Remarks:

(i) For $\frac{h_K}{p_K^2} \le 1$, the estimate becomes $|||e|||_{hp,\beta} \le C\{\sum_{K\in\mathcal{P}_h}\frac{h_K^{2\nu_K-1}}{p_K^{2r-2}}||u||_{r,K}^2\}^{\frac{1}{2}}$

(ii) For p_K =constant, the *a priori* error estimate reduces to the one derived by Johnson and Pitkaranta [2].

(iii) The error estimate reveals that the discontinuous Galerkin method provides some natural control in the $\boldsymbol{\beta}$-derivatives of the approximate solution. The factor $\frac{h_K}{p_K^2}$ means, however, that this control decreases as $\frac{h_K}{p_K^2} \to 0$.

5 Numerical Examples

We verify the estimate in (30) with two examples where $\boldsymbol{\beta} = (0.8, 0.6)^T$, $a(\mathbf{x}) = 1.0$, and $\Omega = (-1,1)\times(-1,1)$.

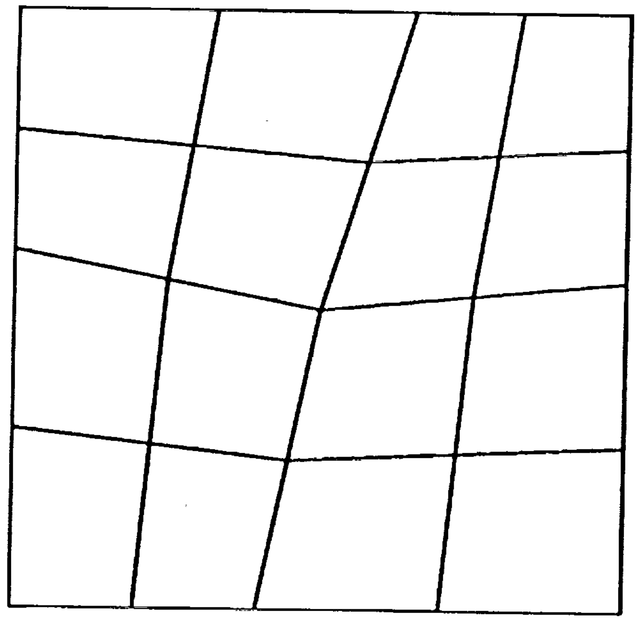

Figure 2: Quadrilateral element mesh used for quasiuniform refinements in Example 1

5.1 Example 1

In the first example, the source term $f(\mathbf{x})$ was chosen so that the exact solution to (1) is the $C^{\infty}(\Omega)$ function

$$u(\mathbf{x}) = 1 + \sin\left(\frac{\pi}{8}(1+x)(1+y)^2\right)$$

with an inflow boundary condition of $g = 1$. The error in the solution obtained with varying h and p is listed in Table 1 for uniform refinements of a mesh consisting of square elements. The error in the solution obtained for quasiuniform refinement of a mesh consisting of quadrilateral elements (see Fig. 2) is listed in Table 2.

To verify the estimate (30), we first consider the case when p_K is fixed and h_K is varied. According to (30), we should get $|||e|||_{hp,\beta} \leq C h_K^{p_K+\frac{1}{2}} ||u||_{r,\Omega}$. This is verified in Fig. 3 where $|||e|||_{hp,\beta}$ is shown as a function of h_K. On the log-log scale, the slope of the lines corresponding to a fixed value of p_K is indeed $p_K + \frac{1}{2}$ for both the uniform and quasiuniform meshes. Next we consider the case when h_K is fixed and p_K is varied. In this case, the estimate

Mesh	$-\log h$	$-log\ \|\|\|u-u_h\|\|\|_{hp,\beta}$			
		$p=1$	$p=2$	$p=3$	$p=4$
2×2	0.000	——	——	1.8323	2.2787
4×4	0.301	0.5552	1.7066	2.5426	3.6065
8×8	0.602	0.9692	2.3909	3.5467	4.9612
16×16	0.903	1.4003	3.1163	4.5834	6.3047
32×32	1.204	1.8412	3.8574	——	——

Table 1: Example 1 _ Error using uniform hp meshes

Mesh	$-\log h$	$-log\ \|\|\|u-u_h\|\|\|_{hp,\beta}$			
		$p=1$	$p=2$	$p=3$	$p=4$
2×2	-0.2116	——	0.8586	1.7402	2.2831
4×4	0.0689	0.5153	1.5930	2.5395	3.4998
8×8	0.347	0.9571	2.3641	3.5814	4.9723
16×16	0.641	1.3913	3.0955	4.6208	6.3196
32×32	0.938	1.8129	3.7870	——	——

Table 2: Example 1 _ Error using nonuniform h and uniform p meshes

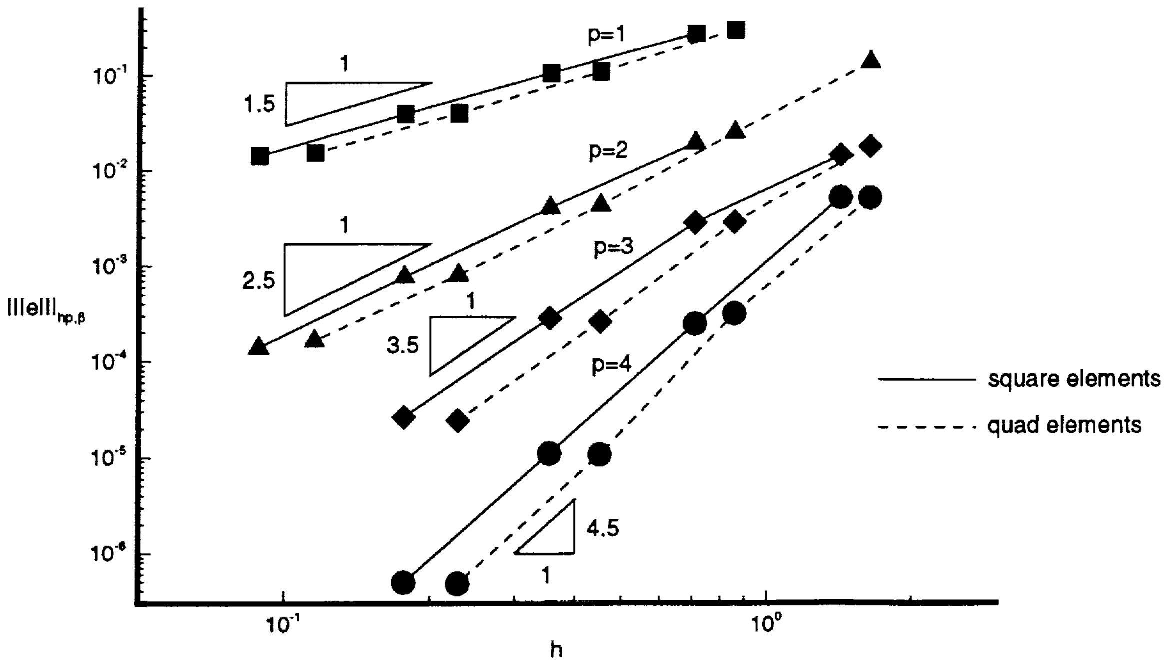

Figure 3: Example 1- Rate of convergence of error for fixed p

(30) reduces to $|||e|||_{hp,\beta} \leq Cp_{_K}{}^{-r+1}||u||_{r,\Omega}$. Since $u \in C^{\infty}(\Omega)$, we should expect exponential rates of convergence. This is confirmed in Fig. 4 where the curves corresponding to $|||e|||_{hp,\beta}$ as a function of $p_{_K}$ have a slope on the log-log scale which increases as $p_{_K}$ increases. These results are combined in Fig. 5 where $|||e|||_{hp,\beta}$ is shown as a function of the total number of unknowns in the solution. The solid lines represent h-refinements for a fixed p and the dashed lines represent p-enrichment for a fixed h. Clearly for smooth solutions, higher-order accuracy is achieved for the same number of unknowns by using higher-order elements.

5.2 Example 2

In this example the source term $f(\mathbf{x})$ was chosen so that $g = 1$ and the exact solution to (1) is the $C^{\infty}(\Omega)$ function

$$u(\mathbf{x}) = 1 + \frac{1}{16}(1+x)(1-x)(1+y)(1-y)\tan^{-1}(\alpha(\xi - \xi_0))$$

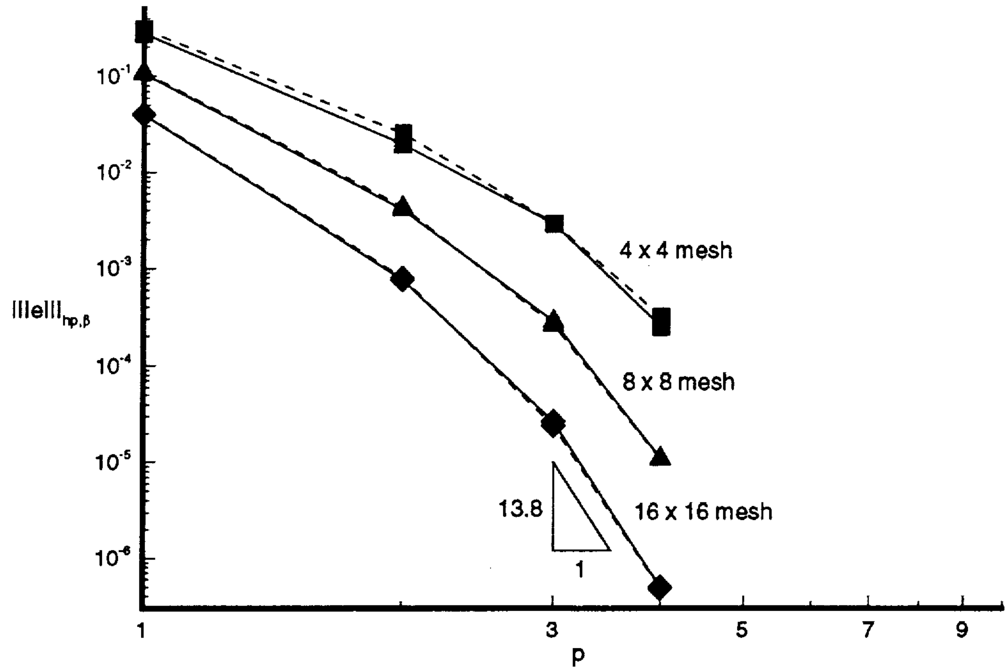

Figure 4: Example 1- Rate of convergence of error for fixed h

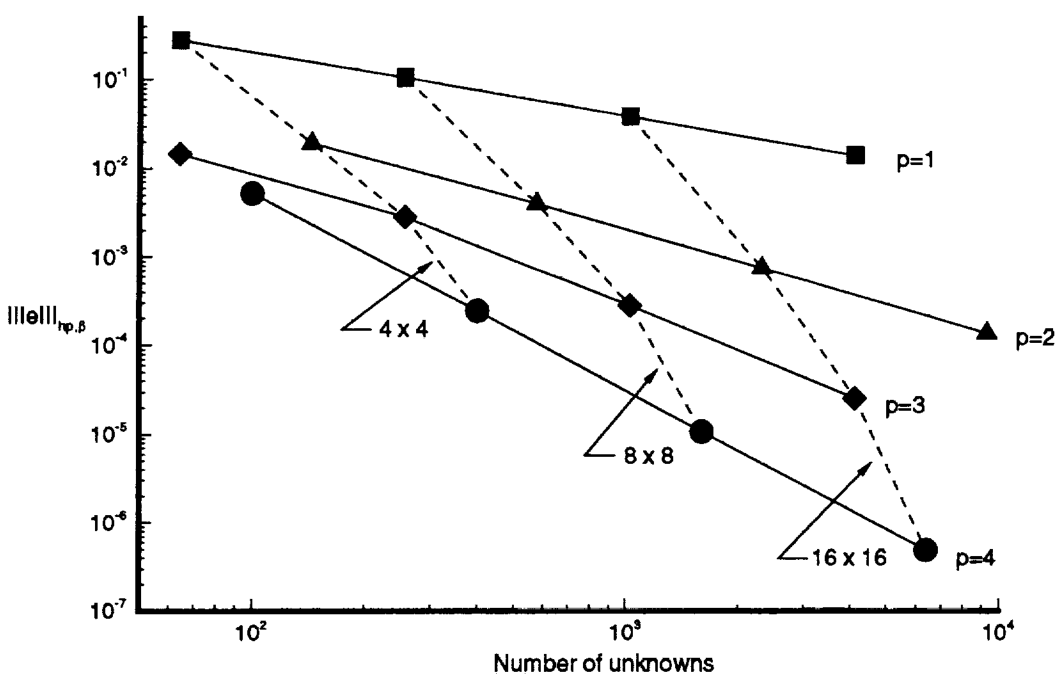

Figure 5: Example 1- Rate of convergence of error with number of unknowns

Mesh	$-\log h$	$-log\ |||u - u_h|||_{hp,\beta}$			
		$p = 1$	$p = 2$	$p = 3$	$p = 4$
2×2	0.000	——	——	1.7957	2.1453
4×4	0.301	——	1.9235	2.5461	3.0656
8×8	0.602	1.5571	2.6252	3.4634	4.2565
16×16	0.903	2.0017	3.3471	4.4822	5.5530
32×32	1.204	2.4512	4.0925	——	——

Table 3: Example 2 - Error using uniform hp meshes

where

$$\xi = \frac{2 + x + y}{2\sqrt{2}}, \quad \alpha = 10, \quad \text{and} \quad \xi_0 = 0.6$$

The error in the discontinuous Galerkin solution obtained on uniform meshes with various values of h and p are listed in Table 3. The error in the solution for uniform h-refinements with fixed p is shown in Fig. 6. The error for uniform p-enrichments with fixed h is shown in Fig. 7. The error as a function of the total number of unknowns is shown in Fig. 8. The rate of convergence of the error given in Theorem 1 is verified for this example.

6 Concluding Remarks

The discontinuous Galerkin method can be viewed as an elementwise application of the standard Galerkin method in which jumps on element boundaries are admitted naturally in the formulation. The method can also be viewed as a natural higher-order extension of finite volume methods, except that the coefficients in the higher-order polynomial representation of the solution are obtained by solving the conservation law and not by some post-processing of solution mean values.

In this paper, we have derived an *a priori* error estimate for an hp-version of the discontinuous Galerkin method. These estimates are derived in a mesh dependent norm in which the coefficients depend upon the local mesh size h_K and a number p_K which can be identified

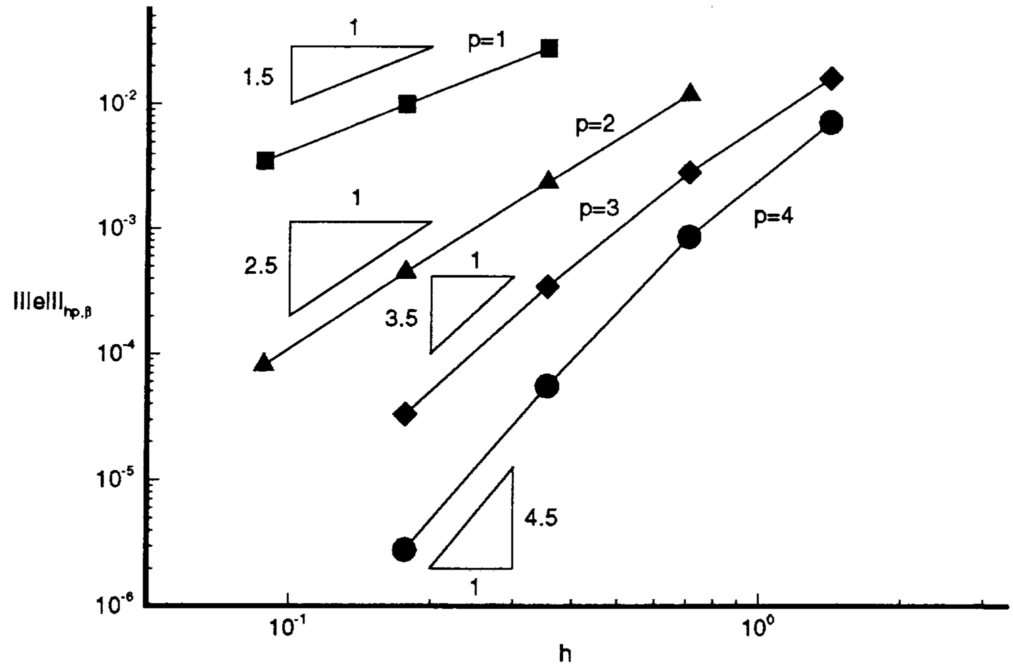

Figure 6: Example 2- Rate of convergence of error for fixed p

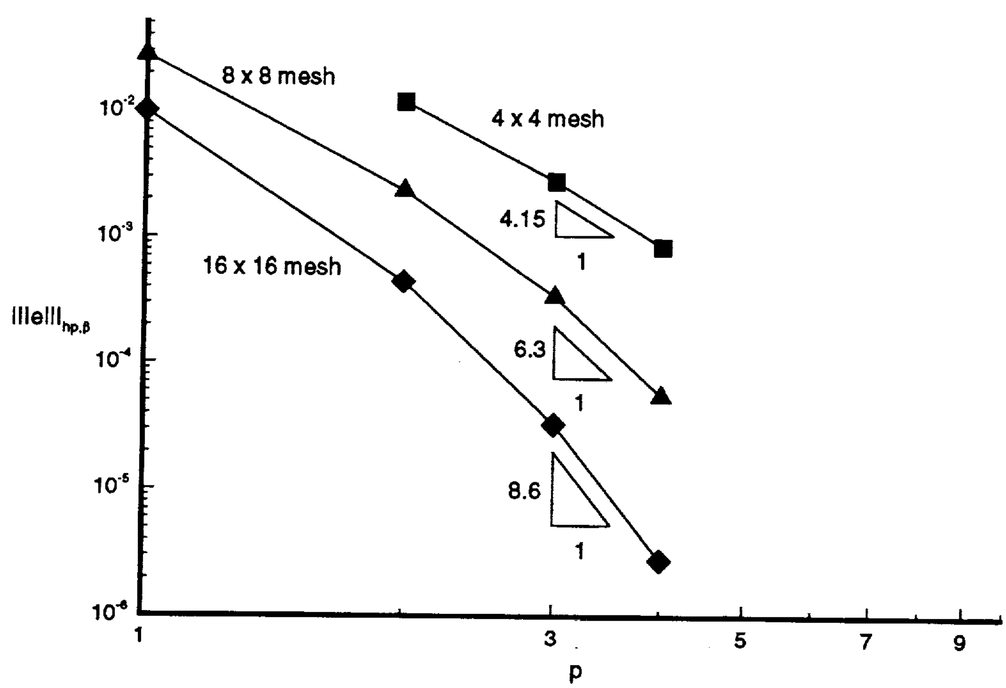

Figure 7: Example 2- Rate of convergence of error for fixed h

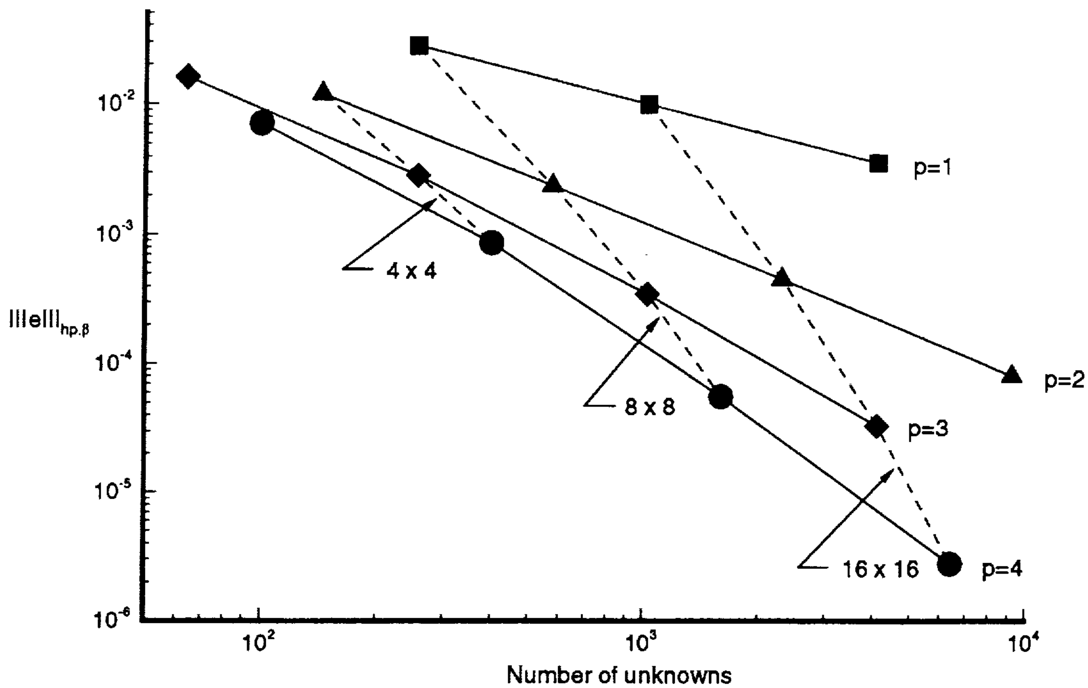

Figure 8: Example 2- Rate of convergence of error with number of unknowns

with the spectral order of the local approximation over each element. The estimate is general in the sense that it is valid for general quadrilateral elements of any degree p_K. Moreover, the estimate is valid for meshes in which the size and the spectral order of the elements vary throughout the mesh.

References

1. P. Lesaint and P. A. Raviart. On a Finite Element Method for Solving the Neutron Transport Equation. In C. de Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 89–123. Academic Press, 1974.

2. C. Johnson and J. Pitkaranta. An Analysis of the Discontinuous Galerkin Method for a Scalar Hyperbolic Equation. *Mathematics of Computations*, 46:1–26, 1986.

3. C. Johnson, U. Navert, and J. Pitkaranta. Finite Element Methods for Linear Hyperbolic Problems. *Computer Methods in Applied Mechanics and Engineering*, 45:285–312,

1984.

4. Cockburn, Hou, and Shu. The Runge-Kutta Local Projection Discontinuous Galerkin Method for Conservation Laws IV: The Multidimensional Case. *Mathematics of Computations*, 54:545–581, 1990.

5. K. S. Bey and J. T. Oden. A Runge-Kutta Discontinuous Galerkin Method for High Speed Flows. In *Proceedings of the 10th AIAA Computational Fluid Dynamics Conference*, Honolulu, Hawaii, 1990.

6. J. Flaherty, K. Devine, and R. Biswas. Parallel, Adaptive Finite Element Methods for Conservation Laws. In *Proceedings of Workshop on Adaptive Methods for Partial Differential Equations*, Rensselaer Polytechnic Institute, Troy, New York, May, 1992.

7. Milo R. Dorr. The Approximation Theory for the *p*-Version of the Finite Element Method. *SIAM Journal of Numerical Analysis*, 21:1180–1207, 1984.

8. P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, 1978.

9. I. Babuška and Manil Suri. The *hp*-Version of the Finite Element Method with Quasiuniform Meshes. *Mathematical Modeling and Numerical Analysis*, Vol. 21:199–238, 1987.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
June 1993

3. REPORT TYPE AND DATES COVERED
Technical Memorandum

4. TITLE AND SUBTITLE
A Priori Error Estimates for an hp-Version of the Discontinuous Galerkin Method for Hyperbolic Conservation Laws

5. FUNDING NUMBERS
WU 506-43-31

6. AUTHOR(S)
Kim S. Bey
J. Tinsley Oden

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-0001

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA TM-108993

11. SUPPLEMENTARY NOTES
Kim S. Bey: Langley Research Center, Hampton, VA
J. Tinsley Oden: The Texas Institute for Computational Mechanics, University of Texas at Austin, Austin, TX

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited

Subject Categories 34 and 64

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
A priori error estimates are derived for hp-versions of the finite element method for discontinuous Galerkin approximations of a model class of linear, scalar, first-order hyperbolic conservation laws. These estimates are derived in a mesh-dependent norm in which the coefficients depend upon both the local mesh size h_K and a number p_K which can be identified with the spectral order of the local approximations over each element.

14. SUBJECT TERMS
A priori error estimates, Discontinuous Galerkin, Hyperbolic Conservation Laws

15. NUMBER OF PAGES
26

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	