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Applications of Computer Algebra to Distributed Parameter Systems

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Abstract

In the analysis of vibrations of continuous elastic systems, one often encounters complicated transcendental equations with roots directly related to the system's natural frequencies. Typically, these equations contain system parameters whose values must be specified before a numerical solution can be obtained. The present paper presents a method whereby the fundamental frequency can be obtained in *analytical form* to any desired degree of accuracy. The method is based upon truncation of rapidly converging series involving inverse powers of the system natural frequencies. A straightforward method to developing these series and summing them in closed form is presented. It is demonstrated how Computer Algebra can be exploited to perform the intricate analytical procedures which otherwise would render the technique difficult to apply in practice. We illustrate the method by developing two analytical approximations to the fundamental frequency of a vibrating cantilever carrying a rigid tip body. The results are compared to the numerical solution of the exact (transcendental) frequency equation over a range of system parameters.

Introduction

The general availability of computer algebra systems has resulted in analyses of complicated problems which heretofore have been regarded as analytically intractable.These tools practically eliminate the tedious error prone manipulations required by hand-derivations and allow the analyst to explore various analytical treatments which would be too costly otherwise. In the same way that digital simulation has revolutionized the numerical treatment of engineering problems, symbolic computation promises to be a powerful tool in analytical investigations.

In the area of multibody dynamics, computer algebra has been used to derive the full nonlinear equations of motion in symbolic form [1]. These equations are typically so complex that they daunt inspection. However, built in translators convert these equations into a higher programming language e.g FORTRAN which results in extremely efficient digital simulations. For sufficiently simple systems, symbolic representations can be of direct use in studying such issues as elastic stability and buckling [2]. Perturbation methods are ideally suited to treatment by symbolic computation [3].

The classical method of determining the natural frequencies of a continuous elastic system results in an eigenvalue problem and associated characteristic equation. In general, this equation is transcendental and is embedded with various system parameters. Thus recourse must be made to numerical methods to solve these equations and the dependencies of the frequencies upon the various parameters can only be revealed through exhaustive computation. It therefore appears desirable to be able to approximate the roots of these equations by analytical expressions which are relatively simple yet accurate. The principal idea behind the method presented in this paper is to find closed form expressions for infinite series, the terms of which involve inverse powers of the natural frequencies. Truncating the series after the first term gives an approximation to the fundamental frequency. By summing sufficiently high powers, this approximation can be made arbitrarily accurate; but the resulting formula increases in complexity. The methods whereby others have addressed this problem are quite varied, ranging all the way from Fourier Series to complicated contour integration and difficult procedures involving integral equations. Hughes [4] obtains numerous modal identities by expanding the Green's function in a series of eigenfunctions. The technique we present is extremely simple and appears to have been applied in a restricted form by Lord Rayleigh [5]. Our result is very general and can be applied to any vibrational system one the characteristic equation is established. The only difficulty in applying the method is the need to develop complicated transcendental functions into Taylor series and manipulating the resulting coefficients. However, with the use of a computer algebra system, this task becomes almost trivial. The method is illustrated by deriving two analytical approximations to the fundamental frequency of a vibrating cantilever carrying a rigid tip body. The accuracy of these results is verified by comparisons with numerical solution of the frequency equation over a range of

parameter values.

Approximations Based Upon Rayleigh's Principle

Before presenting the technique predicated on infinite series, let us consider a "symbolic" solution to a prototype vibration problem employing the celebrated Rayleigh Principle. The elastic system consists of an Euler-Bernoulli beam cantilevered at one end, and carrying a rigid tip body at the other (see Fig. 1.A). The beam has a constant mass density (per unit length) ρ, uniform bending stiffness EI, and nominal length ℓ. A rigid tip body of mass m and inertia J (about P) is attached to the beam tip at $x=\ell$. We denote by c the distance from P to the tip body mass center. The derivation of the partial differential equation of motion and associated eigenvalue problem is given in Appendix A. The system eigenvalues are the solutions β_k to eq.(A.9) and are seen to depend upon the three dimensionless tip body parameters

$$m^* = \frac{m}{\rho\ell}, \quad c^* = \frac{c}{\ell}, \quad J^* = \frac{J}{\rho\ell^3}$$

The relationship between the eigenvalues and the system natural frequencies is given by eq.(A.10).

Let us approximate the beam deflection $u(x,t)$ with a cubic polynomial in x.

$$u(x,t) = \xi^2 q_1(t) + \xi^3 q_2(t)$$

where $\xi = x/\ell$ and the geometric boundary conditions at $x=0$ have been observed. Here $q_1(t), q_2(t)$ are undetermined generalized coordinates. The system kinetic energy T and potential energy V are then discretized into the respective quadratic forms (see eqs.(A.12) & (A.11))

$$T = \left[\frac{\rho\ell}{10} + \frac{m}{2}(1+2c^*) + \frac{2}{\ell^2}(J-mc^2)\right]\dot{q}_1^2 + \left[\frac{\rho\ell}{14} + \frac{m}{2}(1+3c^*)^2 + \frac{9}{2\ell^2}(J-mc^2)\right]\dot{q}_2^2 +$$

$$\left[\frac{\rho\ell}{6} + m(1+2c^*)(1+3c^*) + \frac{6}{\ell^2}(J-mc^2)\right]\dot{q}_1\dot{q}_2$$

$$V = \frac{2EI}{\ell^3}\left(q_1^2 + 3q_2^2 + 3q_1 q_2\right)$$

If we write $T = 1/2\,\dot{q}^T[M]\dot{q}$ and $V = 1/2\,q^T[K]q$, then the system's first two natural frequencies are approximated by the roots ω_i of the characteristic polynomial

$$\det\left([K]-\omega^2[M]\right)=0$$

Expanding this determinant and solving the resultant quadratic is relatively painless if a computer algebra system is invoked. The resulting expression for the fundamental frequency can be written in the form

$$\omega_1 = \sqrt{EI/\rho\ell^4}\ \beta_1^2$$

with

$$\beta_1 = \left[\frac{1260}{(630c^*+210)m^*+630J^*+t_3+51}\right]^{1/4}$$

where $t_3 = 2\sqrt{3}\left(33075J^{*2}+1260J^*+t_2+t_1+208\right)^{1/2}$

and $t_2 = \left[(66150c^*+11025)J^*+4200c^*+1680\right]m^*$

$t_1 = \left(44100c^{*2}+22050c^*+3675\right)m^{*2}$

(1)

This result of course provides an upper bound to the true fundamental frequency.

It should be noted that this method meets with practical difficulties when one attempts to improve the accuracy by retaining additional terms in the expansion of the elastic displacement. The higher degree of the concomitant characteristic polynomial renders an analytical solution impossible. The method to be described in the next section does not have this limitation.

Approximations Based Upon Infinite Series

The current method is based upon truncation of infinite series in the frequencies ω_n such as $\sum_{n=1}^{\infty} 1/\omega_n^2$, $\sum_{n=1}^{\infty} 1/\omega_n^4$ etc. where the sum can be expressed as a relatively simple algebraic function of the system parameters. If the series convergence is sufficiently rapid, then truncating the series after the first term yields a formula which approximates the fundamental frequency ω_1. Clearly, by summing sufficiently high powers of ω_n^{-1} we can approximate the first frequency to any desired degree of accuracy and will always have a *lower* bound. As will be seen, the corresponding formulas become increasingly complex. However, it should be pointed out that the generation of these higher order results can always be carried out in practice unlike the procedure of the last section. Hughes [4] generates series like the above and expresses the sum as a volume integral containing products of the Green's function with the mass density. He notes the difficulty of performing these integrations when the powers of ω_n^{-1} increase. Our method only requires a Taylor series expansion once the transcendental frequency equation is established.

Before considering the case of a transcendental equation, we present an elementary result from the theory of polynomial equations.

Given the polynomial equation

$$1+\alpha_1 z+\alpha_2 z^2+\cdots+\alpha_n z^n=0$$

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with roots $z_i (i=1,2,\cdots,n)$ (over the field of complex numbers), we can show that

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{z_i} = -\alpha_1$

(b) $\displaystyle\sum_{i=1}^{n} \frac{1}{z_i^2} = \alpha_1^2 - 2\alpha_2$

<u>Proof</u>

First note that if 0 is a root, it can be removed leaving a deflated polynomial with no zero roots. Hence there is no loss in generality if we assume all $z_i \neq 0$. The general polynomial with roots $z_1, z_2, \cdots, z_n$ can be written as

$$(z-z_1)(z-z_2)\cdots(z-z_n)=0$$

Expanding and dividing through by the product $(-1)^n z_1 z_2 \cdots z_n$ we obtain

$$1-\left(\frac{1}{z_1}+\frac{1}{z_2}+\cdots+\frac{1}{z_n}\right)z+\left(\frac{1}{z_1 z_2}+\frac{1}{z_1 z_3}+\cdots+\frac{1}{z_{n-1}z_n}\right)z^2+\cdots+\frac{(-1)^n}{z_1 z_2 \cdots z_n}z^n = 0$$

which is of the desired form. It follows that

$-\alpha_1 =$ sum of reciprocals of roots

$\alpha_2 =$ sum of products of the reciprocals of the roots taken 2 at a time

$$\text{Hence, } \alpha_1^2 - 2\alpha_2 = \sum_{i=1}^{n} \frac{1}{z_i^2}.$$

Sums involving higher powers of the inverse roots can also be generated. Thus

$$\sum_{i=1}^{n} \frac{1}{z_i^3} = 3\alpha_1\alpha_2 - 3\alpha_3 - \alpha_1^3,$$

$$\sum_{i=1}^{n} \frac{1}{z_i^4} = \alpha_1^4 - 4\alpha_1^2\alpha_2 + 2\alpha_2^2 + 4\alpha_1\alpha_3 - 4\alpha_4 \text{ etc.}$$

In an effort to adopt these results to the case of a transcendental equation $f(z)=0$ with an infinite number of roots, it is natural to expand $f(z)$ in a power series and *formally* apply the above formulas to this "infinite degree" polynomial. It turns out that this procedure can be proven mathematically valid when f is an entire function of the complex variable z. We will not present the proof here but refer the reader to [6] for the necessary theory of entire (integral) functions.

As a means of illustration, we apply this technique to approximate the fundamental frequency of the beam-tipbody system considered above. Using power series expansions for the trigonometric and hyperbolic functions, the frequency equation (A.9) assumes the form

$$1 - \frac{1}{12}\left(12J^* + 12m^*c^* + 4m^* + 1\right)\beta^4 +$$
$$\frac{1}{5040}\left[\left(420m^* + 168\right)J^* - 420m^{*2}c^{*2} + 56m^*c^* + 8m^* + 1\right]\beta^8 - \cdots = 0 \tag{2}$$

Since the coefficients of β and β^2 are zero, we conclude that $\sum\limits_{n=1}^{\infty}\frac{1}{\beta_n} = 0$ and $\sum\limits_{n=1}^{\infty}\frac{1}{\beta_n^2} = 0$. These two results become immediately obvious, since, if $\beta_k > 0$ is a root of eq.(A.9) then so are: $-\beta_k, i\beta_k, -i\beta_k$. In order to obtain series converging to a nonzero result, write eq.(2) as

$$1 + \alpha_1\beta^4 + \alpha_2\beta^8 + \cdots = 0$$

and form the auxiliary "polynomial"

$$1 + \alpha_1 z + \alpha_2 z^2 + \cdots = 0 \tag{3}$$

If β_k is a root of eq.(2), then $z_k = \beta_k^4$ is a root of eq.(3). This artifice coalesces the quadruple of roots $\{\beta_k, -\beta_k, i\beta_k, -i\beta_k\}$ of eq.(2) into a single root of eq.(3). Applying our method to the auxiliary equation(3), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^4} = \frac{12J^* + 12m^*c^* + 4m^* + 1}{12} \tag{4}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^8} = \{(5880c^{*2} + 3360c^* + 560)m^{*2} + [(10080c^* + 2520)J^* + 728c^* + 264]m^* +$$
$$5040J^{*2} + 504J^* + 33\}/5040 \tag{5}$$

Truncating the above series after the first term leads to the respective approximations

$$\beta_1 = \left(\frac{12}{12J^* + 12m^*c^* + 4m^* + 1}\right)^{1/4} \tag{6}$$

and

$$\beta_1 = \left(\frac{5040}{s_1 + s_2 + s_3}\right)^{1/8} \tag{7}$$

where

$$s_1 = \left(5880c^{*2} + 3360c^* + 560\right)m^{*2}$$

$$s_2 = \left[\left(10080c^* + 2520\right)J^* + 728c^* + 264\right]m^*$$

$$s_3 = 504J^*\left(1 + 10J^*\right) + 33$$

Numerical Results

Verification of the modal identities (4) & (5) is provided in Table 1 below. The eigenvalues β_n were generated by numerically solving the transcendental equation (A.9); the sequences of partial sums appear in the last two columns. The numbers in the last row $\left(n = \infty\right)$ were obtained from the theoretical values appearing on the right hand sides of equations (4) and (5). All values were generated with $m^* = 2.0$, $J^* = 0.028$, and $c^* = 0.1$.

Table 1 – Partial Sums of Series:
Eigenvalues of Beam/tipbody

n	β_n	$\sum_{k=1}^{n} \beta_k^{-4}$	$\sum_{k=1}^{n} \beta_k^{-8}$
1	1.0077	.9699	.9408
2	3.4599	.9769	
3	5.9100	.9777	
4	8.5047	.9779	
5	11.3806	.9780	
∞		.9780 (eq. 4)	.9408 (eq. 5)

The two approximations to the "dimensionless frequency" β_1 based upon series truncation (eqs. (6)&(7)) are tabulated in Table 2 below for the case of a pure tip mass – $J^* = c^* = 0$. The values in the second column ($'\beta_1'$) were obtained by numerical solution of eq.(A.9). As the value of the tip mass increases (relative to the mass of the beam),both approximations improve. As expected,the approximation based upon eq.(7) is superior to that supplied by eq.(6).

Table 2 – Fundamental Frequency Approximations for Beam with Tip Mass

m^*	β_1	Eq. 6 (% error)	Eq. 7 (% error)
0.0	3.5160	3.4641 (1.5)	3.5154 (1.6 E-02)
5.0	0.7569	0.7559 (0.13)	0.7569 (1.4 E-04)
10.0	0.5414	0.5410 (.069)	0.5414 (3.7 E-05)
15.0	0.4437	0.4435 (.047)	0.4437 (1.7 E-05)
20.0	0.3850	0.3849 (.035)	0.3850 (9.7E-06)

Table 3 below is similar in format to Table 2 but was generated with $J^*=1$ and $c^*=0$, which represents a relatively large concentrated inertia at the tip of the beam. In this case we see that the approximations degrade with increasing m^*.

Table 3 – Fundamental Frequency Approximations for Beam with Tip Body
$c^*=0$, $J^*=1$

m^*	β_1	Eq. 6 (% error)	Eq. 7 (% error)
1.0	.8679	.8402 (3.2)	.8669 (.11)
1.3	.8406	.8120 (3.4)	.8396 (.12)
1.5	.8236	.7947(3.5)	.8225 (.13)
1.8	.7995	.7708 (3.6)	.7984 (.14)
2.0	.7845	.7559 (3.6)	.7833 (.14)

Appendix A

We shall presently analyze the free vibration of a uniform cantilevered beam carrying a rigid tip body. Expressions for the potential and kinetic energies as well as the transcendental frequency equation are established.

Fig. 1.A below depicts the system in a deformed state. A uniform beam of mass density ρ (per unit length), bending stiffness EI and length ℓ lies along the x axis when in equilibrium. A rigid body of mass m and moment of inertia J (about P) is attached to the beam tip at P. The distance between P and the rigid body mass center is c, and this directed line segment lies along the beam tip tangent direction (to prevent the tip body from exerting axial loads onto the beam). The small transverse displacement of the beam is denoted by u(x,t).
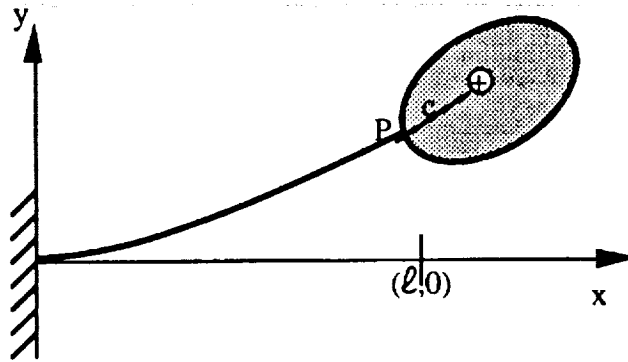

Fig. 1.A – clamped beam with tip body

We shall assume that the displacement of the beam and its slope are small quantities and therefore make the approximation that the angle θ_P between the x axis and the beam tip tangent line at P can be approximated by $u_x(\ell,t)$. Denoting the inertial velocities of P and the tip body mass center by v_P and $v_\oplus$ respectively, we can write

$$v_\oplus = v_P + \omega_P \times c$$

where $\omega_P = u_{xt}(\ell,t)k$ is the angular velocity of the tip body and c is the vector from P to the tip body mass center. Recalling that $|\theta_P|$ is small and neglecting the term $\theta_P \dot{\theta}_P$, we find

$$v_\oplus = \left[\frac{\partial u}{\partial t}(\ell,t) + c\frac{\partial^2 u}{\partial x \partial t}(\ell,t) \right] j \qquad (A.1)$$

The expressions for the absolute translational and rotational accelerations of the tip body

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mass center follow from the above by direct differentiation.

In order to write the boundary conditions for u(x,t) at the endpoint x=ℓ, we consider a free body diagram of the tip body. As indicated in Fig. 2.A, the beam exerts a force S directed along the y axis and a moment M directed along the z axis upon the tip body at the point P.
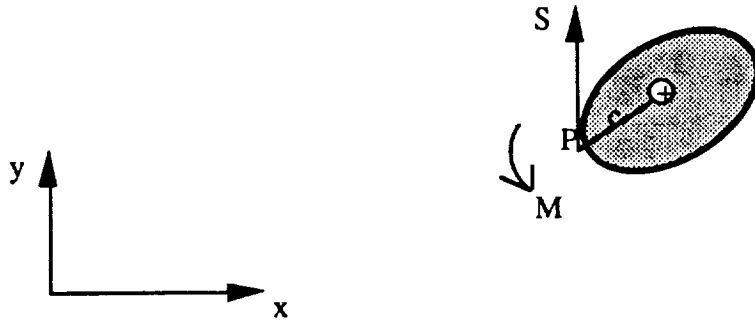

Fig. 2.A free body diagram of tip body

The equation of motion for the tip body along the y axis is

$$S = m\left[\frac{\partial^2 u}{\partial t^2}(\ell,t) + c\frac{\partial^3 u}{\partial x \partial t^2}(\ell,t)\right]$$

From elementary beam theory the shearing force in the beam at x=ℓ is given by $S=EI\ \partial^3 u/\partial x^3|_{x=\ell}$. In conjunction with the above, this supplies one of the required boundary conditions.

$$EI\frac{\partial^3 u}{\partial x^3} - m\left[\frac{\partial^2 u}{\partial t^2} + c\frac{\partial^3 u}{\partial x \partial t^2}\right] = 0 \text{ at } x=\ell \tag{A.2}$$

The second boundary condition at x=ℓ is obtained by considering the rotational motion of the tip body. If we denote by h the angular momentum of the tip body about its mass center, then we have the relation

$$\frac{d}{dt}h = M - c \times S$$

Taking the z component of this equation, neglecting the second order term in ωp and using

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the relation $\mathbf{M}=-EI\partial^2 u/\partial x^2\big|_{x=\ell}\mathbf{k}$, we arrive at the result

$$EI\frac{\partial^2 u}{\partial x^2}+mc\frac{\partial^2 u}{\partial t^2}+J\frac{\partial^3 u}{\partial x\partial t^2}=0 \quad \text{at } x=\ell \qquad (A.3)$$

Since the beam is clamped at x=0, we have the two additional boundary conditions

$$u(0,t)=0 \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t)=0 \qquad (A.4)$$

The partial differential equation for free vibration is the well known relation

$$EI\frac{\partial^4 u}{\partial x^4}+\rho\frac{\partial^2 u}{\partial t^2}=0$$

We now proceed to solve the above homogeneous equation subject to the geometric boundary conditions (A.4) and natural boundary conditions (A.2) & (A.3). Seeking solutions of the form $e^{i\omega t}\varphi(x)$ we are led to the eigenvalue problem

$$\frac{d^4\varphi}{dx^4}-\lambda\varphi=0 \qquad (A.5)$$

$$\varphi'''(\ell)+\frac{m\lambda}{\rho}\big[\varphi(\ell)+c\varphi'(\ell)\big]=0 \qquad (A.6)$$

$$\varphi''(\ell)-\frac{\lambda}{\rho}\big[mc\varphi(\ell)+J\varphi'(\ell)\big]=0 \qquad (A.7)$$

$$\varphi(0)=\varphi'(0)=0 \qquad (A.8)$$

where ($'$) indicates differentiation with respect to x.

It can be shown that all the eigenvalues are positive. The general solution of eq.(A.5) is

$$\varphi(x)=c_1\sin\alpha x+c_2\cos\alpha x+c_3\sinh\alpha x+c_4\cosh\alpha x$$
$$\big(\alpha\equiv\lambda^{1/4}>0\big)$$

In order to have a nontrivial solution satisfying the boundary conditions (A.6),(A.7) & (A.8), the eigenvalues must satisfy the transcendental characteristic equation

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$$m^*(J^* - m^* c_*^2)\beta^4 \left(1 - \cos\beta\cosh\beta\right) + m^*\beta\left(\cos\beta\,\sinh\beta - \sin\beta\,\cosh\beta\right)$$

$$-2m^* c^*\beta^2 \sin\beta\,\sinh\beta - J^*\beta^3\left(\sin\beta\,\cosh\beta + \sinh\beta\,\cos\beta\right) \qquad (A.9)$$

$$+1 + \cos\beta\,\cosh\beta = 0$$

where we have introduced the "dimensionless frequency " $\beta \equiv \alpha\ell$ and the dimensionless tip body parameters are defined by

$$m^* = m/\rho\ell, \quad c^* = c/\ell, \quad J^* = J/\rho\ell^3$$

The natural frequencies are then given by

$$\omega_k = \sqrt{\frac{EI}{\rho\ell^4}}\,\beta_k^2 \qquad (A.10)$$

For purposes of reference, the systems potential energy V is in the form of strain energy stored in the beam and is given by the formula

$$V = \frac{EI}{2} \int_0^\ell \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \qquad (A.11)$$

while the kinetic energy T is the sum of the translational kinetic energy of the beam and tip body with the rotational kinetic energy of the tip body. Employing eq.(A.1) we can write

$$T = \frac{1}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \rho\,dx + \frac{m}{2}\left[\frac{\partial u}{\partial t}(\ell,t) + c\frac{\partial^2 u}{\partial x\partial t}(\ell,t)\right]^2 + \frac{1}{2}(J - mc^2)\left[\frac{\partial^2 u}{\partial x\partial t}(\ell,t)\right]^2 \qquad (A.12)$$

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