IN-71
190206
41P

NASA Contractor Report 191546

On the Use of the Noncentral Chi-Square Density Function for the Distribution of Helicopter Spectral Estimates

Donald P. Garber

Lockheed Engineering & Sciences Company
Hampton, VA   23666-1339

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-0001
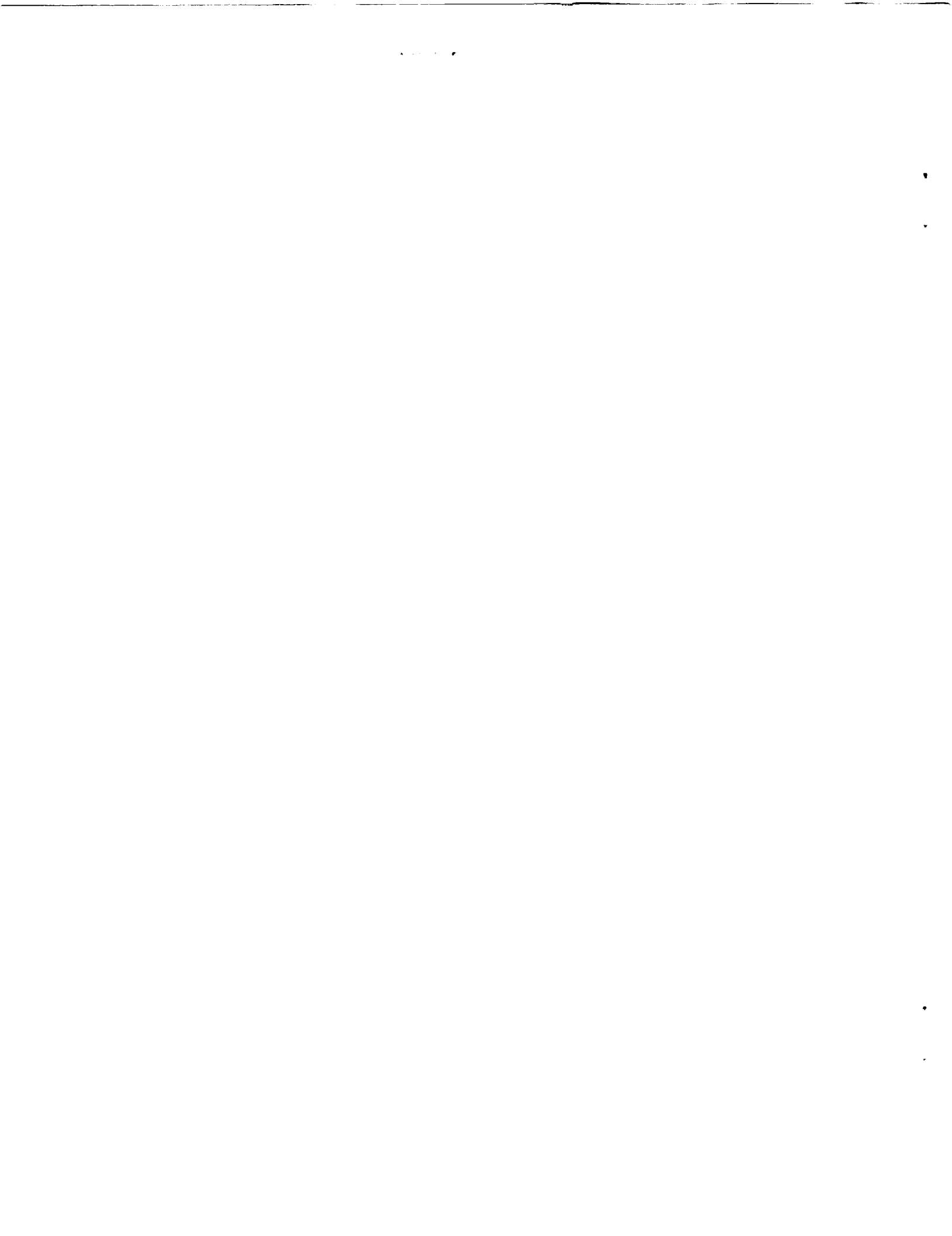
N94-15487

Unclas

G3/71   0190206

(NASA-CR-191546)   ON THE USE OF THE NONCENTRAL CHI-SQUARE DENSITY FUNCTION FOR THE DISTRIBUTION OF HELICOPTER SPECTRAL ESTIMATES (Lockheed Engineering and Sciences Corp.)   41 p

Contents

Abstract

A probability density function for the variability of ensemble averaged spectral estimates from helicopter acoustic signals in Gaussian background noise was evaluated. Numerical methods for calculating the density function and for determining confidence limits were explored. Density functions were predicted for both synthesized and experimental data and compared with observed spectral estimate variability.

Symbol List

$A(t)$	envelope function
a_m	even Fourier coefficient
b_m	odd Fourier coefficient
$c(t)$	band–limited Gaussian composite function
$E[\ ]$	expectation operator
$f()$	chi–square limit function
F	window correction component
$g()$	noncentral chi–square limit function
G	window correction component
$H_\pm$	window correction component
$I_\nu()$	modified Bessel function of order ν
L	lower confidence limit
m	Fourier index
M	number of Gauss–Legendre weights and abscissas
$n(t)$	band–limited Gaussian noise function
N	number of spectra in ensemble average
p	probability density function
P	amplitude of sinusoidal signal
q	index for statistical moments
Q	window bias function
$r(t)$	band–limited Gaussian signal function
R	ratio of sinusoidal energy to Gaussian energy
$S(\omega)$	spectral density function
$s(t)$	periodic signal function
t	time
T	finite Fourier transform integration time
u_j	Gauss–Legendre abscissas
U	upper confidence limit
V	relative variability
w_j	Gauss–Legendre weights
$w(t)$	window function
w_i	window coefficients
W	confidence coefficient
$X_a(t)$	even signal–plus–noise modulation function
$X_b(t)$	odd signal–plus–noise modulation function
$x(t)$	input function to energy detection system
$x_a(t)$	even noise modulation function
$x_b(t)$	odd noise modulation function
y	ensemble averaged spectral estimate
$\bar{y}$	normalized ensemble averaged spectral estimate
$\bar{y}_\ell$	lower limit of integration
$\bar{y}_u$	upper limit of integration
$\tilde{y}$	normalized Bessel function argument
$z(t)$	output function from energy detection system
α	dummy parameter of degenerate hypergeometric function
β	area under tail of probability density function
γ	dummy parameter of degenerate hypergeometric function
δ_i	improvement to approximate root
ϵ	frequency mis–match index
ζ	dummy argument of degenerate hypergeometric function
η	dummy parameter of chi–square limit function
θ	dummy variable of integration
$\vartheta(t)$	pure tone function
μ	mean
$\bar{\mu}$	normalized mean
μ_2	2nd central moment

$\bar{\mu}_2$	normalized 2nd central moment
ξ	argument of chi–square limit function
ρ	parameter for variety of functions and equations
σ^2	energy of random process or sinusoidal signal
ς	approximation to inverse standard normal distribution
τ	time interval
$\phi(t)$	phase function
$\Phi()$	degenerate hypergeometric function
χ_0^2	chi–square distribution
χ_R^2	noncentral chi–square distribution
ψ	uniformly distributed phase random variable
ω	radial frequency
$()$	time–average integration operator

Subscripts:

f	filtered function
n	narrow–band Gaussian noise
r	narrow–band Gaussian signal
s	sinusoidal signal
t	random variable associated with function

Introduction

The variability of a single spectral estimate for a time series consisting of only band–limited Gaussian noise has been shown to follow a chi–square probability density function with two degrees of freedom [1-2]. It has also been shown that the variability is independent of the length of time over which data is collected. Increasing the temporal length of data improves the spectral resolution and reduces the bias of the estimate, but does not reduce the uncertainty inherent in a single spectral estimate. The method generally employed to reduce variability is to collect an ensemble of spectral estimates and then average the ensemble on an energy or mean power basis. The variability of the ensemble average is then given by a chi–square density function

with $2N$ degrees of freedom where N is the number of independent spectra in the ensemble.

When the time series for which a spectral estimate is desired substantially violates the assumption of Gaussian variability, using the chi–square density function to describe spectral uncertainty can be misleading. The acoustic signal generated by a helicopter in flight contains periodic impulsive noise that gives rise to harmonics of the rotor blade passage frequency in the power spectrum. Spectral estimates of helicopter noise at frequencies coinciding with blade passage harmonics can show significantly different variability than that predicted by a chi–square function [3]. Quite simply, the periodic impulsive noise does not follow a Gaussian distribution and, hence, a chi–square density is inadequate to describe the statistical behavior of its spectral estimate.

The purpose of this paper is to show that the mathematical methods necessary to describe the effect of periodic impulsive components on the variability of spectral estimates already exist in the literature in books by Davenport and Root [4], Whalen [5], and Burdic [6], as well as in a variety of scientific papers associated with signal detection methods [7-9]. Those methods are discussed very briefly in this paper and are presented in more detail in Appendix A. A Gaussian approximation to the resultant statistical function is also provided. Numerical methods for evaluating the statistical functions, for integrating the functions, and for determining confidence limits are presented in Appendix B. A FORTRAN 77 computer program that evaluates theoretical confidence limits for chi–square distributions is listed in Appendix C.

Finally, spectral estimate techniques will be discussed briefly and the mathematical model will be compared with spectral estimates of real and synthesized data to examine its validity. Synthesized data will be used to most accurately represent the assumptions and approximations made in the development of the model and to permit an examination of the effects of spectral bias. Comparison with real data will include recordings of tiltrotor hovers at short range and helicopter long range approach flights.

Mathematical Model

Noncentral Chi–Square Distribution

Power spectra may be obtained from time series data using any of three equivalent methods [2]. One approach is to perform a Fourier transform on the autocorrelation of the time series by the Blackman–Tukey procedure. Another is to directly calculate a finite Fourier transform of the original time series data. A third way to obtain a power spectrum, and the approach to be modeled here, is to process the time series data with an

energy detection system [10] composed of three components that operate on the data in series as shown in Figure 1. The first component of the system is a narrow band–pass filter centered on the frequency of interest. The second component is a square–law detector that squares the output of the filter. The output of this detector is proportional to the instantaneous power in the filter band. The third component is an averager or integrator that is equivalent to a low–pass filter. The integrator converts the instantaneous power output from the square–law detector into a time–averaged or mean power that is equivalent to a spectral estimate in the frequency band over the time of integration.

Figure 1. – Energy detector equivalent of spectral estimate.

Correspondence of the energy detection system spectral method with the other spectral methods requires additional constraints. The first is that the width of the band–pass filter must be small compared to the center frequency of the filter. The second constraint is that the width of the low–pass filter must be equal to the width of the band–pass filter. If the integration time of the low–pass filter (which is the inverse of the filter width) is equal to the time length of data used with the other spectral methods, the spectral estimates are equivalent. As will be seen later, the integration time will be allowed to increase without limit for the purposes of the mathematical development. One consequence of this is the elimination of bias from the density functions derived herein. Obviously, the practical calculation of spectral estimates requires a finite data set, but the mathematical ploy is intended to make the derivation tractable and will be tolerated for that purpose. The issue of bias will be raised again, briefly, in the discussion of spectral estimates in the data comparison section.

When analyzing acoustic data from wind tunnel and outdoor experiments involving aircraft, it is convenient to distinguish between that part of the acoustic data arising from the operation of the aircraft and the background, or ambient, part. The background will be referred to as noise while that from the aircraft will be referred to as signal. The signal is further assumed to consist of a periodic and a random part. Certain idealizations of system behavior and acoustic signal and noise characteristics are made to simplify the analysis. The background noise is assumed to be a band–limited Gaussian process and the signal is assumed to be the sum of a periodic part and a band–limited Gaussian process that is independent of the noise process. The

methods described by Davenport and Root [4] are utilized in the ensuing development. The input function to the energy detection system $x(t)$ is given by the sum of the signal and noise as

$$x(t) = s(t) + r(t) + n(t)$$

where $s(t)$ is the periodic part of the signal, $r(t)$ is the Gaussian part of the signal and $n(t)$ is the noise.

The band–pass filter is assumed to pass all frequency components in the band without distortion and reject all other frequencies perfectly. The output from the filter $x_f(t)$ is then given by

$$x_f(t) = s_f(t) + r_f(t) + n_f(t)$$

where $s_f(t)$ is a constant–amplitude sine wave with a constant frequency in the pass band, $r_f(t)$ is a narrow–band Gaussian random process associated with the signal, and $n_f(t)$ is a narrow–band Gaussian random process associated with the background noise. Because the signal and noise random processes are independent, it is possible to define a composite narrow–band Gaussian random process

$$c_f(t) = r_f(t) + n_f(t)$$

where the variance of the composite process, σ_c^2, is equal to the sum of the variances of the narrow–band processes associated with the signal, σ_r^2, and the background noise, σ_n^2. For consistency of notation the energy associated with the sinusoidal part of the signal will be referred to as $\sigma_s^2 = P^2/2$ where P is a constant amplitude.

The integrator is assumed to be an ideal low–pass filter that passes all frequency components in the band without distortion and rejects all other frequencies perfectly. The output from the integrator and, hence, from the energy detection system is given by $z(t)$. The probability density function of a sample of the output, z_t, can be written (see Appendix A for derivation)

$$p(z_t) = \left(\frac{1}{\sigma_c^2}\right) e^{-[z_t + \sigma_s^2]/\sigma_c^2} I_0\left(\frac{2\sigma_s\sqrt{z_t}}{\sigma_c^2}\right)$$

which is referred to as a noncentral chi–square distribution [5]. This probability density function describes the variability inherent in a single spectral estimate when the time series contains a periodic part that is manifested as a harmonic in the frequency band. When there is no periodic part contained in the time series or

when the frequency band under consideration does not include a harmonic of the periodic signal, the density function is simply

$$p(z_t) = \left(\frac{1}{\sigma_c^2}\right) e^{-z_t/\sigma_c^2}$$

which is an exponential function or, equivalently, a chi–square function of two degrees of freedom. These density functions exactly correspond to results obtained by Burdic [6] for a matched filter detector in an active pulse–echo detection system.

When an ensemble of N independent samples of the detection system output is averaged, the new random variable

$$y = \frac{1}{N}\sum_{\ell=1}^{N}(z_t)_\ell$$

has a probability density function that can be obtained by analogy with the results for the multiple pulse matched filter detector given by Burdic [6] or by coordinate transformation of the normalized distribution of Whalen [5]

$$p(y) = \frac{N}{\sigma_c^2}\left(\frac{y}{\sigma_s^2}\right)^{\frac{N-1}{2}} e^{-N(y+\sigma_s^2)/\sigma_c^2} I_{N-1}\left(\frac{2N\sigma_s\sqrt{y}}{\sigma_c^2}\right)$$

This probability density function describes the variability inherent in an ensemble averaged spectral estimate when the time series contains a periodic part that is manifested as a harmonic in the frequency band. When there is no periodic part contained in the time series or when the frequency band under consideration does not include a harmonic of the periodic signal, the density function is simply

$$p(y) = \frac{N}{\sigma_c^2\Gamma(N)}\left(\frac{Ny}{\sigma_c^2}\right)^{N-1} e^{-Ny/\sigma_c^2}$$

which is a chi–square function of $2N$ degrees of freedom. An interesting feature of this equation is that when there is no random component of acoustic signal in the time series, the σ_c^2 term can simply be replaced by σ_n^2. Consequently, the origins of the random components of the time series, at least insofar as they are Gaussian and independent, are unimportant to the shape of the density function. It is also clear that where random Gaussian components alone are present, their relative levels are unimportant and the absolute level has no effect on the relative variability about the mean.

Gaussian Approximation

When the number of spectra included in the average is large, the Central Limit Theorem can be invoked [11] to approximate the noncentral chi–square function with a Gaussian function. To do this, however, a mean and variance are required. The qth moment of the noncentral chi–square distribution is given by

$$E[y^q] = \int_0^\infty y^q p(y)dy$$
$$= \left(\frac{\sigma_c^2}{N}\right)^q \left[\frac{\Gamma(q+N)}{\Gamma(N)}\right] \Phi\left(-q, N; \frac{-N\sigma_s^2}{\sigma_c^2}\right)$$

where the degenerate hypergeometric function Φ is given by [13]

$$\Phi(\alpha, \gamma; \zeta) = 1 + \left(\frac{\alpha}{\gamma}\right)\zeta + \frac{\alpha}{\gamma}\left(\frac{\alpha+1}{\gamma+1}\right)\frac{\zeta^2}{2!} + \dots$$

The mean of an ensemble average spectral estimate is therefore

$$\mu = E[y] = \sigma_c^2 + \sigma_s^2$$

which indicates that, as mentioned in the introduction, allowing the integration length to grow without limit has allowed the derivation of a probability density function that shows no effect of the bias associated with finite data lengths. The variance of an ensemble average is given by

$$\mu_2 = E[y^2] - \mu^2 = (\sigma_c^4 + 2\sigma_c^2\sigma_s^2)/N$$

from which a Gaussian probability density function can now be written

$$p(y) \approx \frac{1}{\sqrt{2\pi\mu_2}} e^{-(y-\mu)^2/2\mu_2}$$

that approximates the noncentral chi–square function for large sample size or when the ratio of periodic energy to Gaussian energy in the spectral band is great. Two parameters useful for quantifying the deviation of a statistical distribution from a Gaussian shape and, hence, the applicability of a Gaussian approximation are skewness

$$\gamma_1 = \frac{2}{\sqrt{N}}\left[\frac{1+3R}{(1+2R)^{3/2}}\right]$$

and (excess) Kurtosis

4

$$\gamma_2 = \frac{6}{N}\left[\frac{1+4R}{(1+2R)^2}\right]$$

where $R = \sigma_s^2/\sigma_c^2$ is the ratio of periodic or tonal energy in the spectral band to the total Gaussian energy (signal and noise) in the band.

By defining a variable that quantifies the relative variability of a spectral estimate about the expected value

$$V = \frac{\sqrt{\mu_2}}{\mu} = \sqrt{\frac{\sigma_c^4 + 2\sigma_c^2\sigma_s^2}{N(\sigma_c^2 + \sigma_s^2)^2}} = \sqrt{\frac{1+2R}{N(1+R)^2}}$$

it is possible to show that when no periodic components are present in the time series, then

$$V = \sqrt{\frac{1}{N}}$$

which serves to reiterate the point made above. When there is a periodic component to the time series that is substantially greater than the total random component the relative variability is approximately

$$V \approx \sqrt{\frac{2\sigma_c^2}{N\sigma_s^2}} = \sqrt{\frac{2}{NR}}$$

and the relative variability decreases as the ratio of tonal energy to random energy increases. Clearly, the relative variability is always less when there is a tonal component present than when there is not. If the ratio of tonal energy to Gaussian energy increases without bound or if the number of samples in the ensemble average increases without bound, both skewness and (excess) Kurtosis approach zero and the Gaussian approximation approaches the noncentral chi–square density function

Confidence Limits

The confidence limits of a density function defined only for positive arguments are determined by integrating under the tails

$$\beta = \int_0^L p(y)dy$$

and either

$$\beta = \int_U^\infty p(y)dy \quad \text{or} \quad 1-\beta = \int_0^U p(y)dy$$

and solving for the lower confidence limit, L, and the upper confidence limit, U, where $W = 1 - 2\beta$ is the confidence coefficient for a two–tailed interval. When there is no sinusoidal component in a spectral estimate, the density function is chi–square with $2N$ degrees of freedom and it can be shown that the limits are given by $L = \xi/N$ where ξ is determined by solving

$$\frac{1}{2}(1 + W) = e^{-\xi}\sum_{k=0}^{N-1}\frac{\xi^k}{k!}$$

and $U = \xi/N$ where ξ is determined by solving

$$\frac{1}{2}(1 - W) = e^{-\xi}\sum_{k=0}^{N-1}\frac{\xi^k}{k!}$$

The expected value of the normalized density function for noise only is just unity and the variance is given by $1/N$. When a sinusoidal component is present in a spectral estimate, the density function is noncentral chi–square with $2N$ degrees of freedom and numerical integration must be performed. The details of the numerical integration scheme are given in Appendix B. Solving for the upper and lower confidence limits involves finding the root of each of the integrations. The numerical methods used for this process are also detailed in Appendix B.

A comparison of the noncentral chi–square, χ_R^2, and chi–square, χ_0^2, distributions is shown in Figure 2 below. For this example, both distributions have four degrees of freedom because they describe an average of two spectra. Both distributions have the same mean (or total energy) but the ratio of tonal energy to broadband energy is $R = 10$ (or 10 dB) for the noncentral distribution. The horizontal scale of the plot is normalized by the noise energy of the noncentral distribution so the total energy of both distributions is given by $1 + R$. The noncentral distribution, denoted by the solid line, is noticably narrower than the central distribution, denoted by the dashed line.

The upper and lower 80% confidence limits, expressed in decibels, are asymmetrically spaced because of both the decibel scale and the asymmetry of the distributions. The plot shows there is an 80% confidence that the actual spectral level is no more than 1.43 dB above and no more than 1.94 dB below the estimated level when the tone–to–noise ratio is 10 dB in the spectral band. This compares with an 80% confidence interval from 2.89 dB above to 5.75 dB below an estimate in a spectral band containing only broadband noise. As the tone–to–noise ratio approaches zero the shape of the noncentral chi–square distribution approaches the

5

shape of the chi–square distribution. As the tone–to–noise ratio increases the noncentral chi–square distribution becomes narrower and the confidence limits approach the estimated spectral level.
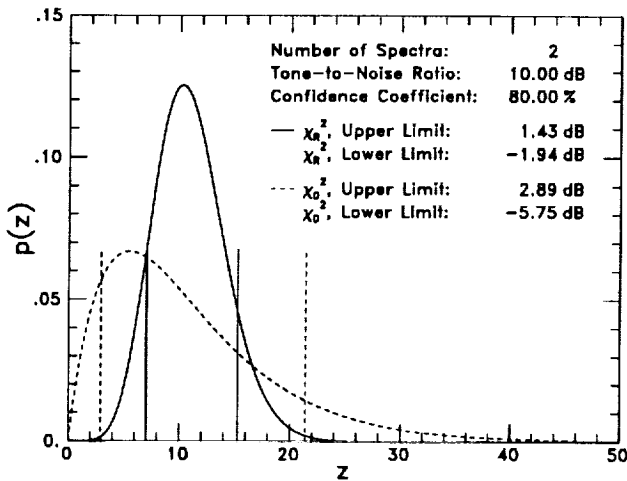

Figure 2. – Comparison of noncentral chi–square, χ_R^2, and chi–square, χ_0^2, distributions.

A comparison of the noncentral chi–square distribution, χ_R^2, and a Gaussian approximation is shown in Figure 3 below. The noncentral chi–square distribution again has four degrees of freedom because it describes an average of two spectra. The tone–to–noise ratio is 5 dB so $R \approx 3.16$. The Gaussian distribution has the same mean (or total energy) as the noncentral distribution. The horizontal scale of the plot is again normalized by the noise energy of the noncentral distribution so the total energy of both distributions is again given by $1 + R$. The variance of the Gaussian distribution is given by $(1 + 2R)/N$. The noncentral distribution, denoted by the solid line, is noticeably more skewed than the Gaussian distribution, denoted by the dashed line, which is symmetric.
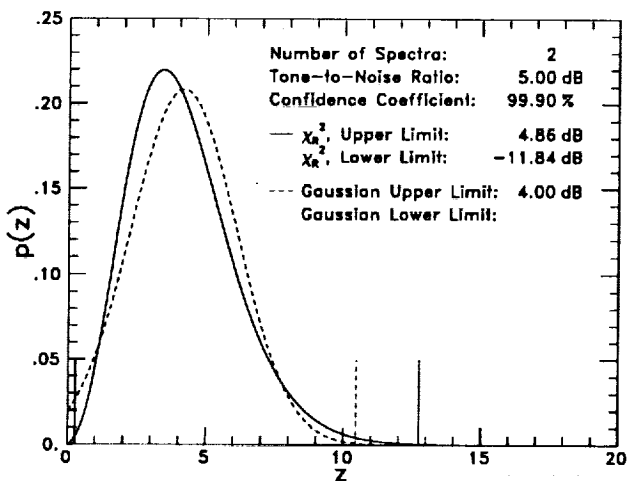

Figure 3. – Comparison of noncentral chi–square, χ_R^2, and Gaussian distributions.

The plot shows there is 99.9% confidence that the actual spectral level is no more than 4.86 dB above and no more than 11.84 dB below the estimated level when the tone–to–noise ratio is 5 dB in the spectral band. The Gaussian distribution has an upper 99.9% confidence limit of 4 dB but an undefined lower confidence limit because the lower tail of the curve extends below zero. As either the tone–to–noise ratio or the number of spectra increases the Gaussian distribution more closely approximates the noncentral chi–square.

A comparison of the 80% confidence limits from the noncentral chi–square distribution, χ_R^2, and chi–square, χ_0^2, distribution is shown in Figure 4 with a tone–to–noise ratio for the noncentral distribution of $R = 10$ (or 10 dB). As the number of spectra included in a spectral estimate increases the confidence limits of both distributions tend to converge toward the estimate. The limits of the noncentral distribution, denoted by the solid line, are always within the limits of the central distribution, denoted by the dashed line.
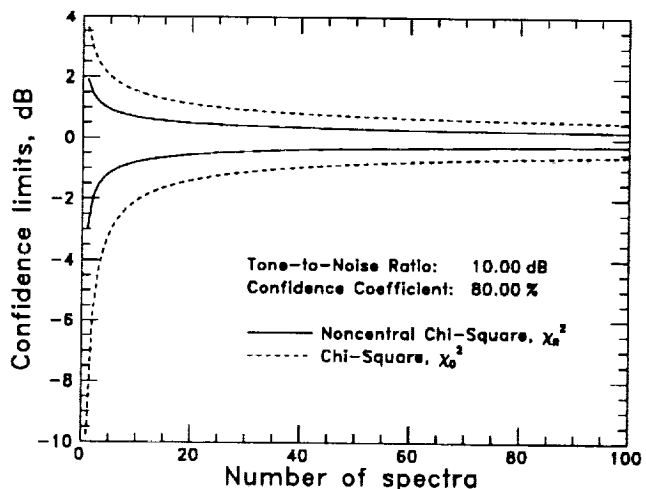
Figure 4. – Comparison of confidence limits from noncentral chi–square, χ_R^2, and chi–square, χ_0^2, distributions.

A comparison of the 99.9% confidence limits from the noncentral chi–square distribution, χ_R^2, and Gaussian approximation is shown in Figure 5 with a tone–to–noise ratio of $R \approx 3.16$ (or 5 dB). As the number of spectra included in a spectral estimate increases the confidence limits of both distributions tend to converge toward the estimate and towards each other. The limits of the noncentral distribution, denoted by the solid line, are always above the limits of the central distribution, denoted by the dashed line.

A comparison of the 90% confidence limits from the noncentral chi–square distribution, χ_R^2, its Gaussian approximation, the chi–square distribution, χ_0^2, and its Gaussian approximation is shown in Figure 6 with a tone–to–noise ratio of $R \approx 3.16$ (or 5 dB). The Gaussian

approximations converge on their respective chi–square distributions as the number of spectra included in an average increases. The convergence tends to be more rapid for the noncentral chi–square distribution and its approximation.
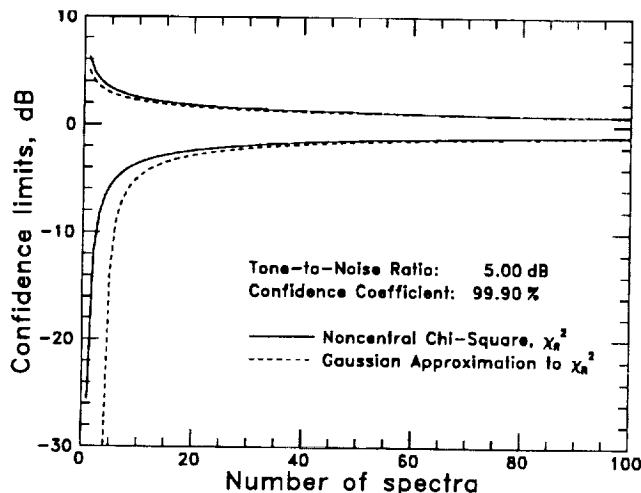

Figure 5. – Comparison of confidence limits from noncentral chi–square, χ_R^2, and Gaussian approximation to χ_R^2.
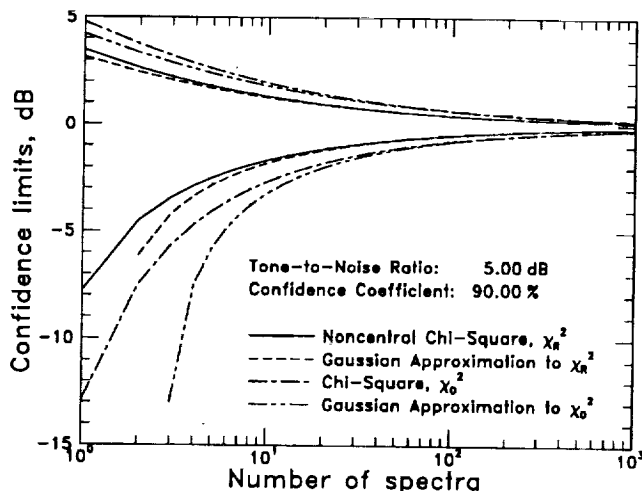

Figure 6. – Comparison of confidence limits from various distributions.

Graphs of 80% and 90% confidence limits are shown in Figures 12 and 13, respectively, for tone–to–noise ratios of $-\infty$, 0, 5, 10, and 20 dB and for number of spectra ranging from 1 to 25.

Comparison of Model with Data

Spectral Estimates

The development of the noncentral chi–square distribution depends on the assumption that the integration time of a Fourier transform is allowed to increase without limit. Practical evaluation of spectral estimates generally entails the Fast Fourier Transform (FFT) which is a discrete finite transform. Both the discretization process and the finite length data record introduce features to spectral estimates that are not considered in the development of the density function presented here. Since the model is to be compared with data that is necessarily finite, the effect of bias on a spectral estimate that is introduced by a finite transform must be considered. The bias in a spectral estimate of random noise has been shown to be roughly proportional to the second derivative of the spectrum and inversely proportional to the square of the period of integration [1]. For this derivation, though, only the bias introduced by a finite transform on a pure tone is considered.

Let a pure tone of frequency ω_o be represented by

$$\vartheta(t) = \sin(\omega_o t + \psi)$$

where ψ is a random phase distributed uniformly on $[-\pi, \pi]$. The magnitude of a spectral estimate at ω_e can then be written as

$$Q(\psi) = |\Theta(\omega_e)|^2$$

where

$$\Theta(\omega_e) = \frac{2}{T}\int_0^T w(t)\vartheta(t)\cos(\omega_e t)dt$$
$$+ \frac{i2}{T}\int_0^T w(t)\vartheta(t)\sin(\omega_e t)dt$$

is the finite Fourier transform of the windowed tone and $w(t)$ is the window function. The expected value of $Q(\psi)$ is then given by

$$Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(\psi)d\psi$$

For a Dirichlet or rectangle window, $w(t) = 1$, with $\omega_e = \omega_o$ the expected value of the magnitude is $Q = 1$ indicating no bias. The expected value of the magnitude of a spectral estimate for a windowed pure tone of a frequency different from the estimate frequency is then an estimate of bias introduced by the frequency mismatch and finite window length.

For a discrete transform the spectral estimates are made only at discrete frequencies

$$\omega_e = k\Delta\omega$$

where $\Delta\omega = 2\pi/T$ is the resolution of an FFT, T is the integration time or length of a data record, and k

7

is a frequency index. The tone frequency may then be specified as

$$\omega_o = (k + \epsilon)\,\Delta\omega$$

where k is the index that refers to the discrete frequency nearest the tone frequency and $\epsilon = (\omega_o - \omega_e)/\Delta\omega$ is an indicial measure of the difference between tone and estimate frequencies such that

$$-\frac{1}{2} \leq \epsilon \leq \frac{1}{2}$$

A number of common window functions [12] may be represented by

$$
\begin{aligned}
w(t) = w_0 &- w_1 \cos\left(\Delta\omega t\right) \\
&+ w_2 \cos\left(2\Delta\omega t\right) \\
&- w_3 \cos\left(3\Delta\omega t\right)
\end{aligned}
$$

where various choices for the coefficients w_i define particular windows. The coefficients may be chosen so that the window is power preserving insofar as the total energy or mean power of a spectral estimate of a random process is unbiased on the average

$$\frac{1}{T}\int_0^T w^2(t)\,dt = 1$$

Coefficients of a number of window functions of this type [12] expressed in a power preserving form are shown in Table 1 below.

Table 1. – Coefficients of selected windows.

Window	w_0	w_1	w_2	w_3
Dirichlet	1.0000	0.0	0.0	0.0
Hanning	0.8165	0.8165	0.0	0.0
Hamming	0.8566	0.7297	0.0	0.0
Blackman	0.7610	0.9060	0.1450	0.0
Kaiser–Bessel 4–sample, $\alpha = 3$	0.7463	0.9236	0.1823	0.0226
Blackman–Harris min 4–sample	0.7063	0.9614	0.2782	0.0230

The bias correction may now be written in terms of the window coefficients, w_i, estimate frequency index, k, and mis–match index, ϵ, as

$$Q = \text{sinc}^2\left(\pi\epsilon\right)\left(\frac{F^2 + G^2}{2}\right)$$

with

$$
\begin{aligned}
F &= w_0\left(\frac{2k + 2\epsilon}{2k + \epsilon}\right) + \epsilon H_+ \\
G &= w_0\left(\frac{2k}{2k + \epsilon}\right) + \epsilon H_-
\end{aligned}
$$

and

$$
\begin{aligned}
H_\pm = -\,&w_1\left(\frac{\epsilon}{\epsilon^2 - 1} \pm \frac{2k + \epsilon}{(2k + \epsilon)^2 - 1}\right) \\
+\,&w_2\left(\frac{\epsilon}{\epsilon^2 - 4} \pm \frac{2k + \epsilon}{(2k + \epsilon)^2 - 4}\right) \\
-\,&w_3\left(\frac{\epsilon}{\epsilon^2 - 9} \pm \frac{2k + \epsilon}{(2k + \epsilon)^2 - 9}\right)
\end{aligned}
$$

For the case where the tone frequency and estimate frequency are the same, $\omega_e = \omega_o$ or $\epsilon = 0$, a window introduces some bias with the correction given, for example, by

$$
\begin{aligned}
20\log_{10}[Q] &= 20\log_{10}[w_0] \\
&\approx -1.76 \text{ dB} \quad for \quad Hanning \\
&\approx -1.34 \text{ dB} \quad for \quad Hamming
\end{aligned}
$$

In other words, the discrete finite transform spectral estimate of a pure tone with a frequency other than one of the discrete estimate frequencies will exhibit a peak lower then the magnitude of the tonal energy (bias) and will show the remaining energy from that tone spread across every discrete frequency of the estimate (leakage). The value Q gives the fraction of the original tonal energy that resides in the frequency bin ω_e nearest the tone frequency ω_o. This fraction of the original tonal energy that remains in the spectral frequency bin is the value that should be used for the energy of the tonal signal, $Q\sigma_s^2$ rather than σ_s^2, in calculating the density function and evaluating confidence limits.

The bias correction may be made for any arbitrary symmetric window in the same manner. The window correction component $H_\pm$ may be expressed as

8

$$\sum_{i=1}(-1)^i w_i \left(\frac{\epsilon}{\epsilon^2 - i^2} \pm \frac{2k + \epsilon}{(2k + \epsilon)^2 - i^2} \right)$$

where the window coefficients, w_i, are determined by discrete Fourier transform of the window.

Synthesized Data

Synthesized data were used to verify the mathematical form of the noncentral chi–square density function and the numerical methods employed in its evaluation. Data records were constructed by adding white noise to sine waves. Each value of white noise for each data record was generated as an independent sample from a standard normal distribution by a commercial pseudo-random number generator and then scaled to the appropriate absolute level. The phase of the sine wave for each data record was generated as an independent sample from a uniform distribution on the interval $(0, 1)$ by a commercial pseudorandom number generator and then transformed to the interval $(-\pi, \pi)$. Each data record was transformed by an FFT after a window was applied and the squared magnitude was then calculated to generate a spectral record. A spectral average was determined from the appropriate number of spectral records and the resulting spectral estimate at the frequency of interest was recorded for statistical analysis.
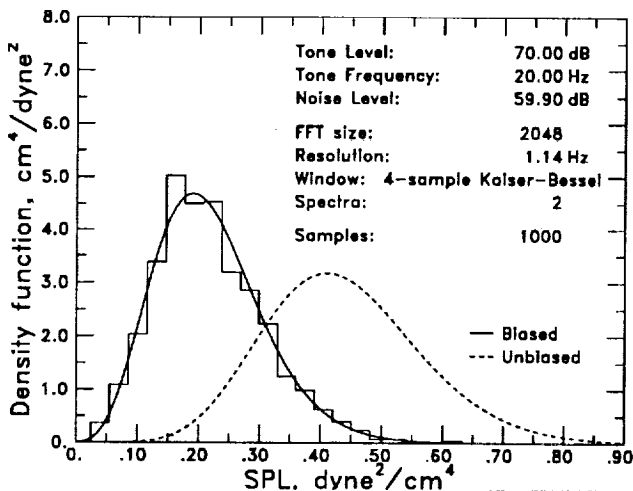

Figure 7. – Comparison of biased and unbiased noncentral chi–square distributions with an estimate obtained from synthesized data.

The results from one realization of this process are shown in Figure 7 above. A pure tone of 20 Hz at a level of 70 dB was added to noise at a level of 90 dB. The sample rate was 2344 per second so a data record size of 2048 points gave a resolution of about 1.14 Hz and yielded an average noise level of about 60 dB in each frequency band. A 4–sample Kaiser-Bessel

$(\alpha = 3)$ window was used, two–record spectral averages were calculated, and 1000 estimates at the frequency band nearest the tone, 19.457 Hz, were recorded. The histogram shows the approximate density function estimated from the synthesized data. The smooth curve shown by the solid line is the noncentral chi–square density function where the window bias correction was made. The smooth curve shown by the dashed line is the noncentral chi–square density function where no window bias correction was made.

The bias corrected theoretical curve agrees very well with the histogram estimate from synthesized data. A parametric evaluation of the agreement between theory and synthesized data is summarized in Table 2 below. The parameters of interest, mean, standard deviation, skewness, and (excess) Kurtosis, are listed in the first column. The values for those parameters estimated from the distribution of spectral estimates are listed in the second column. The corresponding bias corrected theoretical parameters appear in the third column. The t statistic in the fourth column is for testing the null hypothesis that the estimated and theoretical parameters are equal. The critical value at the 50% level for 1000 samples is $t_{.50[999]} = .6744$ so the null hypothesis is not rejected for either the mean or the standard deviation. The critical value at the 50% level for infinite degrees of freedom is $t_{.50[\infty]} = .6742$ so the null hypothesis is also not rejected for either the skewness or (excess) Kurtosis.

Table 2. – Selected parameters of biased noncentral chi–square distribution.

Parameter	Estimate	Theory	t statistic
Mean	0.2182	0.2193	-.3791
Std. Dev.	0.0889	0.0883	0.2910
Skewness	0.6224	0.6417	-.2495
Kurtosis	0.4545	0.5580	-.6698

Given the conflicting assumptions necessary to derive the model and correct for bias, as well as limitations imposed by the random number generator and the use of a discrete transform, it would seem that the noncentral chi–square distribution appropriately describes the distribution of spectral estimates for a pure tone in Gaussian noise and that the numerical evaluation of the distribution is accurately accomplished.

Tiltrotor Hover Data

Acoustic data acquired during an XV–15 tiltrotor hover test were obtained to evaluate the applicability of the noncentral chi–square distribution to helicopter spectral estimates. Data from a single channel were converted to engineering units and spectra were calculated from sequential segments of data with no overlap in the same manner as the synthesized data. Spectral

averages were determined from the appropriate number of spectral records and the resulting spectral estimate at the frequency of greatest magnitude was recorded for statistical analysis.

Because the noise level and tone level were not known a priori, they were estimated from the spectra. The sample rate for these data was 24590 per second so a data record size of 8192 gave a resolution of about 3 Hz. A 4–sample Kaiser–Bessel ($\alpha = 3$) window was used, two–record spectral averages were calculated, and 44 estimates were made from the limited amount of data available. An average spectrum of all the estimates is shown in Figure 8 below. The solid line is the average of all of the spectra while the dashed lines describe the envelope containing the 44 spectral averages of two spectra each. The peak spectral level occurs at 27.02 Hz and the background noise per band was estimated from the spectral average curve to be about 56 dB at that frequency. The tone level in that frequency band was then found by subtracting the noise energy in the band from the total energy in the band. No correction was made for bias because the spectral levels used for this calculation were already biased.
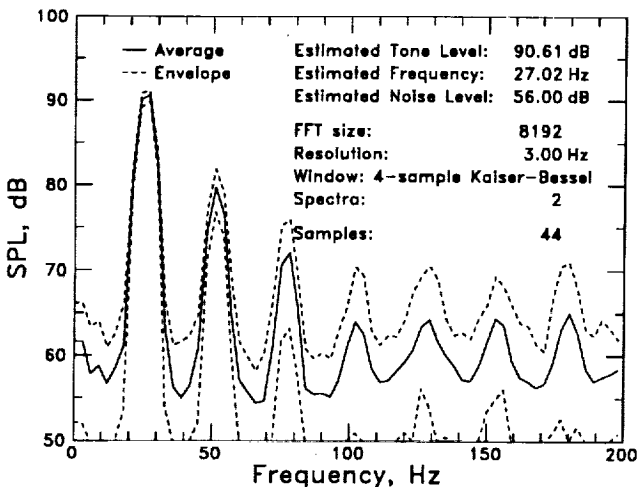

Figure 8. – XV–15 tiltrotor spectral average and envelope of spectra comprising the average.

The estimated tone and noise levels were used to generate the noncentral chi–square density function for comparison with experimental data shown in Figure 9 below. The histogram shows the approximate density function estimated from the hover data. The smooth curve shown by the solid line is the noncentral chi–square density function based on estimated tone and noise levels. The smooth curve shown by the dashed line is the chi–square density function for the total energy contained in the frequency band.

Figure 9. – Comparison of noncentral chi–square and chi–square distributions with an estimate obtained from XV–15 tiltrotor hover data.

The noncentral chi–square curve is much narrower than the histogram estimate from the hover data. On the other hand, the chi–square curve is very much broader than the histogram. While the histogram seems somewhat closer in appearance to the noncentral theoretical distribution, the difference is striking when compared to the excellent agreement with synthesized data.

An examination of assumptions used in deriving the theoretical distribution should indicate the source of disagreement between experiment and theory. Assumptions made in deriving the theoretical function include requirements that the noise be Gaussian and that the tone, or the Fourier component of the periodic signal, maintain constant frequency and amplitude. The hovering tiltrotor was 100 feet above the ground at a horizontal distance of 500 feet from the microphone so propagation induced variability is an unlikely culprit. The high tone–to–broadband ratio and the brevity of the test, 30 seconds, make it unlikely that any non–Gaussian or nonstationary character of the noise would explain the disagreement. The blade passage frequency should be stable so it seems that source amplitude variations may account for the greater than expected variability. Either the source level or the distribution of energy among harmonics varied.

Helicopter Approach Data

Acoustic data acquired during a Sikorsky S–76 helicopter approach were also obtained to evaluate the applicability of the noncentral chi–square distribution to helicopter spectral estimates. Data from a single channel were converted to engineering units and spectra were calculated from segments of data in the same manner as with the other data except for an overlap of about 35% between successive data segments. Spectral

overlapping was used to get as many independent estimates as possible in a short time interval. Harris [12] has suggested that transforms with this level of overlap are essentially independent when good windows are used. Spectral averages were determined from the appropriate number of spectral records and the resulting spectral estimate at the frequency of the second blade passage harmonic was recorded for statistical analysis.

Because the noise level and tone level were not known a priori, they were estimated from the spectra. The sample rate for these data was 2344 per second so a data record size of 2048 gave a resolution of about 1.14 Hz. A 4–sample Kaiser–Bessel ($\alpha = 3$) window was used, three–record spectral averages were calculated, and 35 estimates were made from a small fraction of the available data. The helicopter was approaching at a constant speed so the range was constantly shrinking and the tone level constantly increasing. The data were selected from a fairly short time interval at a relatively long range to minimize the relative effect of the changing range yet give sufficient data for a histogram estimate of the density function.

An average spectrum of all the estimates from these limited data is shown in Figure 10 below. The solid line is the average of all of the spectra while the dashed lines describe the envelope containing the 35 spectral averages of three spectra each. The peak of the second harmonic of the blade passage frequency occurs at 48.07 Hz and the background noise per band was estimated from the spectral average curve to be about 58 dB at that frequency. The tone level in that frequency band was then found by subtracting the noise energy in the band from the total energy in the band. No correction was made for bias because the spectral levels used for this calculation were already biased.
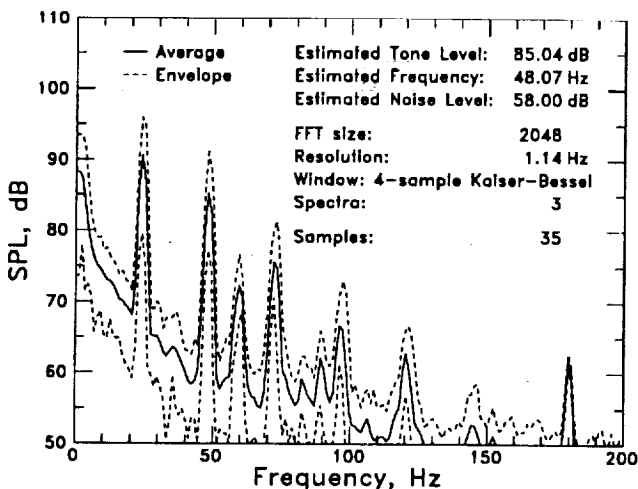
Figure 10. – S–76 helicopter spectral average and envelope of spectra comprising the average.

The estimated tone and noise levels were used to generate the noncentral chi–square density function for

comparison with experimental data shown in Figure 11 below. The histogram shows the approximate density function estimated from the approach data. The smooth curve shown by the solid line is the noncentral chi–square density function based on estimated tone and noise levels. The smooth curve shown by the dashed line is the chi–square density function for the total energy contained in the frequency band.
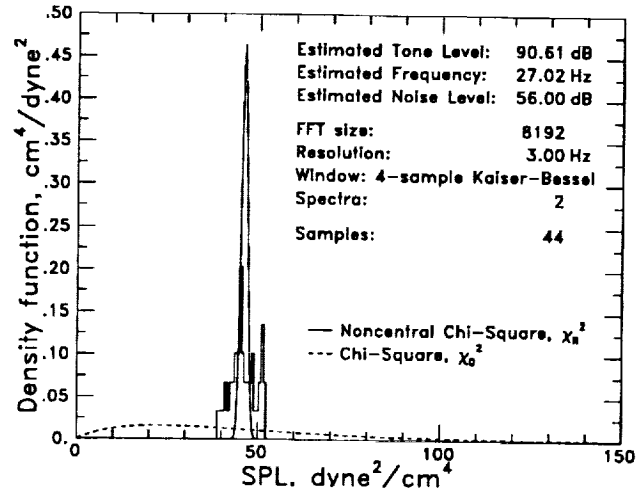
The noncentral chi–square curve is much narrower than the histogram estimate from the hover data. The chi–square curve shows much better agreement with the histogram and would appear to correctly describe the variability of the spectral estimates made from this data set. The helicopter approached the microphone from a considerable distance and it is virtually certain that propagation induced variability plays a significant role in the variability of the received signal. The frequency stability of the source is probably good but the highly directional nature of the sound from a helicopter rotor and variations in aircraft attitude during forward flight at low altitude may cause variations in the sound level emitted in the direction of the microphone. Examination of the envelope of spectra in Figure 10 reveals that tone variability was approximately the same as background variability.
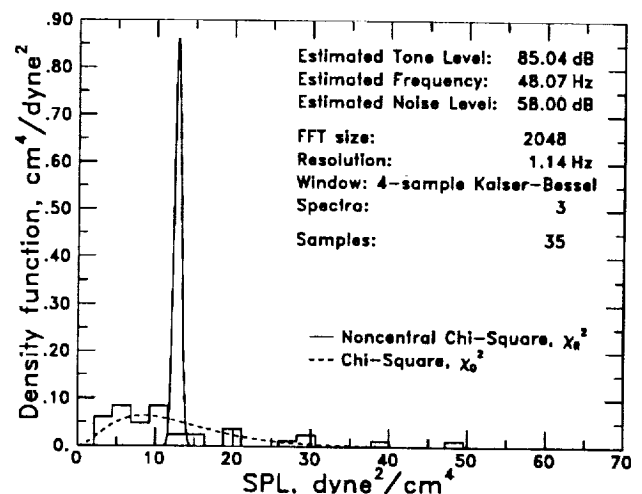
Figure 11. – Comparison of noncentral chi–square and chi–square distributions with an estimate obtained from S–76 helicopter approach data.

Summary and Conclusions

A probability density function for the variability of ensemble average spectral estimates from helicopter acoustic signals in Gaussian background noise was evaluated. A brief summary of the noncentral chi–square and chi–square distributions and Gaussian approximations to them was presented. The details of the development were presented in an appendix. Numerical methods used to calculate the density function and determine

confidence limits were presented in another appendix. A FORTRAN program that implements the numerical methods to calculate confidence limits also appears in an appendix.

Examples were plotted to show differences and similarities between the density functions. Plots were also presented to show differences between confidence limits versus the number of spectra included in an average for various confidence coefficients and tone–to–noise ratios. The noncentral chi–square density function was then compared with synthesized data that closely approximated assumptions made in development of the model, with short range tiltrotor hover data, and with very long range helicopter approach data.

The excellent agreement between synthesized data and theoretical curves indicates that the numerical methods and computer program worked as desired. The somewhat poorer agreement between the tiltrotor hover data and theoretical curves is likely an indication that the assumption of constant source level was violated. In this case the noncentral chi–square distribution was imperfect but showed better agreement with the data than the chi–square distribution. The data from a helicopter approaching an observer showed very poor agreement with the noncentral chi–square distribution. The much better agreement of this data with the chi–square density function is an indication that extremely variable tone levels, whether from source variability or propagation effects, completely invalidate the use of the noncentral chi–square distribution. The noncentral chi–square distribution should give excellent agreement, however, over time scales where the tone levels do not vary significantly, as in wind tunnel measured acoustic data.

Acknowledgements

References

1. Hardin, Jay C.: *Introduction to Time Series Analysis*. NASA RP–1145, March 1986.

2. Bendat, Julius S. and Piersol, Allan G.: *Random Data: Analysis and Measurement Procedures, Second Ed.* John Wiley & Sons, Inc., 1986.

3. Rutledge, Charles K.: *On the Appropriateness of Applying Chi–Square Distribution Based Confidence Intervals to Spectral Estimates of Helicopter Flyover Data*. NASA CR–181692, August 1988.

4. Davenport, Wilbur B., Jr. and Root, William L.: *An Introduction to the Theory of Random Signals and Noise*. McGraw–Hill Book Co., Inc., 1958.

5. Whalen, Anthony D.: *Detection of Signals in Noise* Academic Press, Inc., 1971.

6. Burdic, William S.: *Underwater Acoustic System Analysis*. Prentice–Hall, Inc., 1984.

7. Rice, S. O.: *Statistical Properties of a Sine Wave Plus Random Noise*. Bell Syst. Tech. Jour., Vol. 27, 1948.

8. Reich, Edgar and Swerling, Peter: *The Detection of a Sine Wave in Gaussian Noise*. Journal of Applied Physics, Vol. 24, No. 3, March, 1953.

9. Hinich, Melvin J.: *Detecting a Hidden Periodic Signal When Its Period is Unknown*. IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP–30, No. 5, October 1982.

10. Urick, Robert J.: *Principles of Underwater Sound, Third Ed.* McGraw–Hill Book Co., Inc., 1983.

11. Hoel, Paul G.: *Introduction to Mathematical Statistics, Fourth Ed.* John Wiley & Sons, Inc., 1971.

12. Harris, F. J.: *On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform*. Proc. of the IEEE, Vol. 66, No. 1, 1978.

13. Gradshteyn, Izrail. S. and Ryzhik, Iosif. M.: *Table of Integrals, Series, and Products, Corrected and Enlarged Ed.* Academic Press, Inc., 1980.

14. Abramowitz, Milton and Stegun, Irene A.: *Handbook of Mathematical Functions*. Dover Publications, Inc., 1970.

15. Davis, Philip J. and Rabinowitz, Philip: *Methods of Numerical Integration, Second Ed.* Academic Press, Inc., 1984.
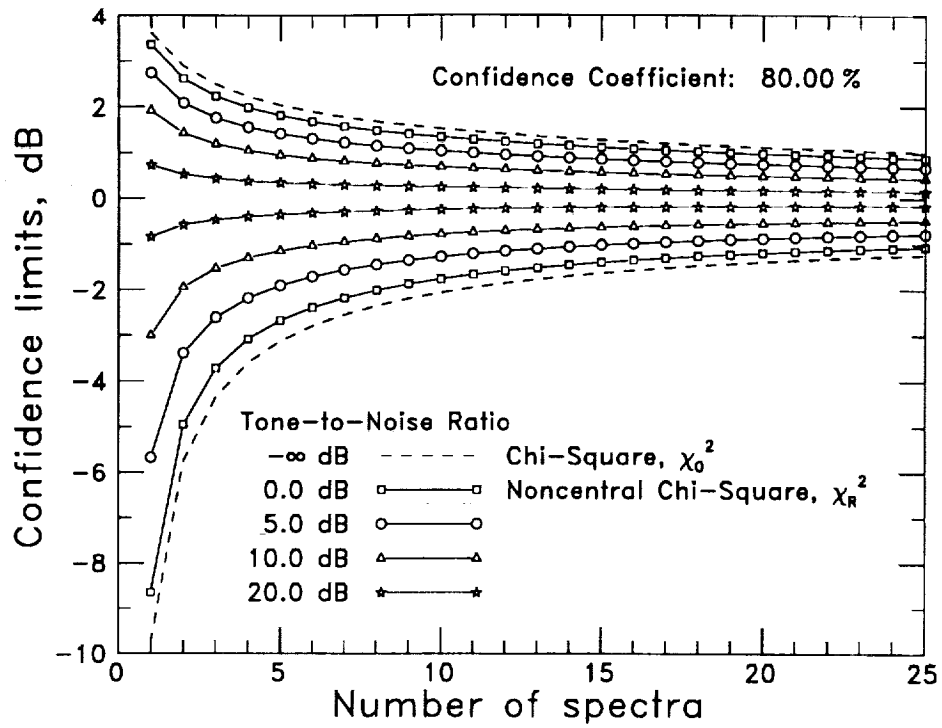
Figure 12. – 80% confidence limits for various tone–to–noise ratios and number of spectra.
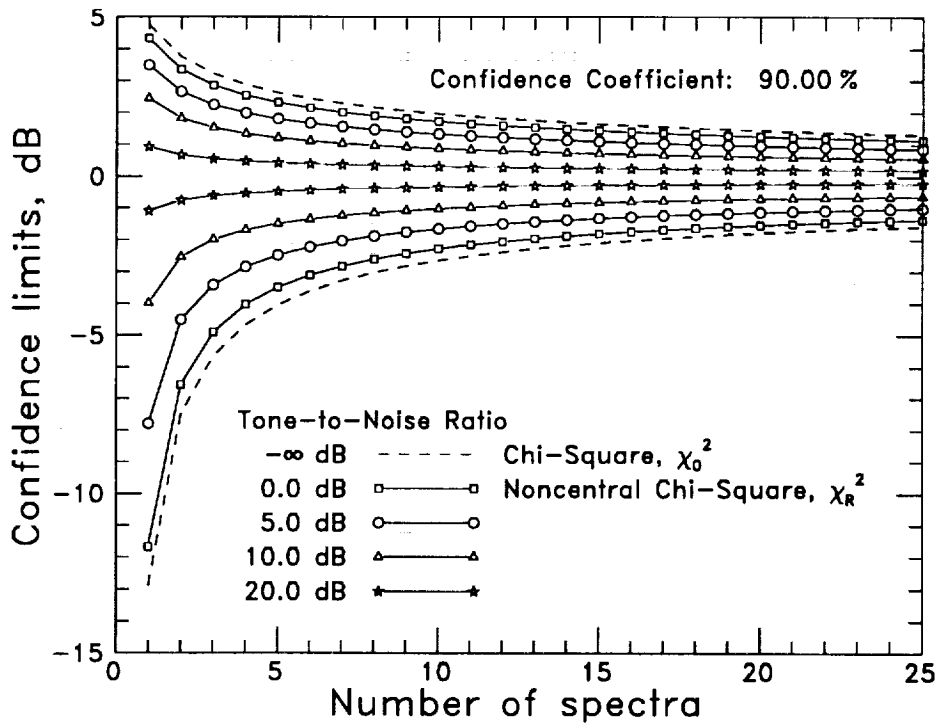
Figure 13. – 90% confidence limits for various tone–to–noise ratios and number of spectra.

Appendix A: Mathematical Model

Single Spectral Estimate

This appendix contains the derivation of the probability density function of a sample of the output of an energy detection system with an input signal consisting of a pure tone and broadband part along with broadband background noise. This derivation continues from the body of the paper. The sinusoidal part of the signal may be written as

$$s_f(t) = P \cos(\omega_s t + \psi)$$

where P is a constant amplitude, ω_s is the constant frequency of the component of the periodic signal contained within the pass band, and ψ is a random variable distributed uniformly on $(0, 2\pi)$. Over a time interval of length τ, where $0 \le t \le \tau$, the composite random process may be expressed as a Fourier series

$$c_f(t) = \sum_{m=1}^{\infty} \left[a_m \cos\left(\frac{2\pi m t}{\tau}\right) + b_m \sin\left(\frac{2\pi m t}{\tau}\right) \right]$$

where the coefficients

$$a_m = \frac{2}{\tau} \int_0^{\tau} c_f(t) \cos\left(\frac{2\pi m t}{\tau}\right) dt$$

$$b_m = \frac{2}{\tau} \int_0^{\tau} c_f(t) \sin\left(\frac{2\pi m t}{\tau}\right) dt$$

are Gaussian random variables that become uncorrelated as τ increases without limit. The composite process may be rewritten as

$$c_f(t) = x_a(t) \cos(\omega_s t) - x_b(t) \sin(\omega_s t)$$

where

$$x_a(t) = \sum_{m=1}^{\infty} \left\{ a_m \cos\left[\left(\frac{2\pi m}{\tau} - \omega_s\right) t \right] + b_m \sin\left[\left(\frac{2\pi m}{\tau} - \omega_s\right) t \right] \right\}$$

$$x_b(t) = \sum_{m=1}^{\infty} \left\{ a_m \sin\left[\left(\frac{2\pi m}{\tau} - \omega_s\right) t \right] - b_m \cos\left[\left(\frac{2\pi m}{\tau} - \omega_s\right) t \right] \right\}$$

so that the filter output may be written as the sum of the expressions for the sinusoidal part and the composite narrow–band process. Thus,

$$x_f(t) = X_a(t) \cos[\omega_s t] - X_b(t) \sin[\omega_s t]$$

where

$$X_a(t) = P \cos(\psi) + x_a(t)$$

$$X_b(t) = P \sin(\psi) + x_b(t)$$

14

The random variables x_{at} and x_{bt} from $x_a(t)$ and $x_b(t)$, respectively, can be shown [4] to be independent and Gaussian with zero mean and a mean square of

$$E\left[x_{at}^2\right] = E\left[x_{bt}^2\right] = \int_{-\infty}^{\infty} S(\omega)d\omega = \sigma_c^2 = \sigma_r^2 + \sigma_n^2$$

where ω is a dummy variable of integration, $S(\omega)$ is the spectral density, σ_c^2 is the variance of the composite narrow-band random process, σ_r^2 is the variance of the narrow-band process associated with the signal, and σ_n^2 is the variance of the narrow-band process associated with the background noise. For consistency of notation the energy associated with the sinusoidal part of the signal will be referred to as $\sigma_s^2 = P^2/2$. The joint probability density function of the random variables x_{at} and x_{bt} is then

$$p(x_{at}, x_{bt}) = \left[\left(\frac{1}{\sigma_c\sqrt{2\pi}}\right) e^{-(x_{at}/\sigma_c)^2/2}\right]\left[\left(\frac{1}{\sigma_c\sqrt{2\pi}}\right) e^{-(x_{bt}/\sigma_c)^2/2}\right]$$

from which the joint probability density function of the random variables X_{at}, X_{bt}, and ψ can be determined:

$$p(X_{at}, X_{bt}, \psi) = \frac{1}{2\pi}\left[\left(\frac{1}{\sigma_c\sqrt{2\pi}}\right) e^{-[(X_{at} - P\cos\psi)/\sigma_c]^2/2}\right]\left[\left(\frac{1}{\sigma_c\sqrt{2\pi}}\right) e^{-[(X_{bt} - P\sin\psi)/\sigma_c]^2/2}\right]$$

If the filter output is expressed in terms of a sinusoidal function of ω_s with an envelope $A(t)$ and phase $\phi(t)$ that are slowly varying functions of time compared with $\cos(\omega_s t)$:

$$x_f(t) = A(t)\cos[\omega_s t + \phi(t)]$$

then the envelope and phase may be written, respectively, as

$$A(t) = \left[X_a^2(t) + X_b^2(t)\right]^{1/2}$$

$$\phi(t) = \tan^{-1}\left[\frac{X_b(t)}{X_a(t)}\right]$$

Since the only nonvanishing terms in the expressions for $x_a(t)$ and $x_b(t)$ are those that fall in the narrow filter band, the frequency components of the envelope and phase are confined to a similar band centered on zero frequency. The joint probability density function of the envelope and phase random variables can be written as

$$p(A_t, \phi_t, \psi) = \left(\frac{A_t}{4\pi^2\sigma_c^2}\right) e^{-[A_t^2 + P^2 - 2A_t P\cos(\phi_t - \psi)]/2\sigma_c^2}$$

The probability density function of the envelope can be determined by integrating over ϕ_t and ψ (with $\theta = \phi_t - \psi$):

$$p(A_t) = \left(\frac{A_t}{\sigma_c^2}\right) e^{-(A_t^2 + P^2)/2\sigma_c^2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_{-\psi}^{2\pi-\psi} e^{(A_t P\cos\theta)/\sigma_c^2}d\theta\right]d\psi\right\}$$

which can be written as

$$p(A_t) = \left(\frac{A_t}{\sigma_c^2}\right) e^{-(A_t^2 + P^2)/2\sigma_c^2}I_0\left(\frac{A_t P}{\sigma_c^2}\right)$$

15

where $I_0()$ is the zero-order modified Bessel function.

The output from the square-law detector is

$$x_f^2(t) = A^2(t)\cos^2[\omega_s t + \phi(t)] = \frac{1}{2}A^2(t) + \frac{1}{2}A^2(t)\cos\{2[\omega_s t + \phi(t)]\}$$

The integrator is assumed to be an ideal low-pass filter that passes all frequency components in the band without distortion and rejects all other frequencies perfectly. The low-pass filter width is the same as the band-pass filter width so that the square of the envelope function, with all its frequency components in the pass band, is passed by the integrator. The high-frequency component of the square-law detector output, with a frequency of $2\omega_s$, is rejected because the band-pass filter width must be small compared with ω_s to satisfy the assumption that $c_f(t)$ is a narrow-band Gaussian random process. The output from the integrator and, hence, from the energy detection system is given by $z(t)$ where

$$z(t) = \left\langle x_f^2(t) \right\rangle = \frac{1}{2}A^2(t)$$

The probability density function of z_t can be obtained by a simple transformation to give an expression

$$p(z_t) = \left(\frac{1}{\sigma_c^2}\right) e^{-[z_t + \sigma_s^2]/\sigma_c^2} I_0\left(\frac{2\sigma_s\sqrt{z_t}}{\sigma_c^2}\right)$$

which is referred to as a noncentral chi-square distribution [5]. This is the probability density function that describes the variability inherent in a single spectral estimate when the time series contains a periodic part that is manifested as a harmonic in the frequency band.

Appendix B: Numerical Methods

Density Function Evaluation

Although the density functions appear to be relatively clear and concise, their numerical evaluation is difficult. When there is no periodic signal present the density function is the product of a constant, a power, and an exponential. In general the function exhibits a region where the power component dominates, a region where the exponential component dominates, and a region where the power and exponential components approximately balance. Because the magnitudes of the individual components can exceed the capacity of a computer to easily represent them while their product does not, the density function is evaluated numerically by expressing it in terms of an exponential only. Normalizing by the noise energy to simplify the expression gives

$$p(\bar{y}) = e^{\ln N + (N-1)\ln(N\bar{y}) - N\bar{y} - \ln(\Gamma(N))}$$

where $\bar{y} = y/\sigma_c^2$. The Gamma function is calculated by an asymptotic approximation [14] when N is sufficiently large. This expression is used even when a periodic signal is present if the ratio of tonal energy to broadband energy is less than 50 dB.

When a periodic signal of significant magnitude is present the inclusion of the modified Bessel function enormously complicates the situation. Again normalizing by the noise energy to simplify the expression, the density function with a periodic signal is

$$p(\bar{y}) = N \left(\frac{\bar{y}}{R} \right)^{\frac{N-1}{2}} e^{-N(\bar{y}+R)} I_{N-1} \left(2N\sqrt{R\bar{y}} \right)$$

There are a variety of equivalent forms for a modified Bessel function of integer order but they each present difficulties for numerical evaluation. Four different forms, all from Abramowitz and Stegun [14], were chosen for evaluating the probability density function. The first equation uses a polynomial approximation to I_0 that has a different form for each of two different regions

$$\begin{aligned} p(\bar{y}) \approx e^{-(\bar{y}+R)}[1 &+ 3.5156229\rho^2 + 3.0899424\rho^4 \\ &+ 1.2067492\rho^6 + 0.2659732\rho^8 \\ &+ 0.0360768\rho^{10} + 0.0045813\rho^{12}] \end{aligned}$$

for $\rho < 1$ and

$$\begin{aligned} p(\bar{y}) \approx e^{-\left(\sqrt{\bar{y}}-\sqrt{R}\right)^2}[0.398942281 &+ 0.01328592\rho^{-1} + 0.00225319\rho^{-2} \\ &+ 0.00157565\rho^{-3} + 0.00916281\rho^{-4} \\ &- 0.02057706\rho^{-5} + 0.02635537\rho^{-6} \\ &- 0.01647633\rho^{-7} + 0.00392377\rho^{-8}]/\sqrt[4]{4R\bar{y}} \end{aligned}$$

for $\rho \geq 1$ where $\rho = 8\sqrt{R\bar{y}}/15$.

The second equation uses a polynomial approximation to I_1 that also has a different form for each of two different regions

$$\begin{aligned} p(\bar{y}) \approx 8\bar{y}e^{-2(\bar{y}+R)}[0.5 &+ 0.87890594\rho^2 + 0.51498869\rho^4 \\ &+ 0.15084934\rho^6 + 0.02658733\rho^8 \\ &+ 0.00301532\rho^{10} + 0.00032411\rho^{12}] \end{aligned}$$

for $\rho < 1$ and

17

$$p(\bar{y}) \approx \sqrt[4]{\bar{y}/R^3}\; e^{-2(\sqrt{\bar{y}}-\sqrt{R})^2}[0.398942281 - 0.03988024\rho^{-1} - 0.00362018\rho^{-2}$$
$$+\, 0.00163801\rho^{-3} - 0.01031555\rho^{-4}$$
$$+\, 0.02282967\rho^{-5} - 0.02895312\rho^{-6}$$
$$+\, 0.01787654\rho^{-7} - 0.00420059\rho^{-8}]$$

for $\rho \geq 1$ where $\rho = 16\sqrt{R\bar{y}}/15$.

The third equation uses a uniform asymptotic expansion when both $N > 2$ and $N + \log_{10}(R) > 3$

$$p(\bar{y}) \approx e^{-N(\bar{y}+R)+\ln N + \frac{\nu}{2}(\ln \bar{y}-\ln R) - \frac{1}{2}\ln(2\pi\nu) - \frac{1}{4}\ln(1+\tilde{y}^2)+\nu\eta+\ln\left(1+\sum_{k=1}^{6} u_k(\rho)/\nu^k\right)}$$

where $\nu = N - 1$, $\tilde{y} = 2(N/\nu)\sqrt{R\bar{y}}$, $\rho = 1/\sqrt{1+\tilde{y}^2}$, and the parameter η takes the simple form

$$\eta = \sqrt{1+\tilde{y}^2} + \ln \frac{\tilde{y}}{1+\sqrt{1+\tilde{y}^2}}, \qquad \text{if } \tilde{y} \leq 2,$$

or the somewhat more complicated form

$$\eta = \tilde{y}\sqrt{1+\frac{1}{\tilde{y}^2}} + \ln \frac{1}{\frac{1}{\tilde{y}} + \sqrt{1+\frac{1}{\tilde{y}^2}}}, \qquad \text{if } \tilde{y} > 2,$$

and

$$u_{k+1}(\rho) = \frac{1}{2}\rho^2(1-\rho^2)u_k'(\rho) + \frac{1}{8}\int_0^\rho (1-5\rho^2)u_k(\rho)d\rho$$

where $u_0(\rho) = 1$.

The fourth, and final, equation uses an ascending series representation of the modified Bessel function such that

$$p(\bar{y}) \approx \sum_{k=0}^{K} e^{(N+2k)\ln N + (N+k-1)\ln \bar{y} + k\ln R - N(\bar{y}+R) - \ln[\Gamma(k+1)\Gamma(k+N)]}$$

where K is determined when the Kth term is sufficiently small compared with the sum.

Integration

A Gauss–Legendre numerical integration scheme takes the form

$$\int_{\bar{y}_\ell}^{\bar{y}_u} p(\bar{y})d\bar{y} \approx \left(\frac{\bar{y}_u - \bar{y}_\ell}{2}\right) \sum_{j=1}^{M} w_j p\left(\left(\frac{\bar{y}_u + \bar{y}_\ell}{2}\right) + u_j\left(\frac{\bar{y}_u - \bar{y}_\ell}{2}\right)\right)$$

where the M weights, w_j, and abscissas, u_j, are found on the interval $(-1, 1)$ by approximating the roots of Legendre polynomials [15]. Because the probability density function sometimes exhibits significant values only over a finite interval, a further approximation can be made by replacing the extreme limits of integration so that the confidence limits are calculated by solving

$$\beta \approx \int_{L^*}^{L} p(\bar{y})d\bar{y} \approx \left(\frac{L - L^*}{2}\right) \sum_{j=1}^{M} w_j p\left(\left(\frac{L + L^*}{2}\right) + u_j\left(\frac{L - L^*}{2}\right)\right)$$

and

$$1 - \beta \approx \int_{L^*}^{U} p(\bar{y}) d\bar{y} \approx \left(\frac{U - L^*}{2} \right) \sum_{j=1}^{M} w_j p \left(\left(\frac{U + L^*}{2} \right) + u_j \left(\frac{U - L^*}{2} \right) \right)$$

for L and U. The lower limit of integration, L^*, is arbitrarily set either to zero or to a value twelve standard deviations below the mean, whichever is greater

$$L^* = \max \left(0, \bar{\mu} - 12\sqrt{\bar{\mu}_2} \right)$$

where the normalized mean and variance are given by $\bar{\mu} = 1 + R$ and $\bar{\mu}_2 = (1 + 2R)/N$.

Root Location

Solving for the upper and lower confidence limits of a chi–square density function involves finding the root of each of two equations that exhibit the same mathematical form

$$0 = f(\xi) = \eta - e^{-\xi} \sum_{k=0}^{N-1} \frac{\xi^k}{k!}$$

where $\eta = (1 + W)/2$ for the lower limit and $\eta = (1 - W)/2$ for the upper limit. Because both the first and second derivative of this function exist in analytic form

$$f'(\xi) = e^{-\xi} \left(\frac{\xi^{N-1}}{(N-1)!} \right)$$

and

$$f''(\xi) = f'(\xi) \left(\frac{N-1}{\xi} - 1 \right)$$

rapid convergence to the root can be achieved, given an acceptable initial guess, by using a refinement of Newton's method [15]

$$\xi_{i+1} = \xi_i - \frac{f(\xi_i)}{f'(\xi_i)} \Delta_i$$

with

$$\Delta_i = 1 + \frac{f(\xi_i)}{f'(\xi_i)} \frac{f''(\xi_i)}{2f'(\xi_i)}$$

where successive improvements are made to the approximate root until some convergence criterion is met. The function need only be evaluated once at each step and additional advantages are gained by observing that $f'(\xi)$ is the last term in the summation in $f(\xi)$ and also that only the ratio of $f''(\xi)$ to $f'(\xi)$ must be calculated. The second order correction term Δ_i can cause instability so, in practice, its value is restricted to the range $(0.1, 2.0)$. Because the summation is an approximation to the exponential function

$$e^{\xi} \approx \sum_{k=0}^{N-1} \frac{\xi^k}{k!}$$

19

the function $f(\xi)$ is evaluated numerically by

$$f(\xi) = \eta - \sum_{k=0}^{N-1} e^{-\xi + k \ln \xi - \ln(k!)}$$

to minimize round–off difficulties.

An initial guess for determining the root is provided by an approximation to the inverse chi–square function given by [14]

$$\xi \approx N \left(1 - \frac{1}{9N} + \varsigma \sqrt{\frac{1}{9N}} \right)^3$$

where ς is a rational approximation to the inverse standard normal distribution given by [14]

$$\varsigma = \pm \left(\rho - \frac{2.515517 + 0.802853\rho + 0.010328\rho^2}{1 + 1.432788\rho + 0.189269\rho^2 + 0.001308\rho^3} \right)$$

The sign of ς depends on whether the tail is to the right or left and

$$\rho = \sqrt{-2 \ln \left(\frac{1}{2}(1 - W) \right)}$$

Because no simple expressions for the derivatives of the noncentral chi–square confidence limit functions exist, a secant method is used in which the first derivative term of Newton's method is replaced by a secant approximation. The root of

$$0 = g(L) = \beta - \left(\frac{L - L^*}{2} \right) \sum_{j=1}^{M} w_j p \left(\left(\frac{L + L^*}{2} \right) + u_j \left(\frac{L - L^*}{2} \right) \right)$$

which is the lower confidence limit, is found by making successive improvements to the approximate root

$$L_{i+1} = L_i + \delta_i$$

where

$$\delta_i = -\delta_{i-1} \left[\frac{g(L_i)}{g(L_i) - g(L_{i-1})} \right]$$

until some convergence criterion is met. The correction term δ_i can yield an approximate root outside the range for which the density function is defined so, in practice, its value is restricted to the range $(-.99L_i, (\bar{\mu} - L_i)/2)$. The same method is used for the upper confidence limit using the appropriate function with the exception that there are no restrictions to the range of δ_i.

An initial guess for the upper root is provided by the rational approximation to an inverse standard normal distribution, ς, given above translated and scaled by mean and variance equal to those of the noncentral chi–square distribution

$$U_1 = \bar{\mu} + \varsigma \sqrt{\bar{\mu}_2}$$

An initial guess for the lower root is provided in nearly the same way when R is large, except that the value is not allowed to be too small

$$L_1 = \max(\bar{\mu}/10, \mu + \varsigma\sqrt{\bar{\mu}_2})$$

The approximation to the inverse chi-square function, given above, is used when R is less than $1/10$. The first increment to the initial guess of the root, δ_1, is arbitrarily set to $\bar{\mu}_2/10$.

Appendix C: FORTRAN Program

The FORTRAN program CHISQR calculates theoretical confidence limits for a chi-square distribution and a noncentral chi-square distribution. An example of program output is appended.

```
      program chisqr
C
C-----------------------------------------------------------------------
C
C CHISQR calculates confidence limits for chi-square and noncentral
C chi-square probability density functions and compares them with
C Gaussian approximations
C
C-----------------------------------------------------------------------
C
      integer no
      parameter (no=51)
      real gla(no),glw(no),glx(no)
      integer io,n
      real aln,alo,arl,aru,csll,csul,del,eps,fn,fo,gcll,gcul
      real gncll,gncul,nccsul,nccsll,sdn,snn,snr,snt
      real splr,spn,tt,vv,w,xp,x1max,x1min
      double precision dn,lnu,nu,snrl,snrd,ldn,a0,a1,lngn
      common /gla/ gla
      common /glw/ glw
      common /glx/ glx
      common /dp/ n,snr,dn,lnu,nu,snrl,snrd,ldn,a0,a1
      common /lngn/ lngn
      real xion,glt
      double precision lgamma
      eps=1.e-6
      sdn=12.
C
C-----------------------------------------------------------------------
C
C Calculate Gauss-Legendre integration weights and abscissas on (-1,1)
C
      call gl(no,glx,glw)
      do io=1+(no+1)/2,no
         glx(io)=-glx(no+1-io)
         glw(io)=glw(no+1-io)
      enddo
C
C-----------------------------------------------------------------------
C
C Enter the number of spectra, tone-to-noise ratio, and confidence level
C
      write(6,699)
      write(6,601) 'Enter the number of averages     : '
      read(5,*) n
C
      write(6,600)
      write(6,601) 'Enter the tone-to-noise ratio (dB): '
      read(5,*) splr
C
```

```
      write(6,600)
      write(6,601) 'Enter the confidence level (%)    : '
      read(5,*) w
C
C-----------------------------------------------------------------
C
C First guess from standard normal distribution inverse
C
      aru=.5*(1.+.01*w)
      arl=.5*(1.-.01*w)
      tt=sqrt(-2.*alog(arl))
      xp=tt-(2.515517+tt*(.802853+tt*.010328))
     &      /(1.+tt*(1.432788+tt*(.189269+tt*.001308)))
C
C-----------------------------------------------------------------
C
C Calculate tone and noise values
C
      snr=10.**(.1*splr)
      snrd=10.d0**(.1d0*splr)
      spn=1.+snr
C
C Standard deviations of normal distributions
C
      snn=1./sqrt(float(n))
      snt=sqrt((1.+2.*snr)/n)
C
C Calculate preliminary values for EDSPDF
C
      dn=dble(n)
      ldn=dlog(dn)
      a0=dn*(ldn-snrd)
      nu=dn-1.d0
      if (n.gt.1) lnu=dlog(nu)
      snrl=dlog(snrd)
      a1=2.d0*ldn+snrl
      lngn=lgamma(n)
C
C Confidence limits for noise by chi-square distribution
C
      csul=10.*alog10(xion(arl,n))
      csll=10.*alog10(xion(aru,n))
C
C Confidence limits by Gaussian approximation
C
      gncll=10.*alog10(amax1(1.258925412e-10,(1.-xp*snt/spn)))
      gncul=10.*alog10(1.+xp*snt/spn)
      gcll=10.*alog10(amax1(1.258925412e-10,(1.-xp*snn)))
      gcul=10.*alog10(1.+xp*snn)
C
C-----------------------------------------------------------------
```

```
C
C Integration limits
C
      x1max=spn+sdn*snt
      x1min=amax1(0.,spn-sdn*snt)
C
C-----------------------------------
C
C Search for confidence limits of noncentral chi-square using a
C modified secant method for the lower limit and a secant method
C for the upper limit
C
C-----------------------------------
C
C First guess at lower limit
C
      aln=amax1(.1*spn,spn-xp*snt)
C
C First guess if chi-square
C
      if (splr.le.-10.) then
         vv=1./(9.*n)
         aln=amax1(0.,((1.-vv-xp*sqrt(vv))**3))
      endif
C
C---------
C
C First function evaluation
C
      fn=glt(x1min,aln)-arl
      del=(spn-aln)*.05
C
C---------
C
C Repeat until there is an answer
C
    1 continue
         alo=aln
         aln=alo+del
C
C Next function evaluation
C
         fo=fn
         fn=glt(x1min,aln)-arl
C
C Next delta
C
         if (fo.eq.fn) goto 2
         del=amax1(-.99*aln,amin1(.5*(spn-aln),del*fn/(fo-fn)))
         if (abs(del).lt.aln*eps) goto 2
```

25

```
      goto 1
C
C Convergence
C
    2 continue
      nccsll=10.*alog10(aln/spn)
C
C-----------------------------------
C
C First guess at upper limit
C
      aln=spn+xp*snt
C
C---------
C
C First function evaluation
C
      fn=glt(x1min,aln)-aru
      del=snt*.1
C
C---------
C
C Repeat until there is an answer
C
    3 continue
          alo=aln
          aln=alo+del
C
C Next function evaluation
C
          fo=fn
          fn=glt(x1min,aln)-aru
C
C Next delta
C
          if (fo.eq.fn) goto 4
          del=del*fn/(fo-fn)
          if (abs(del).lt.aln*eps) goto 4
      goto 3
C
C Convergence
C
    4 continue
      nccsul=10.*alog10(aln/spn)
C
C-----------------------------------
C
C Write results
C
      write(6,602) 'Noise'
      write(6,600)
```

```fortran
      write(6,603) '     Chi-Square     ','     Gaussian     '
      write(6,603) '     ----------     ','     ----------     '
      write(6,604) 'Upper limit',csul,' dB',gcul,' dB'
      write(6,604) 'Lower limit',csll,' dB',gcll,' dB'
C
      write(6,602) 'Tone + Noise,  ',splr,' dB  T/N Ratio'
      write(6,600)
      write(6,603) 'Noncentral Chi-Square ','     Gaussian     '
      write(6,603) '     ----------     ','     ----------     '
      write(6,604) 'Upper Limit',nccsul,' dB',gncul,' dB'
      write(6,604) 'Lower Limit',nccsll,' dB',gncll,' dB'
      write(6,699)
C
      stop ' '
C
C----------------------------------------------------------------------
C
C Formats
C
  600 format(x)
  601 format(x,a,$)
  602 format(////,x,a,2x,f5.1,a)
  603 format(16x,a,a)
  604 format(x,a,3x,(5x,f9.4,a,5x),(5x,f9.4,a))
  699 format(/)
C
      end
```

```fortran
      real function edspdf(z)
C
C-----------------------------------------------------------------------
C
C Noncentral chi-squared probability density function of order 2n
C
C-----------------------------------------------------------------------
C
      real z
      integer n
      real snr
      double precision dn,lnu,nu,snrl,snrd,ldn,a0,a1,lngn
      common /dp/ n,snr,dn,lnu,nu,snrl,snrd,ldn,a0,a1
      common /lngn/ lngn
      integer k
      real qeta,qim,qin,qio,qip,qrho,qt,qti,qt2,qx,qzeta
      double precision eps
      double precision zz,z0
      double precision b0,b1
      double precision lnxi,lnxim,pdf,xi,lncf
      double precision s
      double precision zn,zs,eta,us,den,z2
      double precision t,tt,ttt,tttt,ttttt
      eps=1.d-16
C
C-----------------------------------------------------------------------
C
C Evaluate PDF using polynomial approximation when N=1
C                                      (Abramowitz and Stegun p 378)
C
      if (n.eq.1) then
         if (z.eq.0.) then
            edspdf=exp(-snr)
         else
            qeta=sqrt(snr)
            qzeta=sqrt(z)
            qx=2.*qeta*qzeta
            qt=qx/3.75
            if (qt.lt.1.) then
               qt2=qt*qt
               qio=1.
     &              +qt2*(3.5156229
     &              +qt2*(3.0899424
     &              +qt2*(1.2067492
     &              +qt2*(0.2659732
     &              +qt2*(0.0360768
     &              +qt2*(0.0045813))))))
               edspdf=qio*exp(-(snr+z))
            else
```

```
               qti=1./qt
               qim=0.39894228
     &             +qti*(0.01328592
     &             +qti*(0.00225319
     &             +qti*(-.00157565
     &             +qti*(0.00916281
     &             +qti*(-.02057706
     &             +qti*(0.02635537
     &             +qti*(-.01647633
     &             +qti*(0.00392377))))))))
             qrho=qzeta-qeta
             edspdf=qim*exp(-qrho*qrho)/sqrt(qx)
           endif
         endif
         return
       endif
C
C----------------------------------------------------------------------
C
C Evaluate PDF using polynomial approximation when N=2
C                                       (Abramowitz and Stegun p 378)
C
       if (n.eq.2) then
         if (z.eq.0.) then
           edspdf=0.
         else
           qeta=sqrt(snr)
           qzeta=sqrt(z)
           qx=4.*qeta*qzeta
           qt=qx/3.75
           if (qt.lt.1.) then
             qt2=qt*qt
             qip=0.5
     &             +qt2*(0.87890594
     &             +qt2*(0.51498869
     &             +qt2*(0.15084934
     &             +qt2*(0.02658733
     &             +qt2*(0.00301532
     &             +qt2*(0.00032411))))))
             edspdf=8.*z*qip*exp(-2.*(snr+z))
           else
             qti=1./qt
             qin=0.39894228
     &             +qti*(-.03988024
     &             +qti*(-.00362018
     &             +qti*(0.00163801
     &             +qti*(-.01031555
     &             +qti*(0.02282967
     &             +qti*(-.02895312
     &             +qti*(0.01787654
```

```
     &                +qti*(-.00420059)))))))))
                  qrho=qzeta-qeta
                  edspdf=sqrt(qzeta/(snr*qeta))*qin*exp(-2.*qrho*qrho)
               endif
           endif
           return
        endif
C
C-------------------------------------------------------------------
C
C Evaluate PDF when N>2
C
        if (z.eq.0.) then
           if (n.gt.1) then
              edspdf=0.
           else
              edspdf=exp(-snr)
           endif
           return
        endif
        zz=dble(z)
        if (snrd.gt.1.d-5) then
           if (dn+dlog10(snrd).gt.3.d0) then
C
C Uniform asymptotic approximation (Abramowitz and Stegun p 378)
C
              zn=2.d0*(n/nu)*dsqrt(snrd*zz)
              z2=zn*zn
              if (zn.gt.2.d0) then
                 zs=zn*dsqrt(1.d0+1.d0/z2)
              else
                 zs=dmax1(1.d0,dsqrt(1.d0+z2))
              endif
              t=1.d0/zs
              tt=t*t
              ttt=t*tt
              tttt=tt*tt
              ttttt=tt*ttt
              if (zn.gt.2.d0) then
                 eta=zs+dlog(1.d0/(1.d0/zn+dsqrt(1.d0+1.d0/z2)))
              else
                 eta=zs+dlog(zn/(1.d0+dsqrt(1.d0+z2)))
              endif
              us=1.d0
              den=nu
              us=us+t*(.125d0
     &               +tt*(-.2083333333333333d0))/den
              den=den*nu
              us=us+tt*(.0703125d0
     &               +tt*(-.4010416666666667d0
     &               +tt*(.3342013888888889d0)))/den
```

```fortran
          den=den*nu
          us=us+ttt*(.0732421875d0
     &          +tt*(-.8912109375d0
     &          +tt*(1.846462673611111d0
     &          +tt*(-1.025812596450617d0))))/den
          den=den*nu
          us=us+tttt*(.112152099609375d0
     &          +tt*(-2.3640869140625d0
     &          +tt*(8.78912353515625d0
     &          +tt*(-11.20700261622299d0
     &          +tt*(4.669584423426248d0)))))/den
          den=den*nu
          us=us+ttttt*(.2271080017089844d0
     &          +tt*(-7.368794359479632d0
     &          +tt*(42.53499874538846d0
     &          +tt*(-91.81824154324002d0
     &          +tt*(84.63621767460074d0
     &          +tt*(-28.21207255820025d0))))))/den
          den=den*nu
          us=us+ttt*ttt*(.5725014209747314d0
     &            +tt*(-26.49143048695155d0
     &            +tt*(218.1905117442116d0
     &            +tt*(-699.5796273761326d0
     &            +tt*(1059.990452528d0
     &            +tt*(-765.2524681411817d0
     &            +tt*(212.5701300392171d0)))))))/den
          xi=dn*((nu/dn)*eta-(zz+snrd))
          xi=xi+ldn-.5d0*dlog(zs)-.5d0*lnu
          xi=xi+.5d0*nu*(dlog(zz)-snrl)
          xi=xi-0.918938533204673d0
          xi=xi+dlog(us)
          edspdf=sngl(dexp(xi))
        else
C
C Ascending series (Abramowitz and Stegun p 375)
C
          z0=dlog(zz)
          b0=a0+nu*z0-dn*zz
          b1=a1+z0
          pdf=0.d0
          k=0
          lncf=lngn
          lnxim=b0-lncf
    1     continue
            lnxi=b0+b1*k-lncf
            if ((lnxi.lt.lnxim).and.(pdf.eq.0.d0)) goto 3
            if (lnxi.lt.-300.d0) goto 2
              xi=dexp(lnxi)
              pdf=pdf+xi
              if (xi.le.eps*pdf) goto 3
    2       continue
```

31

```
                k=k+1
                lncf=lncf+dlog(dble(k))+dlog(dble(n+k-1))
                goto 1
      3         continue
                edspdf=sngl(pdf)
            endif
        else
C
C Broadband approximation when tone-to-broadband ratio is small
C
          if (n.eq.1) then
             edspdf=sngl(dn*dexp(-zz))
          else
             if (z.eq.0.) then
                edspdf=0.
             else
                s=zz*dn
                edspdf=sngl(dexp(ldn+nu*dlog(s)-(s+lngn)))
             endif
          endif
        endif
C
        return
        end
```

```fortran
      double precision function lgamma(n)
C
C-------------------------------------------------------------------
C
C Natural logarithm of the gamma function, G(n)
C
C-------------------------------------------------------------------
C
      integer n
      integer i
      double precision nn,in,ins
C
C-------------------------------------------------------------------
C
C Evaluate gamma function (Abramowitz and Stegun p 257)
C
      if (n.lt.100) then
         lgamma=0.
         if (n.gt.2) then
            do i=3,n
               lgamma=lgamma+dlog(i-1.d0)
            enddo
         endif
      else
         nn=dble(n)
         in=1.d0/nn
         ins=in*in
         lgamma=(nn-.5d0)*dlog(nn)-nn
         lgamma=lgamma+0.918938533204673d0
         lgamma=lgamma+in*(+8.333333333333333d-2
     &                 +ins*(-2.777777777777778d-3
     &                 +ins*(+7.936507936507937d-4
     &                 +ins*(-5.952380952380952d-4))))
      endif
C
      return
      end
```

```fortran
      real function xion(a,n)
C
C----------------------------------------------------------------------
C
C                              k
C                  -x  N-1   x
C Solve:    A = e     SUM ----      for x given A and N, return x/N.
C                     k=0  k!
C
C----------------------------------------------------------------------
C
      real a
      integer n
      double precision delt,eps,fxio,xin,xio,lfnk,d,xiol
      real vv,t,xp
      integer k
C
C----------------------------------------------------------------------
C
C Modified second order Newton's method when N>1 (Davis and Rabinowitz
C p 114) (first estimate by Abramowitz and Stegun p 941 via p 933)
C
      xin=-dlog(dble(a))
      if (n.gt.1) then
         vv=1./(9.*n)
         if (a.lt.0.5) then
            t=sqrt(-2.*dlog(dble(a)))
            xp=t-(2.515517+t*(.802853+t*.010328))
     &          /(1.+t*(1.432788+t*(.189269+t*.001308)))
            xin=n*(1.-vv+xp*sqrt(vv))**3
         else
            t=sqrt(-2.*dlog(1.d0-a))
            xp=t-(2.515517+t*(.802853+t*.010328))
     &          /(1.+t*(1.432788+t*(.189269+t*.001308)))
            xin=n*(1.-vv-xp*sqrt(vv))**3
         endif
C
         eps=1.d-8
    1    continue
            xio=xin
            xiol=dlog(xio)
            fxio=dexp(-xio)
            lfnk=0.d0
            do k=1,n-1
               lfnk=lfnk+dlog(dble(k))
               fxio=fxio+dexp(k*xiol-lfnk-xio)
            enddo
            delt=(fxio-a)*dexp(xio+lfnk+(1-n)*xiol)
            d=dmax1(.1d0,dmin1(2.d0,1.d0-delt*(.5d0*((n-1)/xio-1.d0))))
            xin=xio+d*delt
```

34

```fortran
      if (dabs(d*delt).gt.dabs(xio*eps)) goto 1
      endif
      xion=sngl(xin/n)
C

      return
      end
```

```fortran
      real function glt(xmin,xmax)
C
C-------------------------------------------------------------------
C
C Adjust Gauss-Legendre abcissas and integrate the PDF
C
C-------------------------------------------------------------------
C
      integer no
      parameter (no=51)
      real gla(no),glw(no),glx(no)
      common /gla/ gla
      common /glw/ glw
      common /glx/ glx
      real xmin,xmax
      integer io
      real hr,ta
      real edspdf
C
C-------------------------------------------------------------------
C
C Adjust Gauss-Legendre abcissas for interval (xmin,xmax)
C
      hr=.5*(xmax-xmin)
      ta=xmax+xmin
      do io=1,(no+1)/2
         gla(io)=xmin+hr*(1.+glx(io))
      enddo
      do io=1+(no+1)/2,no
         gla(io)=ta-gla(no+1-io)
      enddo
C
C Integrate the PDF over (xmin,xmax)
C
      glt=0.
      do io=1,no
         glt=glt+glw(io)*edspdf(gla(io))
      enddo
      glt=hr*glt
C
      return
      end
```

```fortran
      subroutine gl(n,x,w)
C
C------------------------------------------------------------------------
C
C Calculate Gauss-Legendre integration abscissas and weights on (-1,1)
C
C------------------------------------------------------------------------
C
      real x(1),w(1)
      integer i,k,m,n
      real den,dp,dpn,d1,d2pn,d3pn,d4pn,e1
      real fx,h,p,pk,pkm1,pkp1,t,t1,u,v,x0
C
C------------------------------------------------------------------------
C
C Find Gauss-Legendre integration abscissas and weights on (-1,1)
C (Davis and Rabinowitz p 487)
C
      m=(n+1)/2
      e1=n*(n+1)
      do i=1,m
         t=(4*i-1)*3.1415926536/(4*n+2)
         x0=(1.-(1.-1./n)/(8.*n*n))*cos(t)
         pkm1=1.
         pk=x0
         do k=2,n
            t1=x0*pk
            pkp1=t1-pkm1-(t1-pkm1)/k+t1
            pkm1=pk
            pk=pkp1
         enddo
         den=1.-x0*x0
         d1=n*(pkm1-x0*pk)
         dpn=d1/den
         d2pn=(2.*x0*dpn-e1*pk)/den
         d3pn=(4.*x0*d2pn+(2.-e1)*dpn)/den
         d4pn=(6.*x0*d3pn+(6.-e1)*d2pn)/den
         u=pk/dpn
         v=d2pn/dpn
         h=-u*(1.+.5*u*(v+u*(v*v-d3pn/(3.*dpn))))
         p=pk+h*(dpn+.5*h*(d2pn+h/3.*(d3pn+.25*h*d4pn)))
         dp=dpn+h*(d2pn+.5*h*(d3pn+h*d4pn/3.))
         h=h-p/dp
         x(i)=x0+h
         fx=d1-h*e1*(pk+.5*h*(dpn+h/3.*(d2pn+.25*h*(d3pn+.2*h*d4pn))))
         w(i)=2.*(1.-x(i)*x(i))/(fx*fx)
      enddo
      if (m+m.gt.n) x(m)=0.
      return
      end
```

Example run of FORTRAN program CHISQR:

```
Enter the number of averages     : 5

Enter the tone-to-noise ratio (dB): 5

Enter the confidence level (%)    : 80




Noise

                    Chi-Square          Gaussian
                    ----------          ----------
Upper limit          2.0377 dB           1.9679 dB
Lower limit         -3.1290 dB          -3.6978 dB




Tone + Noise,      5.0 dB  T/N Ratio

                Noncentral Chi-Square    Gaussian
                    ----------          ----------
Upper Limit          1.4144 dB           1.3758 dB
Lower Limit         -1.9070 dB          -2.0253 dB


STOP
```

REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY(Leave blank)	2. REPORT DATE October 1993	3. REPORT TYPE AND DATES COVERED Contractor Report	

4. TITLE AND SUBTITLE
On the Use of the Noncentral Chi-Square Density Function for the Distribution of Helicopter Spectral Estimates

5. FUNDING NUMBERS
C NAS1-19000
WU 505-63-70-02

6. AUTHOR(S)
Donald P. Garber

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Lockheed Engineering & Sciences Company
144 Research Drive
Hampton, VA 23666-1339

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA CR-191546

11. SUPPLEMENTARY NOTES
Langley Technical Monitor: William L. Willshire, Jr.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 71

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
A probability density function for the variability of ensemble averaged spectral estimates from helicopter acoustic signals in Gaussian background noise was evaluated. Numerical methods for calculating the density function and for determining confidence limits were explored. Density functions were predicted for both synthesized and experimental data and compared with observed spectral estimate variability.

14. SUBJECT TERMS
Spectral analysis; Acoustics; Aircraft noise

15. NUMBER OF PAGES
40

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT

NSN 7540-01-280-5500