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ON THE DESIGN OF STRUCTURAL COMPONENTS USING MATERIALS WITH TIME-DEPENDENT PROPERTIES

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The application of the elastic-viscoelastic correspondence principle is presented as a design tool for structural design engineers for composite material applications. The classical problem of cantilever beams is used as the illustration problem. Both closed-form and approximate numerical solutions are presented for several different problems. The application of the collocation method is presented as a viable and simple design tool to determine the time-dependent behavior and response of viscoelastic composite beams under load.

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LIST OF SYMBOLS

A, A_{13}	constants
B_i	constant
b	width of beam
B_{13}, B_{23}, B_{33}	constants
c	distance from neutral axis to outermost surface of beam cross section
$D_{ij};\ ij = 1, 2, 6$	bending stiffnesses
$D^*_{ij};\ ij = 1, 2, 6$	inverted bending stiffnesses
$\frac{d}{dx}, \frac{d}{ds}$	differential operators
E	Young's modulus
$E(s)$	associated elastic relaxation modulus
$E_m(t)$	relaxation modulus for composite material matrix
$E_{11}(t)$	time-dependent apparent Young's modulus in direction of fibers
E_f	Young's modulus of fibers
$\overline{E}(s)$	relaxation modulus in the Laplace domain
$E(t)$	relaxation modulus
$E_{22}(t)$	time-dependent apparent Young's modulus perpendicular to fibers
$f_i\,;\ i = 1, 2, \ldots 9$	constants
$F_a(s)$	$s^3 + f_1 s^2 + f_2 s + f_3$
$f(t)$	function of time
$G_a(s)$	$L_1 s^3 + L_2 s^2 + L_3 s + L_4$
$G_{12}(t)$	time-dependent apparent shear modulus
$G_m(t)$	time-dependent shear modulus for matrix material
$g(t)$	function of t
$\overline{g}(s)$	$g(t)$ in the Laplace domain

h_v	constant
I	moment of inertia about centroidal axis
i,j	counters
K	bulk modulus for viscoelastic material
k	ply number
L	length of beam
L_1, L_2, L_3, L_4	constants
M	applied bending moment
M_i ; $i=1,2,3,4$	constants
m	mass, constant
M_x, M_y, M_{xy}	in-plane bending moments
n	constant
$N(t)$	function of t
P_o	applied load at $t=0$
$P(t)$	time-dependent applied load
P	applied load
p	applied load per unit width of beam
Q	constant; $\left(4L^3/bt^3\right)$
$\overline{Q}_{ij}$; $i,j=1,2,6$	transformed reduced stiffnesses
$\overline{S}_{ij}$; $i,j=1,2,6$	transformed compliances
s	Laplace parameter
$S(x)$	function of x; $\left[6(L-x)/bt^2\right]$
t	beam thickness
t_o	initial time
v,w	counters
V_f	fiber volume fraction

V_m	matrix volume fraction
w	deflection
$w(t)$	time-dependent deflection
$w(s)$	deflection in the Laplace domain
x	length coordinate
y	width coordinate
z	thickness coordinate
α_v , α_w	constants
$\delta(t)$	time-dependent deflection
$\delta(x,t)$	deflection as a function of length and time
ε_b	bending strain
$\varepsilon_b(t)$	time-dependent bending strain
$\varepsilon_x , \varepsilon_y , \varepsilon_{xy}$	in-plane strains
$\varepsilon_{yz} , \varepsilon_{xz}$	transverse shear strains
$\bar{\varepsilon}_b(s)$	time-dependent bending strain in Laplace domain
Γ_1 , Γ_2	constants
$\gamma_i \; ; \; i = 1, 2, 3$	exponential constants
$\gamma_x , \gamma_y , \gamma_{xy}$	in-plane curvatures
λ	constant
$\lambda_i \; ; \; i = 1, 2, 3$	exponential constants
$v(s)$	associated elastic Poisson's ratio
$\bar{v}(s)$	time-dependent Poisson's ratio in Laplace domain
$v(t)$	time-dependent Poisson's ratio
v_f	Poisson's ratio for fibers
$v_m(t)$	time-dependent Poisson's ratio for matrix material
$v_{21}(t)$	time-dependent minor Poisson's ratio
$v_{12}(t)$	time-dependent major Poisson's ratio

ω	natural frequency
σ_b	bending stress
σ_x , σ_y , σ_{xy}	in-plane stresses

TECHNICAL MEMORANDUM

ON THE DESIGN OF STRUCTURAL COMPONENTS USING MATERIALS WITH TIME-DEPENDENT PROPERTIES

I. INTRODUCTION

With the increased use of polymer matrix composite materials for aerospace applications, design engineers are faced with the need for knowledge of environmental effects on the functional behavior of structures. A variety of epoxies are widely used as a binding agent or matrix for fiber reinforced composites. Thermoset polymers are often used over thermoplastic polymers because of their better thermal stability and chemical resistance. A great advantage of thermoset polymers is their higher resistance to creep and stress relaxation. Typical thermoset matrix materials are epoxies, polyesters, and vinyl esters. Although they have relatively better creep characteristics than thermoplastics, they will still "relax" as a function of time, load, temperature, and other environmental factors. The understanding of this viscoelastic behavior when designing with composite materials is the topic of this report.

During the typical preliminary design phase of composite material structures, basic classical lamination theory (CLT) solutions are used to obtain stress, strain, deflections, natural frequency, and buckling strength of the structure analyzed. With the increased emphasis on long-term aerospace structures (10 to 30 years useful life), it is of great importance that viscoelastic effects also be included in the early design phase in order to obtain knowledge of the structural/functional behavior of the hardware after an extended period of time. With analytical representation of the material characteristics, the composite design could possibly require modification of initial dimensions and geometry in order to meet critical functional requirements at the end of their useful life.

This report presents one of the widely used methods, namely the elastic-viscoelastic correspondence principle, in the analysis of viscoelastic structural materials. For this report, the concern is with the basic time dependency of the viscoelastic material properties. The goal is to present the design engineer with an effective method to analyze, in closed form or numerically, and to design time-dependent structures. The problem investigated is one of the most basic problems in structural design, "beams."

It is the author's goal to provide information for structural design engineers to consider during the inception and preliminary design of any structure with time-dependent properties. The methods presented are proven and widely accepted yet simple to understand and apply.

II. THE ELASTIC BEAM PROBLEM

The classical problem of pure bending of beams with constant cross section is chosen, since the basic equations for stress, strain, deflection, and natural frequency have been well established.[1–3] For purposes of this report, a cantilever beam with a single load applied at its free end will be considered. The solution derivation will have the usual assumptions. These are:

1. The beam is thin. This implies that the thickness is much smaller than any of the other physical dimensions.

2. The deflection of the beam in the direction of the applied load is small compared to the beam thickness. (This assumption has been shown to be applicable even in the case of relatively large resulting deflections.)[4]

3. The in-plane strains ε_x, ε_y, and ε_{xy} are small compared to unity.

4. Transverse shear strains ε_{yz} and ε_{xz} are negligible.

5. The material obeys Hooke's law.

6. Rotatory inertia terms are negligible.

7. There are no body forces.

8. The material is isotropic.

A. Determination of Stress and Strain

The bending stress in a one-dimensional beam can be determined from the following equation,

$$\sigma_b = \frac{Mc}{I} \tag{1}$$

where M is the bending moment, c is the distance from the neutral axis to the outermost surface, and I is the moment of inertia of the beam cross section. For the beam configuration shown in figure 1, the bending moment can be expressed as:

$$M = -P(L-x), \tag{1a}$$

where

$P = pb$

$c = -t/2$ (for the top (tension) surface of the beam)

$I = bt^3/12$ (for a rectangular cross section).

Substituting these values into equation (1), one obtains:

$$\sigma_b = \frac{6P(L-x)}{bt^2} \quad . \tag{2}$$

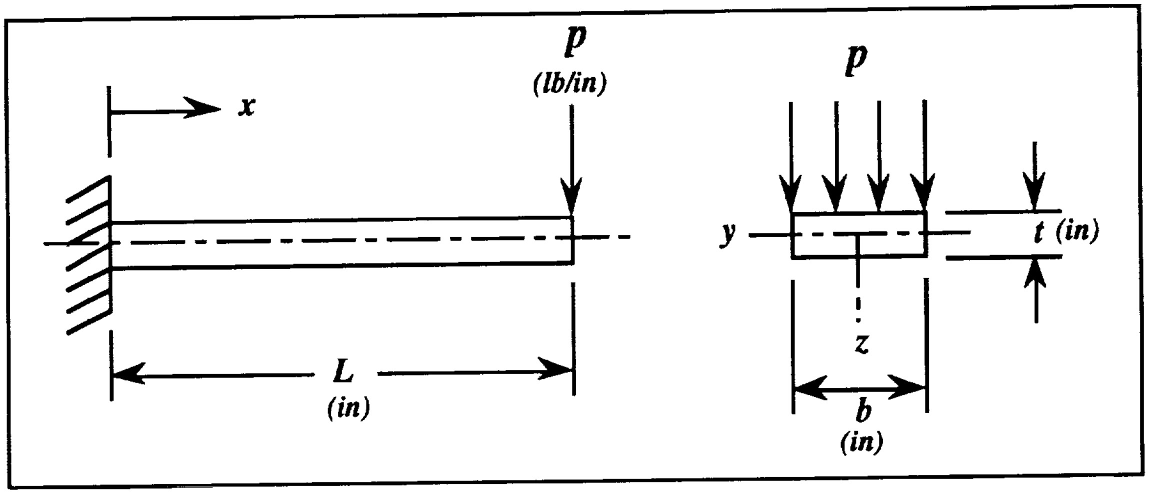

Figure 1. Cantilever beam configuration.

Following Hooke's, law one can express the axial strain caused by the applied bending moment by simply dividing equation (2) by the modulus of elasticity (Young's modulus) of the beam material. In this manner one obtains:

$$\varepsilon_b = \frac{6P(L-x)}{Ebt^2} \quad . \tag{3}$$

B. Determination of Deflection

The formulation which links the curvature of the central line of the beam cross section with the applied bending moment is called the Euler-Bernoulli law. This is expressed in differential form as:

$$-\frac{d^2w}{dx^2} = \frac{M}{EI} \quad . \tag{4}$$

With the knowledge that the slope and deflection at the fixed end of the beam are zero, equation (4) can be integrated twice to yield the following expression for the deflection at the free end of the beam:

$$w = \frac{4PL^3}{Ebt^3} \quad . \tag{5}$$

C. Determination of Natural Frequency

For the one-dimensional beam problem, classical linear elastic beam theory yields the following frequency equation in the absence of in-plane forces:

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = 0 \quad . \tag{6}$$

With the knowledge that the slope and deflection vanish at the fixed end and the shear force and moment are zero at the free end, this equation can be used to obtain the various bending vibration modes of the beam. The resulting characteristic equation of the beam is:

$$\cos\lambda L \ \cosh\lambda L = -1 \ , \tag{7}$$

where

$$\lambda^4 = \frac{m\omega^2}{EI} \ . \tag{8}$$

Solving equation (7) for λL and substituting the results into equation (8) yields the following expression for the first or natural frequency of the beam:

$$\omega = 3.51601\sqrt{\frac{EI}{mL^4}} \ . \tag{9}$$

III. THE SINGLE-PHASE VISCOELASTIC BEAM PROBLEM

A. Material Properties Characterization

In order to obtain a solution to any structural design problem where the material analyzed has time-dependent properties such as plastics, elastomers, and resin-based matrix composite materials, an analytical expression for the relaxation modulus $E(t)$ and Poisson's ratio $\nu(t)$ must be developed. The relaxation modulus is obtained from standard stress relaxation tests. In these tests, the viscoelastic material is subjected to a constant strain. Under the influence of this strain, the material will relax; and the stress will gradually decrease. The stress is measured at specific time intervals, and the relaxation modulus is plotted as a function of time. These data are then curve fitted to a function which can be readily manipulated to perform the necessary analysis for the solution of the problem. A very common function used is the Prony series curve fit due to Gaspard Francois Clair Marie Riche de Prony (1755–1839). A form of this function is:

$$f(t) \ = \ A + \sum_{i=1}^{n} B_i e^{-\gamma_i t} \ . \tag{10}$$

Recent investigations have produced methods for obtaining the coefficients and exponents of this function automatically with the aid of computers.[5 6] For purposes of this report, the following function will be used for the relaxation modulus:

$$E(t) = A + B_1 e^{-\gamma_1 t} + B_2 e^{-\gamma_2 t} + B_3 e^{-\gamma_3 t} \ , \tag{11}$$

where

$$\begin{aligned} A &= 180{,}000 & \gamma_1 &= 1{,}000 \\ B_1 &= 5{,}000 & \gamma_2 &= 10 \\ B_2 &= 5{,}000 & \gamma_3 &= 0.10 \ . \\ B_3 &= 170{,}000 \end{aligned}$$

The time-dependent Poisson's ratio is obtained from the relaxation modulus data and from the knowledge of the bulk modulus of elasticity of the material. The bulk modulus of elasticity can be expressed as:

$$K = \frac{E(t)}{3(1-2\,\nu(t))} \quad . \tag{12}$$

Solving for $\nu(t)$ one obtains:

$$\nu(t) = \frac{1}{2} - \frac{E(t)}{6K} \quad . \tag{13}$$

Substituting equation (11) into equation (13) and expanding, one obtains the following expression:

$$\nu(t) = A_{13} + B_{13}e^{-\gamma_1 t} + B_{23}e^{-\gamma_1 t} + B_{33}e^{-\gamma_1 t} \quad , \tag{14}$$

where

$$A_{13} = \frac{1}{2} - \frac{A}{6K} \qquad B_{23} = -\frac{B_2}{6K}$$

$$B_{13} = -\frac{B_1}{6K} \qquad B_{33} = -\frac{B_3}{6K} \quad .$$

Figures 2 and 3 show plots of the relaxation modulus and Poisson's ratio for the Prony functions of equations (11) and (14), respectively.

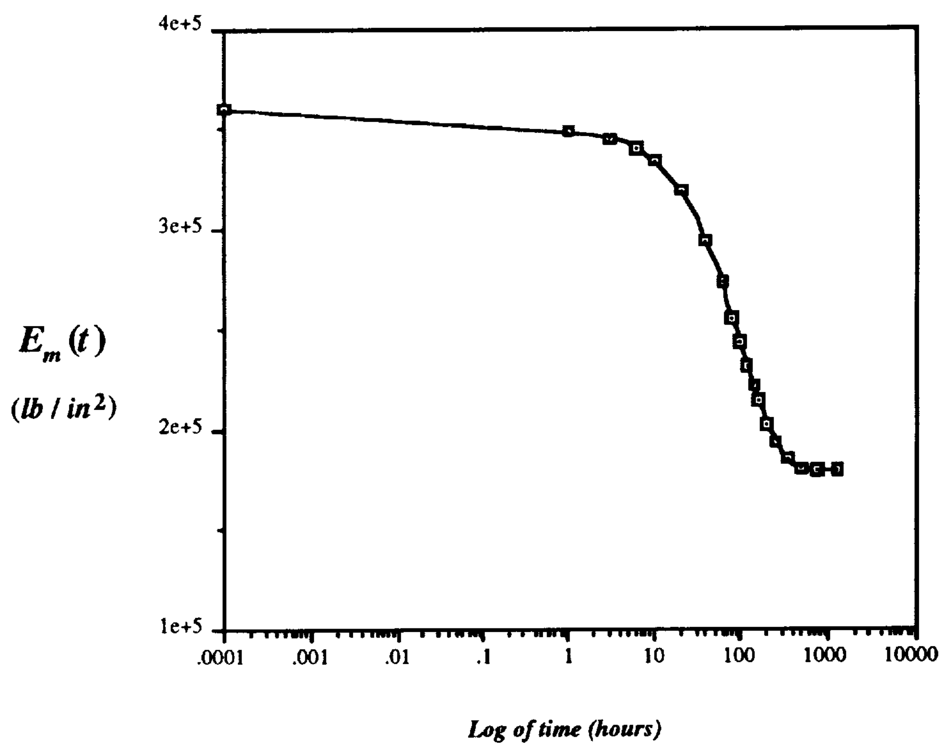

Figure 2. Relaxation modulus of a single-phase viscoelastic material.

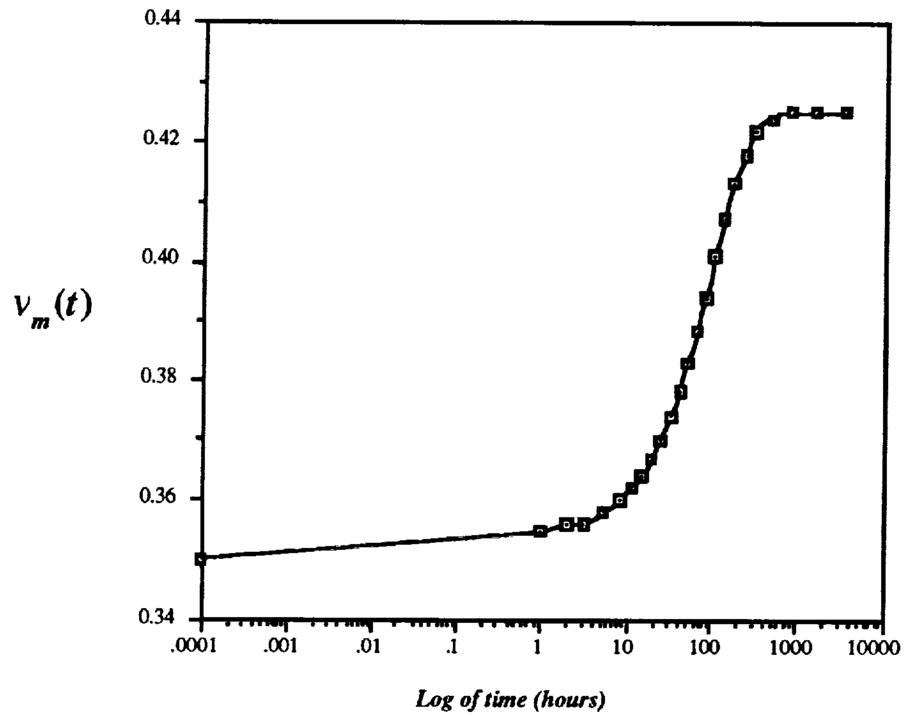

Figure 3. Poisson's ratio of a single-phase viscoelastic material.

B. Elastic-Viscoelastic Correspondence Principle

A very effective method of solution of time-dependent structural problems is obtained by applying the Laplace transform to the time-dependent functions. This removes the time variable, and the analysis problem for the viscoelastic body is converted to an "associated elastic" problem. This method allows the solution of the viscoelastic problem by simply expressing the constitutive equations and boundary conditions as functions of the Laplace transform parameter s. Once a solution to the "associated elastic" problem is obtained in the Laplace domain, it can be inverted into the original time domain, and a solution to the original viscoelastic problem is developed. For a single-phase material exhibiting properties that are time dependent, the solution of the beam problem becomes fairly straightforward. In fact, by using the "associated elastic" problem approach which is officially known as the "elastic-viscoelastic correspondence principle," this problem and many others can be solved in closed form. This method has been successfully used and documented by many authors.[7-9] It will be used in this report to demonstrate its simplicity and usefulness to the design engineer when designing structures with time-dependent materials.

The solutions derived in this report assume linearly viscoelastic materials. This means that, for any time interval, the time-dependent functions can be assumed proportionally linear to the applied constant load. Also a material is assumed linearly viscoelastic if the combined effects of two or more simultaneously applied loads or displacements can be expressed as the sum of the individual effects when the same loads or displacements are applied separately.

Equation (11) can be expressed in the Laplace domain as:

$$\overline{E}(s) = \frac{A}{s} + \frac{B_1}{s+\gamma_1} + \frac{B_2}{s+\gamma_2} + \frac{B_3}{s+\gamma_3} \ . \tag{15}$$

The "associated elastic" expression for the relaxation modulus can be obtained by multiplying equation (15) by the Laplace parameter s.[8] In this manner, one obtains:

$$E(s) \ = \ s\overline{E}(s) = \ A + \frac{sB_1}{s+\gamma_1} + \frac{sB_2}{s+\gamma_2} + \frac{sB_3}{s+\gamma_3} \ . \tag{16}$$

Following this same procedure, the expression for the "associated elastic" Poisson's ratio is expressed as:

$$\nu(s) \ = \ s\overline{\nu}(s) = \ A_{13} + \frac{sB_{13}}{s+\gamma_1} + \frac{sB_{23}}{s+\gamma_2} + \frac{sB_{33}}{s+\gamma_3} \ . \tag{16a}$$

C. Determination of Time-Dependent Strain

Any applied loads, either constant or varying with time, must also be transformed into the Laplace domain. So if, in the time domain, equation (3) is expressed as:

$$\varepsilon_b(t) = \frac{S(x)P(t)}{E(t)} \ , \tag{17}$$

where

$$S(x) \ = \ \frac{6(L-x)}{bt^2} \qquad \text{(where } t \text{ is thickness)} \ ,$$

then, in the Laplace domain, equation (17) becomes:

$$\varepsilon_b(s) \ = \ \frac{S(x)\left(\frac{P_o}{s}\right)}{E(s)} \ = \ S(x)\left[\frac{P_o}{s^2\,\overline{E}(s)}\right] \ . \tag{18}$$

Substituting equation (16) into (18) and combining all terms, one obtains, after some algebraic manipulations:

$$\varepsilon_b(s) \ = \ S(x)P_o\left(\frac{s^3 + f_1 s^2 + f_2 s + f_3}{L_1 s^4 + L_2 s^3 + L_3 s^2 + L_4 s}\right) \ , \tag{19}$$

where

$$f_1 = \gamma_1 + \gamma_2 + \gamma_3 \qquad f_6 = \gamma_1 + \gamma_3 \qquad L_1 = A + B_1 + B_2 + B_3$$
$$f_2 = \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1 \qquad f_7 = \gamma_1\gamma_3 \qquad L_2 = Af_1 + B_1 f_4 + B_2 f_6 + B_3 f_8$$
$$f_3 = \gamma_1\gamma_2\gamma_3 \qquad f_8 = \gamma_1 + \gamma_2 \qquad L_3 = Af_2 + B_1 f_5 + B_2 f_7 + B_3 f_9$$
$$f_4 = \gamma_2 + \gamma_3 \qquad f_9 = \gamma_1\gamma_2 \qquad L_4 = Af_3$$
$$f_5 = \gamma_2\gamma_3$$

The ratio of polynomials in equation (19) can be expressed as:

$$\frac{s^3 + f_1 s^2 + f_2 s + f_3}{s\left(L_1 s^3 + L_2 s^2 + L_3 s + L_4\right)} = \frac{F_a(s)}{sG_a(s)} \ . \tag{20}$$

The roots of the cubic equation $G_a(s)$ can be obtained, allowing the expression to be written as:

$$G_a(s) = (s + \lambda_1)(s + \lambda_2)(s + \lambda_3) \ . \tag{21}$$

It is important to point out that for the physical problem, these roots should never have an imaginary component. This knowledge can be used as a check to verify that the numerical calculations have been performed appropriately. In fact, all roots in equation (21) should be real and negative for a material with a modulus that can be characterized as exponentially decaying.

One should notice that equation (20) is a quotient of two polynomials with no common factors, and the degree of the numerator is lower than that of the denominator. This is the classical fraction that can be solved in a straightforward manner by application of Heaviside's partial fraction expansion.[10] Following Heaviside's procedure, the total derivative of the denominator of equation (20) can be expressed as:

$$\frac{d}{ds}[sG_a(s)] = s\frac{d[G_a(s)]}{ds} + \frac{d[s]}{ds}G_a(s) \ , \tag{22}$$

or

$$\frac{d}{ds}[sG_a(s)] = 4L_1 s^3 + 3L_2 s^2 + 2L_3 s + L_4 \ . \tag{23}$$

Equation (20) can now be expressed as a sum of partial fractions as:

$$\frac{F_a(s)}{sG_a(s)} = \frac{M_1}{s} + \frac{M_2}{s+\lambda_1} + \frac{M_3}{s+\lambda_2} + \frac{M_4}{s+\lambda_3} \ , \tag{24}$$

where

$$M_1 = \left.\frac{F_a(s)}{\frac{d}{ds}[sG_a(s)]}\right|_{s=0}$$

$$M_2 = \left. \frac{F_a(s)}{\frac{d}{ds}\left[s\,G_a(s)\right]} \right|_{s=\lambda_1}$$

$$M_3 = \left. \frac{F_a(s)}{\frac{d}{ds}\left[s\,G_a(s)\right]} \right|_{s=\lambda_2}$$

$$M_4 = \left. \frac{F_a(s)}{\frac{d}{ds}\left[s\,G_a(s)\right]} \right|_{s=\lambda_3} \quad .$$

Substituting equation (24) into equation (19) yields the expression for axial strain due to bending in the Laplace domain:

$$\varepsilon_b(s) = S(x)P_o\left(\frac{M_1}{s} + \frac{M_2}{s+\lambda_1} + \frac{M_3}{s+\lambda_2} + \frac{M_4}{s+\lambda_3}\right) \quad . \tag{25}$$

Taking the inverse transformation of equation (25) yields the final expression for the time-dependent axial strain due to bending:

$$\varepsilon_b(t) \; = \; S(x)\,P_o\left(M_1 + M_2 e^{-\lambda_1 t} + M_3 e^{-\lambda_2 t} + M_4 e^{-\lambda_3 t}\right) \quad . \tag{26}$$

D. Determination of Time-Dependent Deflection

Equation (5) is the expression for maximum deflection of the cantilever beam under study. Following the same procedure as for the viscoelastic strain calculations, the deflection can be written as a function of the time-dependent variables:

$$w(t) = Q\left[\frac{P(t)}{E(t)}\right] \quad , \tag{27}$$

where

$$Q = \frac{4L^3}{bt^3} \quad \text{(where } t \text{ is thickness)} \quad .$$

In the Laplace domain, equation (27) can be written as:

$$w(s) = \; Q\left[\frac{P_o}{s^2\bar{E}(s)}\right] \quad . \tag{28}$$

One should notice that the only difference between equation (28) and equation (18) is in the replacement of the term $S(x)$ with the constant Q. Since both these terms are independent of time, it is a straightforward matter to express the time-dependent deflection as in equation (25). This expression is:

$$w(t) = QP_o\left(M_1 + M_2 e^{-\lambda_1 t} + M_3 e^{-\lambda_2 t} + M_4 e^{-\lambda_3 t}\right) \quad . \tag{29}$$

IV. THE TWO-PHASE VISCOELASTIC BEAM PROBLEM

The formulation for strain and deflection obtained in section III is applicable for linearly viscoelastic materials which are isotropic. Although many plastics can be analyzed in this manner, the majority of the viscoelastic materials used for aerospace structural applications are two phase. This means that they are composed of one phase that exhibits time-dependent properties and another phase that does not. This is the case for many composite materials, in particular the epoxy- or resin-based two-phase systems. For example, graphite/epoxy, boron/epoxy, and silicon-carbide/epoxy are considered two-phase composites.

A. Micromechanics Determination of Material Properties

In order to identify the time-dependent material properties of the two-phase composite, we must look at the interaction between the time-dependent and time-independent components at a microscopic level. This heterogeneous look at the composite system is known as micromechanics. In this report, the classical stiffness approach to micromechanics is used.[11] There are some basic restrictions on the composite material. For example, the composite ply (lamina) resulting from the constituent parts must be macroscopically homogeneous and macroscopically orthotropic. It must be linearly viscoelastic and initially stress-free. For the constituents, the fibers are homogeneous, linearly elastic, isotropic, regularly spaced, and perfectly aligned; the matrix is homogeneous, linearly viscoelastic, and isotropic. In addition, the bonds between the fibers and the matrix are assumed to be perfect (no voids). Although these restrictions are seemingly stringent, modern manufacturing methods combined with material characterization at a macroscopic level $\left(E_{11}, E_{22}, G_{12}, etc.\right)$ can be used to "back-out" the necessary constituent characteristics $\left(E_m(t), E_f, \nu_m(t), etc.\right)$.

The binder or matrix of the composite material used in this report has a relaxation modulus described by equation (11). In this manner, one has:

$$E_m(t) = A_1 + B_1 e^{-\gamma_1 t} + B_2 e^{-\gamma_2 t} + B_3 e^{-\gamma_3 t} \quad . \tag{30}$$

Following the micromechanics approach to stiffness, a "rule of mixtures" expression for the apparent time-dependent Young's modulus in the direction of the fibers can be obtained. This is:

$$E_{11}(t) = E_f V_f + E_m(t) V_m \quad , \tag{31}$$

where

E_f = Young's modulus for an isotropic fiber

V_f = fiber volume content for the composite material

$E_m(t)$ = relaxation modulus for the matrix

V_m = matrix volume content for the composite material.

In the direction transverse to the fibers, the apparent Young's modulus is expressed as:

$$E_{22}(t) = \frac{E_f E_m(t)}{E_f V_m + E_m(t) V_f} \quad . \tag{32}$$

Several approaches for the accurate determination of the apparent in-plane shear modulus $G_{12}(t)$ have been investigated. Using a variational analysis approach, Foye[12] developed an expression for a square array of fibers in the laminate. This represented the best closed-form estimates of this orthotropic constant for a unidirectional composite ply. It is expressed, in this report, as follows:

$$G_{12}(t) = \frac{G_m(t)}{2}\left[\frac{(4-\pi)+\pi N(t)}{4} + \frac{4N(t)}{(4-\pi)N(t)+\pi}\right] \quad , \tag{33}$$

where

$$G_m(t) = \frac{E_m(t)}{2\left[1+\nu_m(t)\right]} \quad , \tag{34}$$

$$N(t) = \frac{G_f\left(\pi+4V_f\right)+G_m(t)\left(\pi-4V_f\right)}{G_f\left(\pi-4V_f\right)+G_m(t)\left(\pi+4V_f\right)} \quad , \tag{35}$$

$$G_f = \frac{E_f}{2\left(1+\nu_f\right)} \quad . \tag{36}$$

Poisson's ratio for the matrix material will be described by equation (14) or:

$$\nu_m(t) = A_{13} + B_{13} e^{-\gamma_1 t} + B_{23} e^{-\gamma_2 t} + B_{33} e^{-\gamma_3 t} \quad . \tag{37}$$

The major Poisson's ratio for the unidirectional composite lamina can be written using the rule of mixtures in the same manner as the time-dependent Young's modulus $E_{11}(t)$:

$$\nu_{12}(t) = \nu_f V_f + \nu_m(t) V_m \quad , \tag{38}$$

where

$$\nu_f = \text{ Poisson's ratio for isotropic fibers.}$$

The minor Poisson's ratio is defined as usual from the symmetric properties of the compliance matrix:

$$\nu_{21}(t) = \nu_{12}(t)\frac{E_{22}(t)}{E_{11}(t)} \quad . \tag{39}$$

In order to obtain the time-dependent strain values of the individual plies within the laminated beam, one can use the elastic viscoelastic correspondence principle. Due to the fact that the determination of strains in composite laminates is critically dependent on stiffness parameters for each ply, as well as stiffness parameters for the laminate, a closed-form solution to the time-dependent strains is considerably more elaborate than what is expressed in equation (26). In fact, one will realize that even for the simplest problems (cantilever laminated beam), although a closed-form solution is possible, it is not time nor cost effective to expect a design engineer to obtain them. For more complex mathematical models, the function to be inverted is often known only for discrete positive real values of the transform parameter therefore making it very difficult if not impossible to obtain an exact solution. A more effective approach is the application of numerical inversion methods to obtain the approximate transformed solution. A widely used and effective method of inversion is the collocation technique due to Schapery.[13]

B. The Collocation Method

This numerical inversion technique is readily applicable to a general class of problems that have a solution of the form:[9]

$$f(t) = \Gamma_1 + \Gamma_2 t + g(t) \quad , \tag{40}$$

where Γ_1 and Γ_2 are constants, and $g(t)$ is the transient component of the solution. The transient component is normally expressed, approximately, as a sum of exponential functions or:

$$g(t) = \sum_{\nu=1}^{m} h_\nu e^{-t/\alpha_\nu} \quad , \tag{41}$$

where h_ν and α_ν are constants.

The time-dependent axial strain due to bending can be written according to equations (40) and (41) as:

$$\varepsilon_b(t) = \Gamma_1 + \Gamma_2 t + \sum_{\nu=1}^{m} h_\nu e^{-t/\alpha_\nu} \quad . \tag{42}$$

After the material experiences the creep that is characteristic of viscoelastic materials, it is assumed that the long-term value of strain is approximately constant. In reality, this long-term strain is not constant, but for many materials, its rate of change is very small. With this assumption, the linear time-dependent component of equation (42) vanishes, yielding:

$$\varepsilon_b(t) = \Gamma_1 + \sum_{v=1}^{m} h_v \, e^{-t/\alpha_v} \quad . \tag{43}$$

In the Laplace domain, equation (43) can be expressed as:

$$s\,\bar{\varepsilon}_b(s) = s\,\bar{\Gamma}_1(s) + s\,\bar{g}(s) \quad , \tag{44}$$

where

$$s\bar{\Gamma}_1(s) = s\left[\frac{\Gamma_1}{s}\right] = \Gamma_1 \quad ,$$

and

$$\bar{g}(s) = \sum_{v=1}^{m} \frac{h_v}{(s + 1/\alpha_v)} \quad .$$

Equation (44) is now written as:

$$s\,\bar{\varepsilon}_b(s) = \Gamma_1 + \sum_{v=1}^{m} \frac{s\,h_v}{(s + 1/\alpha_v)} \quad . \tag{45}$$

In order to try to minimize the error of the approximation given by $\bar{g}(s)$, the transform of the approximate solution should be equal to the transform of the exact solution, at least at the m discrete values of s:

$$\bar{g}(s)_{exact} \Bigg|_{s=1/\alpha_w} = \bar{g}(s)_{approx} \Bigg|_{s=1/\alpha_w} \quad , \tag{46}$$

where

$$w = 1, 2, 3, \ldots m \quad .$$

Considering equation (46), equation (45) can be expressed as:

$$s\,\bar{\varepsilon}_b(s) = \Gamma_1 + \sum_{v=1}^{m} \frac{s\,h_v}{(1/\alpha_w + 1/\alpha_v)} \qquad (w = 1, 2, 3 \ldots, m) \quad . \tag{47}$$

At $t = 0$, equation (43) can be solved for the constant Γ_1. It is written as:

$$\Gamma_1 = \varepsilon_b(t_o) - \sum_{v=1}^{m} h_v \quad . \tag{48}$$

Letting $s = 1/\alpha_w$ on the right-hand side of equation (47) and substituting equation (48) into equation (47) yields, after rearranging:

$$\sum_{v=1}^{m} \frac{h_v}{(1+\alpha_v/\alpha_w)} = \varepsilon_b(t_o) - s\,\bar{\varepsilon}_b(s)\Big|_{s=1/\alpha_w} \quad . \tag{49}$$

The expression for α_j is:[9]

$$\alpha_j = e^{(7-2j)} \qquad (j = 1,2,3\ldots,m) \ . \tag{50}$$

Substituting equation (50) into the left-hand side of equation (49) yields:

$$\sum_{v=1}^{m} \frac{h_v}{\left[1+e^{2(w-v)}\right]} = \varepsilon_b(t_o) - s\,\bar{\varepsilon}_b(s)\Big|_{s=1/\alpha_w} \quad . \tag{51}$$

All quantities in the set of equations (51) are either known or can be determined at the discrete values of the Laplace parameter s except the constants h_v. Solving for the h_v's gives the necessary information to evaluate equation (48). A final substitution into equation (43) yields the expression for the time-dependent strain.

C. Determination of Time-Dependent Strain

For the problem of a multilayered viscoelastic composite beam, the time-dependent response is readily obtained using an appropriate numerical method. The collocation method is used in this section to obtain the axial strain of the cantilever beam due to bending.

For symmetric laminates under bending loads, the stresses in the k^{th} ply of the beam can be expressed as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_x \\ \gamma_y \\ \gamma_{xy} \end{Bmatrix} , \tag{52}$$

where $\bar{Q}_{ij}$ are the transformed reduced stiffnesses of the ply and γ_j are the curvatures.[11 14] From the moment-curvature constitutive relations for bending of composite laminates, one can write:

$$\begin{Bmatrix} \gamma_x \\ \gamma_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} D^*_{11} & D^*_{12} & D^*_{16} \\ D^*_{12} & D^*_{22} & D^*_{26} \\ D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} , \tag{53}$$

where D_{ij}^* are the components of the inverted bending stiffness matrix. For one-dimensional beam problems, the following assumption is made:

$$M_y = M_{xy} = 0 \ . \tag{54}$$

Expressions for the transformed reduced stiffnesses can be found in Jones[11] and Whitney[14] and will not be repeated here. Substituting equations (53) and (54) into (52) yields the expressions relating stresses to the ply and laminate stiffnesses, and the applied moment. For the axial direction (maximum bending stress direction), the stress is expressed as:

$$\sigma_x^{(k)} = z^{(k)} M_x \left(\overline{Q}_{11}^{(k)} D_{11}^* + \overline{Q}_{12}^{(k)} D_{12}^* + \overline{Q}_{16}^{(k)} D_{16}^* \right) \ . \tag{55}$$

From the classical lamination theory, the bending stiffnesses are expressed as:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} \left(z_k^3 - z_{k-1}^3 \right) \ . \tag{56}$$

Once the stresses have been determined, one can express the strains in terms of the stresses by transformation of the strain-stress relations from principal material directions to body coordinates. The resultant expression is:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} , \tag{57}$$

where $\overline{S}_{ij}$ are the components of the transformed compliance matrix for the k^{th} ply. Once again expressions for the components of the transformed compliance matrix can be found in Jones[11] and Whitney[14] and will not be repeated here.

Using the elastic-viscoelastic correspondence principle, the solution for the time-dependent strains can be obtained. The procedure is as follows:

1. Determine the analytical expressions for the time-dependent relaxation modulus and Poisson's ratio (equations (30) and (37)).

2. Obtain the Laplace transforms of these functions and determine the associated elastic expressions (equations (16) and (16*a*)).

3. Calculate the values of the longitudinal and transverse properties for the unidirectional ply by transforming equations (31), (32), (33), (38) and (39) into the Laplace domain and substituting the values from step 2.

4. Calculate the reduced stiffness matrix, compliance matrix, transformed reduced stiffness matrix, and transformed compliance matrix in terms of the unidirectional ply properties from step 3.

5. With the information in step 4, calculate the laminate bending stiffness matrix (equation (56)).

6. Identify the applied loads (moments) in the Laplace domain. For a constant moment, this is simply dividing the moment by the Laplace parameter.

7. Calculate stresses and strains using equations (52) and (57).

8. Express the calculated strains as the "associated elastic" solution by simply multiplying the calculated strains from step 7 by the Laplace parameter s.

9. Solve for the constants, h_ν, from equation (51). In expanded form, equation (51) can be written as:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{1+e^{-2}} & \frac{1}{1+e^{-4}} & \frac{1}{1+e^{-6}} & \frac{1}{1+e^{-8}} & \frac{1}{1+e^{-10}} \\ \frac{1}{1+e^{2}} & \frac{1}{2} & \frac{1}{1+e^{-2}} & \frac{1}{1+e^{-4}} & \frac{1}{1+e^{-6}} & \frac{1}{1+e^{-8}} \\ \frac{1}{1+e^{4}} & \frac{1}{1+e^{2}} & \frac{1}{2} & \frac{1}{1+e^{-2}} & \frac{1}{1+e^{-4}} & \frac{1}{1+e^{-6}} \\ \frac{1}{1+e^{6}} & \frac{1}{1+e^{4}} & \frac{1}{1+e^{2}} & \frac{1}{2} & \frac{1}{1+e^{-2}} & \frac{1}{1+e^{-4}} \\ \frac{1}{1+e^{8}} & \frac{1}{1+e^{6}} & \frac{1}{1+e^{4}} & \frac{1}{1+e^{2}} & \frac{1}{2} & \frac{1}{1+e^{-2}} \\ \frac{1}{1+e^{10}} & \frac{1}{1+e^{8}} & \frac{1}{1+e^{6}} & \frac{1}{1+e^{4}} & \frac{1}{1+e^{2}} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e^{-5}} \\ \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e^{-3}} \\ \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e^{-1}} \\ \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e} \\ \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e^{3}} \\ \varepsilon_b(t_o) - s\bar{\varepsilon}_b(s)\big|_{s=e^{5}} \end{Bmatrix} .$$

10. Calculate the constant Γ_1 from equation (48).

11. Obtain the final expression for time-dependent strain by substituting the constants from steps 9 and 10 into equation (43).

D. Determination of Time-Dependent Deflection

For the one-dimensional beam, the Euler-Bernoulli equation for a composite orthotropic laminate is expressed as:

$$-\frac{d^2w}{dx^2} = D_{11}^{*}(t)M_x \ . \qquad (58)$$

For a cantilever beam of constant cross section and thickness, the bending stiffness parameter $D_{11}^*(t)$ is independent of x, and integrating equation (58) twice yields the expression for the deflection of the beam. In this manner, one obtains:

$$w = -\frac{D_{11}^*(t) P_o L^3}{3} . \tag{59}$$

The procedure to calculate the time-dependent deflection now follows the one described in section IV.C with the following difference.

After step 7, the load P, like the moment, is divided by the Laplace parameter. Equation (59) is then calculated for the value of the Laplace parameter. This is repeated for each value of Laplace parameters chosen for the analysis. Equation (51) is again solved with the values of deflection used instead of the values of strain. The constants α_v and h_v are again calculated, and a final expression for the time-dependent deflection is then obtained.

E. Determination of Time-Dependent Natural Frequency

The natural frequency of a one-dimensional composite orthotropic beam can be determined from the elastic beam frequency equation by simply making the following substitution into equation (6):

$$EI = \frac{b}{D_{11}^*(t)} . \tag{60}$$

With this substitution, the resulting expression for natural frequency is:

$$\omega = 3.51601 \sqrt{\frac{b}{D_{11}^*(t) m L^4}} . \tag{61}$$

Again the steps to obtain the time-dependent natural frequency are the same as in the time-dependent strain and deflection with one exception. Notice that the expression for natural frequency is independent of applied load. Equation (61), once expressed in the Laplace domain, becomes the associated elastic expression. In other words, $\overline{\omega}(s) = \omega(s)$. The values of the constants α_v and h_v are again calculated, and the final expression for time-dependent natural frequency also takes the form of equation (51).

V. NUMERICAL EXAMPLES

A. Single-Phase Viscoelastic Beam Problem

As a first example of the determination of time-dependent strains and deflection of a cantilever beam of a linear viscoelastic material, the case of a single-phase material will be investigated. This problem is by no means a new one. It is presented here to demonstrate a simple application of the "correspondence principle" and its usefulness to the design engineer.

Figures 2 and 3 show plots of the relaxation modulus and Poisson's ratio, respectively, for the material described in equations (11) and (14). The constants for both equations are defined in section III.A. The material presented here is a fictitious one, but the curves represent typical ones for actual viscoelastic materials.

Appendix A. contains the computer program "RELAX" which identifies the step-by-step procedure to determine the time-dependent strain and deflection of a linear viscoelastic cantilever beam of constant cross section. For this example, the applied load is 25 lb at the tip (free end) of the beam. The length of the beam is 29.25 inches, the width is 5 inches, and the thickness is 1 inch. The program produces the following results for equations (26) and (29):

$$\lambda_1 = -0.00514282$$

$$\lambda_2 = -986.113154$$

$$\lambda_3 = -9.859137$$

$$M_o = 5.555555E-6$$

$$M_1 = -2.698453E-6$$

$$M_2 = -3.911196E-8$$

$$M_3 = -4.023282E-8 \cdot$$

The plots of strain and deflection from this run of "RELAX" are shown in figures 4 and 5, respectively. "RELAX" also calculates the deflected shape of the beam as a function of time. This shows how the beam relaxes with time, thus yielding a deflection that increases as time passes. The points in time selected for the plot are $t=0$ h, $t=10$ h, $t=100$ h, and $t=$ 1,000 h. This is plotted in figure 6.

B. Two-Phase Viscoelastic Beam Problem

Many composite materials have, as constituents, a time-dependent component (polymer matrix) and fibers or particulates that are made from materials that have properties that are relatively insensitive to creep or relaxation (graphite, boron, silicone-carbide, etc.). In these cases, the problem of determining the time-dependent response to applied loads becomes more involved, and in many cases, closed-form solutions are not available. The intent of this section is to use the correspondence principle and apply the collocation method[13] to a laminated composite cantilever beam loaded at the free end to determine the strain, deflection, and natural frequency as a function of time. The application of the procedure to different problems has been well documented.[8 9] The approach is given in section IV and is applied and explained in a step-by-step manner in the computer program "VISCOBM" found in appendix B.

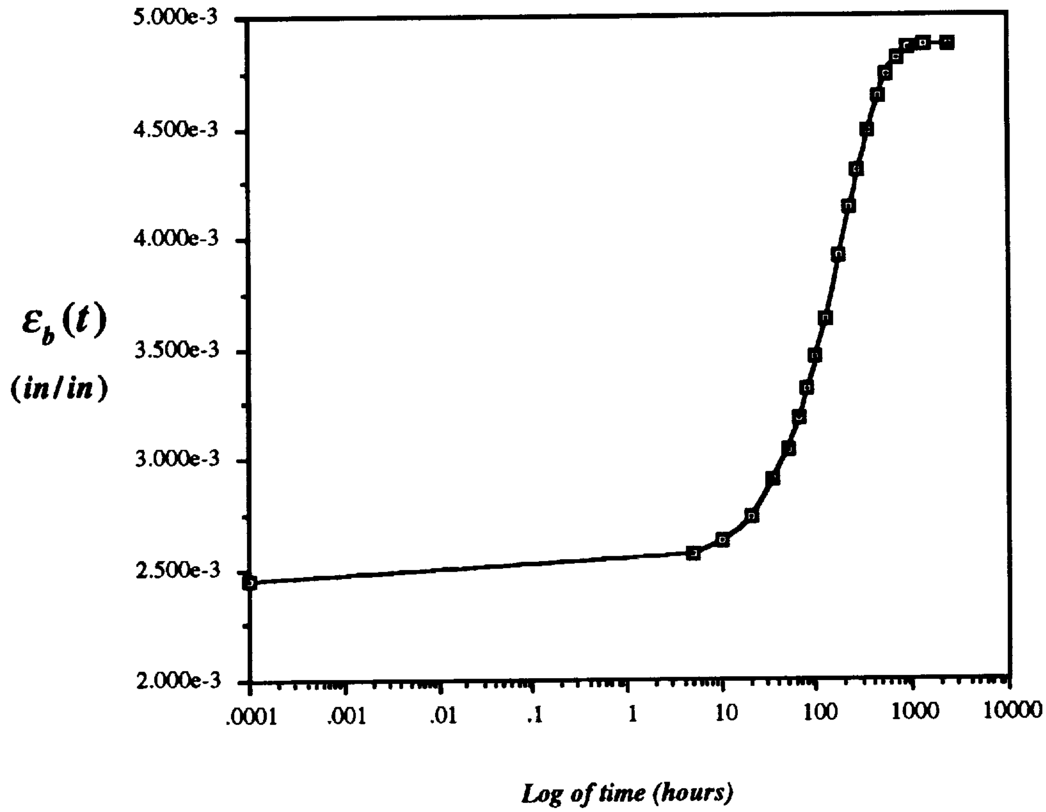

Figure 4. Time-dependent bending strain for a single-phase viscoelastic beam.

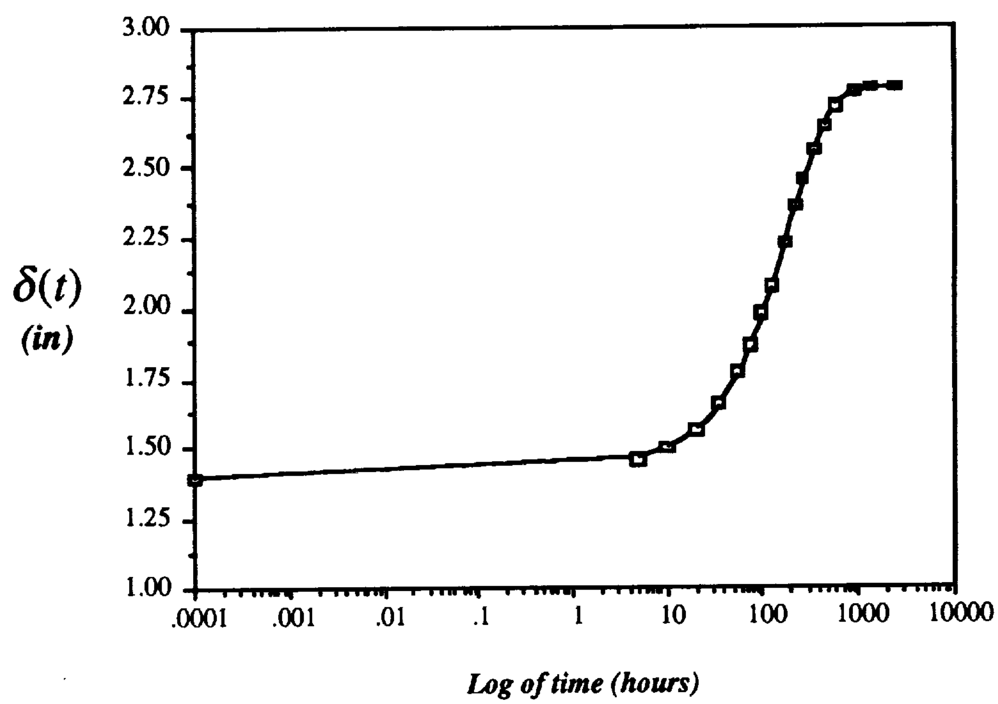

Figure 5. Time-dependent deflection for a single-phase viscoelastic beam.

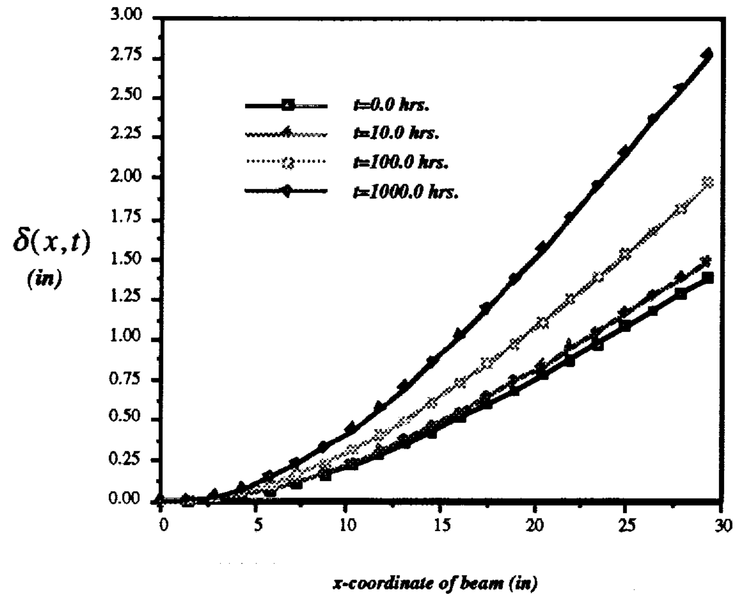

Figure 6. Deflected shape of a single-phase viscoelastic cantilever beam at various times.

One will notice, when using the collocation method, that the accurate determination of the initial response ($t = 0$) is critical to the proper solution of the viscoelastic problem. For this purpose, a computer program "VIST0" is included in appendix C. This program is nothing more than "VISCOBM" at time $t = 0$. It is important to point out that it is not necessary to create a separate program for this condition since running "VISCOBM" at $t = 0$ would accomplish the same results. It is done this way here as a means of simply demonstrating the method.

Figure 7 shows the calculated time-dependent strain for various ply lay-up angles of a four-ply symmetric beam. It is interesting to see how the lay-up angle plays a significant role in the long-term strain values. Figure 8 shows the same trends for deflection. Figure 9 illustrates that increasing the fiber content of the composite not only increases the stiffness of the beam but it also reduces the long-term effects of relaxation in the beam. Figure 10 shows the significant drop in natural frequency of the beam as a function of time.

VI. LIMITATIONS OF THE CORRESPONDENCE PRINCIPLE

A very important limitation to the application of the correspondence principle must be explained. The knowledge of the state of the boundary conditions must be known in order to apply this method effectively.[15]

If the interface between the surfaces where the stress is prescribed and where the displacements are prescribed changes with time, the correspondence principle is not applicable. The conditions of each surface, however, can be time-dependent. The illustration in figure 11 shows how the shaded area (interface boundary area) in a viscoelastic medium changes as a function of time for a spherical indentor. For a cylindrical indentor, the interface boundary does not change. An example of a changing interface boundary is the increasing inner diameter of a solid propellant motor as it is consumed during operation.

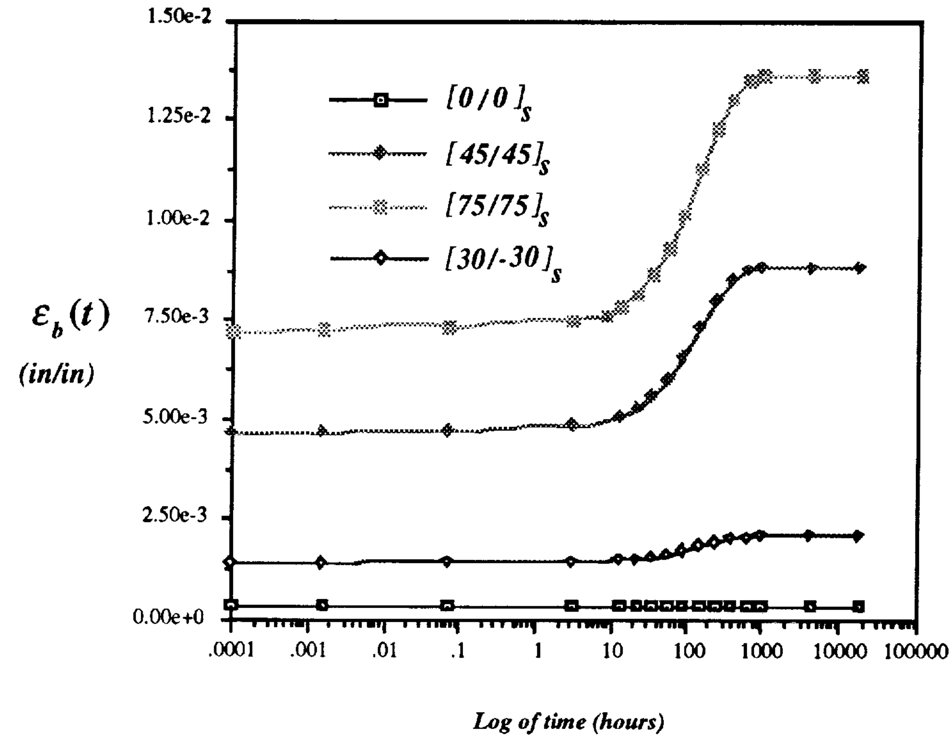

Figure 7. Time-dependent bending strain of outer ply in a two-phase viscoelastic cantilever beam for various laminate configurations.

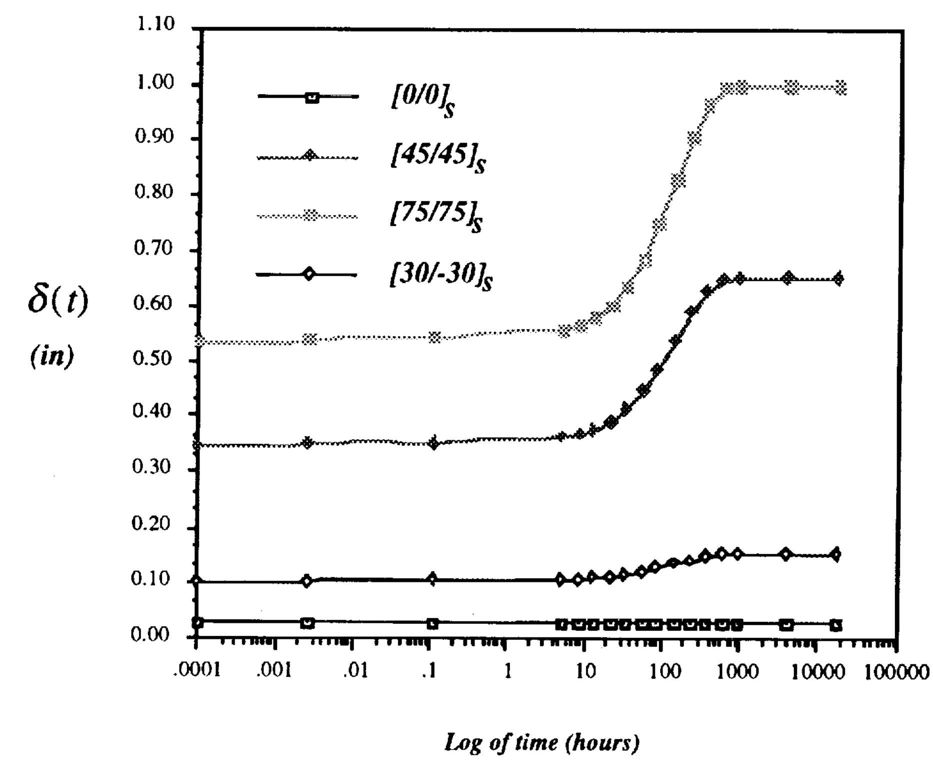

Figure 8. Normalized time-dependent deflection of a two-phase viscoelastic cantilever beam for various laminate configurations.

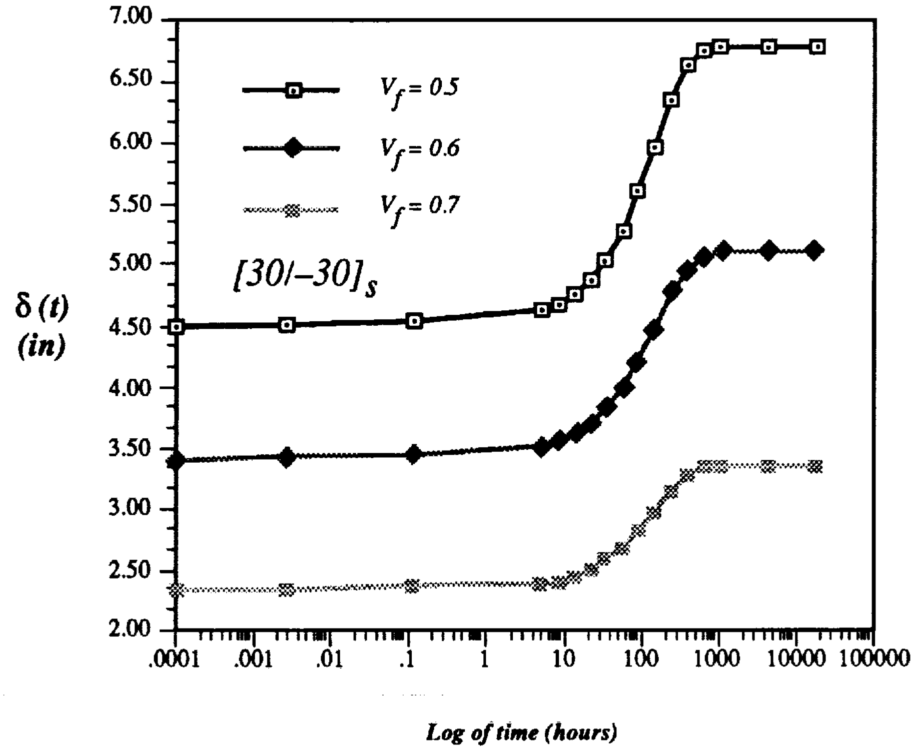

Figure 9. Time-dependent deflection of a two-phase cantilever beam for various fiber volume fractions.

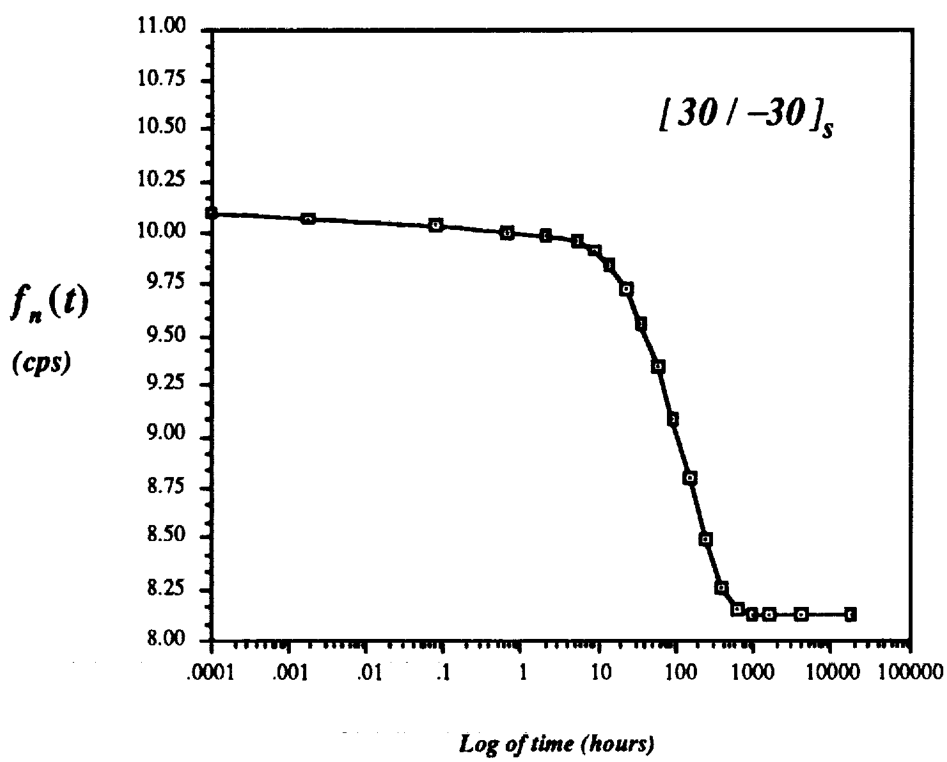

Figure 10. Time-dependent natural frequency of a two-phase viscoelastic cantilever beam.

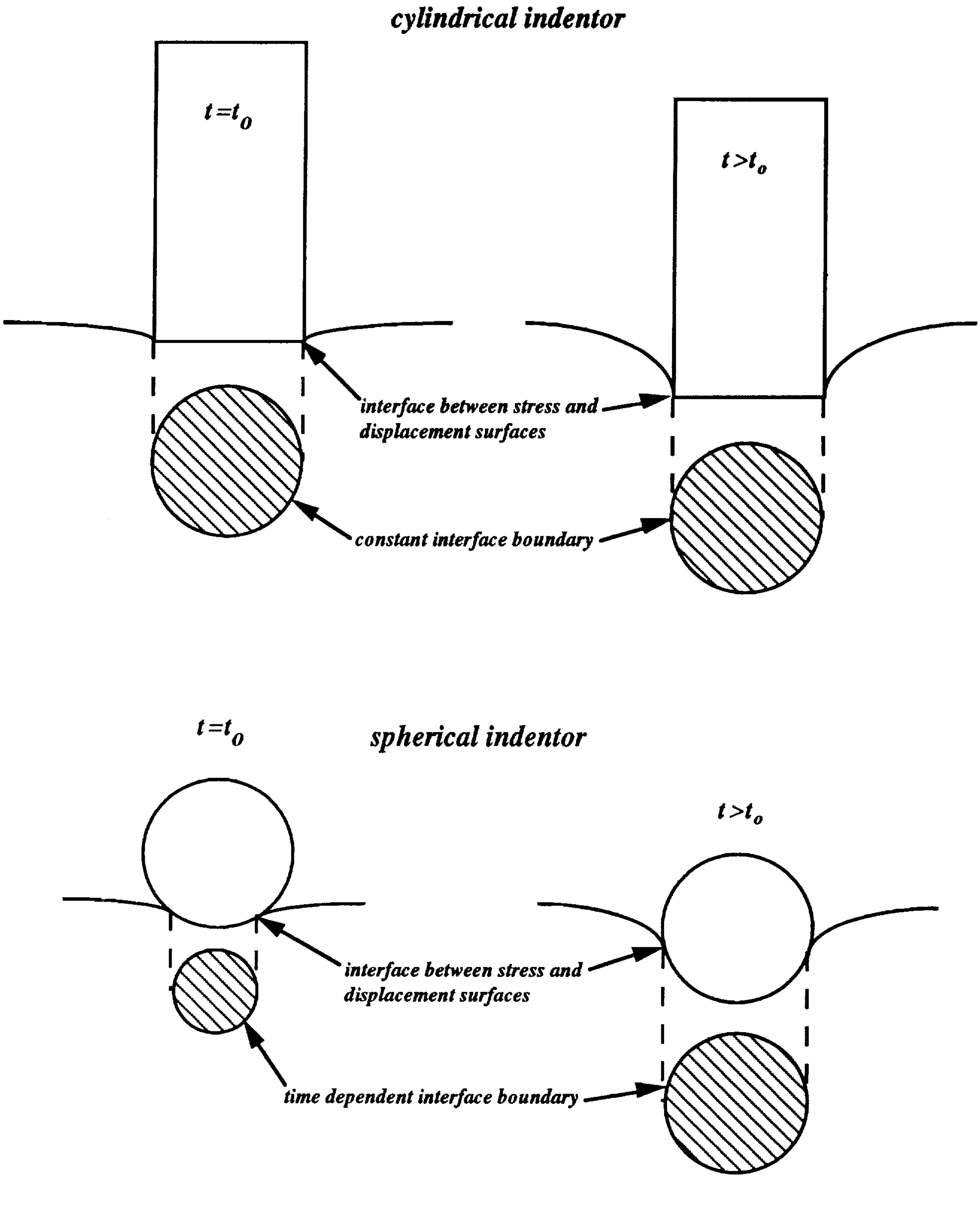

Figure 11. Difference between time-dependent and time-independent boundary conditions.

VII. CONCLUSIONS

The elastic-viscoelastic correspondence principle is a very effective and straightforward tool for determining the time-dependent response of viscoelastic structures to applied loads. The design engineer can use this knowledge to optimize parameters such as ply lay-up angle, fiber volume content, and configuration variables. This can help to minimize or maximize the effects of creep or relaxation depending on the functional use of the hardware designed. The examples presented offer the design engineer the capability to understand the effect that a viscoelastic material can have on the performance of structures.

Although not included here, temperature and exposure to environmental effects (ultraviolet light, ozone, atomic oxygen, etc.) can play an important role in the degradation of the material properties. These effects should be included in the development of the basic material properties such as the relaxation modulus, creep compliance, and Poisson's ratio.

As composite materials become more widely used in aerospace applications, the effects of long-term exposure to the space environment can become a significant factor in the geometry and specifications of the hardware. With the method presented here, the design engineer has the tools necessary to alter initial designs which have not taken into consideration the time-dependency factor. This will provide for more structurally sound and efficient structures.

APPENDIX A

COMPUTER PROGRAM "RELAX"

```
C***********************************************************************
C                                R   E   L   A   X
C***********************************************************************
C            PROGRAM TO DETERMINE THE TIME DEPENDENT STRAIN AND
C                DEFLECTIONS OF A SINGLE PHASE VISCOELASTIC
C                              CANTILEVER BEAM
C***********************************************************************
       IMPLICIT DOUBLE PRECISION(A-H,O-Z)
       DIMENSION XL(4),XLMD(3),RI(3),FA(3),GA(3),XM(3)
       OPEN(UNIT=7,FILE='C:\FORTRAN\RELAX.DAT')
       OPEN(UNIT=8,FILE='C:\FORTRAN\STRAIN.DAT')
C----------------------------------------------------------------------
C                  IDENTIFY BEAM GEOMETRY AND APPLIED LOAD
C----------------------------------------------------------------------
       P = 25.00
       XLTH = 29.25
       WIDE = 5.00
       THK = 1.00
C----------------------------------------------------------------------
C            CALCULATE MOMENT OF INERTIA AND SPECIFIC CONSTANTS
C----------------------------------------------------------------------
       XINRT = (WIDE*(THK**3))/12.
       F1XZ = 12*P*XLTH*(THK/2)/(WIDE*(THK**3))
       Q = 4.*P*(XLTH**3)/(WIDE*(THK**3))
       T = 0.0001
C----------------------------------------------------------------------
C        IDENTIFY MATERIAL CONSTANTS FOR RELAXATION MODULUS FUNCTION
C----------------------------------------------------------------------
       A = 180000.
       B1 = 5000.
       B2 = 5000.
       B3 = 170000.
       GAMA1 = 1000.
       GAMA2 = 10.
       GAMA3 = .01
C----------------------------------------------------------------------
C                        CALCULATE RELAXATION MODULUS
C----------------------------------------------------------------------
       DO 20 I = 1,1301
       E = A + B1*EXP(-GAMA1*T) + B2*EXP(-GAMA2*T) + B3*EXP(-GAMA3*T)
       WRITE(7,*)T,E
       T = T + 1
    20 CONTINUE
C----------------------------------------------------------------------
C           CALCULATE CONSTANTS FOR CALCULATION OF STRAIN AND
C                       DEFLECTION IN THE LAPLACE DOMAIN
C----------------------------------------------------------------------
       F1 = GAMA1 + GAMA2 + GAMA3
       F2 = GAMA1*GAMA2 + GAMA2*GAMA3 + GAMA3*GAMA1
       F3 = GAMA1*GAMA2*GAMA3
       F4 = GAMA2 + GAMA3
       F5 = GAMA2*GAMA3
       F6 = GAMA1 + GAMA3
       F7 = GAMA1*GAMA3
```

```
      F8 = GAMA1 + GAMA2
      F9 = GAMA1*GAMA2
      XL(1) = A + B1 + B2 + B3
      XL(2) = A*F1 + B1*F4 + B2*F6 + B3*F8
      XL(3) = A*F2 + B1*F5 + B2*F7 + B3*F9
      XL(4) = A*F3
C-----------------------------------------------------------------------
C OBTAIN THE ROOTS OF THE Ga(s) TERM IN THE DENOMINATOR OF EQUATION (20)
C-----------------------------------------------------------------------
      CALL CUBIC(XL,XLMD,RI)
      WRITE(7,10)XLMD(1),XLMD(2),XLMD(3)
   10 FORMAT(' THE REAL ROOTS OF Ga(s) ARE',3E12.5)
      WRITE(7,11)RI(1),RI(2),RI(3)
   11 FORMAT(' THE IMAGINARY ROOTS OF Ga(s) ARE',3E12.5)
C-----------------------------------------------------------------------
C       CALCULATE THE "M" CONSTANTS OF EQUATION (24) BY HEAVISIDE'S
C                   PARTIAL FRACTION EXPANSION THEOREM
C-----------------------------------------------------------------------
      XM0 = F3/XL(4)
      XXM0 = F3/XXL(4)
      WRITE(7,12)XM0
   12 FORMAT(' THE CONSTANT Mo IS',E12.5)
      DO 21 I = 1,3
      FA(I) = XLMD(I)**3+F1*XLMD(I)**2+F2*XLMD(I)+F3
      GA(I) = 4.*XL(1)*XLMD(I)**3+3.*XL(2)*XLMD(I)**2+2.*XL(3)*XLMD(I)+
     *XL(4)
      XM(I) = FA(I)/GA(I)
      WRITE(7,13)I,XM(I)
   13 FORMAT(' M',I1,' =',E12.5)
   21 CONTINUE
C-----------------------------------------------------------------------
C           CALCULATE THE TIME DEPENDENT STRAIN PER EQUATION (26)
C-----------------------------------------------------------------------
      T = .0001
      DO 22 I = 1,2500,5
      STRN = F1XZ*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
C-----------------------------------------------------------------------
C        CALCULATE THE TIME DEPENDENT DEFLECTION PER EQUATION (29)
C-----------------------------------------------------------------------
      DEFL = Q*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
      WRITE(8,30)T,STRN,DEFL
   30 FORMAT(11E12.5)
      T = T + 5
   22 CONTINUE
C-----------------------------------------------------------------------
C          CALCULATE THE DEFLECTED SHAPE OF THE CANTILEVER BEAM
C      AT SEVERAL DIFFERENT TIMES (0 hrs, 10 hrs, 100 hrs & 1000 hrs)
C-----------------------------------------------------------------------
      X = 0.0
      DO 23 I = 1,21
      Q1 = (P*XLTH*(X**2)/2. - P*(X**3)/6.)/XINRT
      Q1X = (P*XLTH*(X**2)/2. - P*(X**3)/6.)
```

```
C-----------------------------------------------------------------------
C                              T = 0.0 hrs.
C-----------------------------------------------------------------------
      T=0
      D0  = Q1*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
C-----------------------------------------------------------------------
C                              T = 10.0 hrs.
C-----------------------------------------------------------------------
      T=10
      D10  = Q1*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
C-----------------------------------------------------------------------
C                               T = 100.0 hrs.
C-----------------------------------------------------------------------
      T=100
      D100  = Q1*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
C-----------------------------------------------------------------------
C                               T = 1000.0 hrs.
C-----------------------------------------------------------------------
      T=1000
      D1000  = Q1*(XM(1)*EXP(XLMD(1)*T)+XM(2)*EXP(XLMD(2)*T)+
     *XM(3)*EXP(XLMD(3)*T)+XM0)
      WRITE(9,30)X,D0,D10,D100,D1000
      X = X + 1.4625
   23 CONTINUE
      STOP
      END
```

APPENDIX B

COMPUTER PROGRAM “VISCOBM”

```
C***********************************************************************
C        PROGRAM TO DETERMINE THE TIME DEPENDENT STRAINS,DEFLECTION
C  AND NATURAL FREQUENCY OF A COMPOSITE BEAM SUBJECTED TO BENDING LOADS
C***********************************************************************
       IMPLICIT DOUBLE PRECISION (A-H,O-Z)
       DIMENSION ALPHA(4)
       DIMENSION E(4,22)
       DIMENSION XNU(4,21),G(4,12)
       DIMENSION ZT(5)
       DIMENSION D(6,6),EPSX(4),EPSY(4),EPSXY(4)
       DIMENSION SIGX(4),THK(4),SIGY(4),SIGXY(4)
       DIMENSION SL(6),EMS(6),PMS(6)
       DIMENSION STS1(200),STS2(200),STS3(200),STS4(200)
       DIMENSION H1(6),H2(6),H3(6),H4(6)
       DIMENSION F1(6),F2(6),F3(6),F4(6),ALAP(6,6)
       DIMENSION ALAP2(6,6),ALAP3(6,6),ALAP4(6,6)
       DIMENSION QR11(4),QR22(4),QR12(4),QR66(4),QR26(4),QR16(4)
       DIMENSION SR11(4),SR22(4),SR12(4),SR66(4),SR26(4),SR16(4)
       DIMENSION S11(4),S22(4),S12(4),S66(4)
       DIMENSION Q11(4),Q22(4),Q12(4),Q66(4)
       DIMENSION XIN(31),CAL(31),XLAMD(4),F5(6),ALAP5(6,6),H5(6)
       DIMENSION F6(6),ALAP6(6,6),H6(6)
C-----------------------------------------------------------------
C                       IDENTIFY OUTPUT FILES
C-----------------------------------------------------------------
       OPEN(UNIT=12,FILE='C:\FORTRAN\ACTUAL.DAT')
       OPEN(UNIT=13,FILE='C:\FORTRAN\NORMAL.DAT')
C-----------------------------------------------------------------
C                      INPUT WIDTH AND LENGTH OF BEAM
C-----------------------------------------------------------------
       WIDTH = 4.
       XLB = 29.25
C-----------------------------------------------------------------
C      INPUT VALUES OF THE LAPLACE PARAMETER AT DISCRETE POINTS
C-----------------------------------------------------------------
       SL(1) =    .0067379
       SL(2) =    .0497871
       SL(3) =    .3678794
       SL(4) =   2.7182818
       SL(5) =  20.0855369
       SL(6) = 148.4131591
C-----------------------------------------------------------------
C        ***********************************************
C        *     IDENTIFY LAMINA MATERIAL PROPERTIES     *
C        ***********************************************
C-----------------------------------------------------------------
C      INPUT VOLUME FRACTION FOR MATRIX AND FIBERS (VM & VF)
C      INPUT BULK MODULUS OF ELASTICITY FOR MATRIX (XK)
C      INPUT POISSON'S RATIO AND YOUNG'S MODULUS FOR FIBERS (XNUF & EF)
C      INPUT TOTAL NUMBER OF PLIES (NLAY)
C      INPUT DENSITY OF A PLY (RHO)
C-----------------------------------------------------------------
       VM = .38
       VF = .62
```

```
      XK = 400000.
      EF = 33.33E6
      XNUF = .2
      NLAY = 4
      PI = 3.14159
      RHO = .057
C-------------------------------------------------------------------
C                 CALCULATE SHEAR MODULUS FOR FIBERS
C-------------------------------------------------------------------
      GF = EF/(2*(1+XNUF))
C-------------------------------------------------------------------
C      INPUT PREVIOUSLY CALCULATED VALUE OF PLY BENDING STRAINS,
C      DEFLECTION AND NATURAL FREQUENCY AT INITIAL TIME (T=0.0)
C                          (FROM VIST0.FOR)
C-------------------------------------------------------------------
C                   THIS CASE IS 4 PLY  [30/-30]symmetric
C-------------------------------------------------------------------
      EPS01 =  .00140303
      EPS02 =  .000701517
      EPS03 =  .00000000
      EPS04 = -.000701517
      DEFT0 =  3.201
      FREQ0 =  10.1
C-------------------------------------------------------------------
C      INPUT PRONY COEFFICIENTS FOR RELAXATION MODULUS
C      AND TIME DEPENDENT POISSON'S RATIO
C-------------------------------------------------------------------
C                    RELAXATION MODULUS COEFFICIENTS
C-------------------------------------------------------------------
      A1 = 180000.
      B1 = 5000.
      B2 = 5000.
      B3 = 170000.
C-------------------------------------------------------------------
C                     POISSON'S RATIO COEFFICIENTS
C-------------------------------------------------------------------
      A13 = .425
      B13 = -.0020833
      B23 = -.0020833
      B33 = -.070833
C-------------------------------------------------------------------
C      CALCULATE THE RELAXATION MODULUS AND TIME DEPENDENT
C      POISSON'S RATIO FOR THE COMPOSITE SYSTEM MATRIX MATERIAL
C      IN THE LAPLACE DOMAIN
C-------------------------------------------------------------------
C             EMS(I) = RELAXATION MODULUS FOR MATRIX
C             PMS(I) = POISSON'S RATIO FOR MATRIX
C             SL(I) = LAPLACE PARAMETER
C-------------------------------------------------------------------
C
      DO 1000 LAP = 1,6
      EMS(LAP) = A1 + SL(LAP)*B1/(SL(LAP)+1000.) + SL(LAP)*B2/(SL(LAP)+
     1          10.) + SL(LAP)*B3/(SL(LAP)+.01)
      PMS(LAP) = A13 + SL(LAP)*B13/(SL(LAP)+1000.) +
```

```
     1            SL(LAP)*B23/(SL(LAP)+10.) + SL(LAP)*B33/(SL(LAP)+.01)
C-----------------------------------------------------------------------
C              CALCULATE SHEAR MODULUS FOR THE MATRIX
C-----------------------------------------------------------------------
      GM = EMS(LAP)/(2*(1+PMS(LAP)))
C-----------------------------------------------------------------------
C     CALCULATE LONGITUDINAL AND TRANSVERSE PROPERTIES OF A
C     UNIDIRECTIONAL LAMINA (USING MICROMECHANICS APPROACH)
C-----------------------------------------------------------------------
      DO 10 I = 1,NLAY
      E(I,11) = EF*VF + EMS(LAP)*VM
      E(I,22) = EF*EMS(LAP)/(VM*EF + VF*EMS(LAP))
      GNUM = GF*(PI+4*VF) + GM*(PI-4*VF)
      GDEN = GF*(PI-4*VF) + GM*(PI+4*VF)
      GN = GNUM/GDEN
      G121 = ((4-PI)+PI*GN)/4
      G122 = 4*GN/((4-PI)*GN+PI)
      G(I,12) = (GM/2)*(G121 + G122)
      XNU(I,12) = XNUF*VF + PMS(LAP)*VM
      XNU(I,21) = XNU(I,12)*E(I,22)/E(I,11)
   10 CONTINUE
C---------------------------------------------------------------------
C     INPUT THE ANGULAR ORIENTATION OF EACH LAMINA WITH RESPECT TO
C     THE LONGITUDINAL AXIS OF THE LAMINATE
C---------------------------------------------------------------------
      ALPHA(1) =  30.0
      ALPHA(2) =  -30.0
      ALPHA(3) =  -30.0
      ALPHA(4) =  30.0
C---------------------------------------------------------------------
C     CALCULATE THE REDUCED STIFFNESS COEFFICIENTS FOR EACH LAMINA
C---------------------------------------------------------------------
      DO 11 I = 1,NLAY
      Q11(I) = E(I,11)/(1.-XNU(I,12)*XNU(I,21))
      Q12(I) = XNU(I,12)*E(I,22)/(1.-XNU(I,12)*XNU(I,21))
      Q22(I) = E(I,22)/(1.-XNU(I,12)*XNU(I,21))
      Q66(I) = G(I,12)
C---------------------------------------------------------------------
C     CALCULATE THE COMPLIANCE MATRIX COEFFICIENTS FOR EACH LAMINA
C---------------------------------------------------------------------
      S11(I) = 1/E(I,11)
      S12(I) = -XNU(I,12)/E(I,11)
      S22(I) = 1/E(I,22)
      S66(I) = 1/G(I,12)
C---------------------------------------------------------------------
C     CALCULATE THE TRANSFORMED REDUCED STIFFNESSES FOR EACH LAMINA
C---------------------------------------------------------------------
      ALPHA(I) = ALPHA(I)*PI/180.
      QR11(I) = Q11(I)*(COS(ALPHA(I)))**4
     1          +2.*(Q12(I)+2.*Q66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+Q22(I)*(SIN(ALPHA(I)))**4
C
      QR12(I) = (Q11(I)+Q22(I)-4.*Q66(I))*((SIN(ALPHA(I)))**2)*
     1          ((COS(ALPHA(I)))**2)+Q12(I)*((SIN(ALPHA(I)))**4+
```

```
     2              (COS(ALPHA(I)))**4)
C
      QR16(I) = (Q11(I)-Q12(I)-2.*Q66(I))*((COS(ALPHA(I)))**3)*
     1          SIN(ALPHA(I))+(Q12(I)-Q22(I)+2.*Q66(I))*COS(ALPHA(I))*
     2          (SIN(ALPHA(I)))**3
C
      QR22(I) = Q11(I)*(SIN(ALPHA(I)))**4
     1          +2.*(Q12(I)+2.*Q66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+Q22(I)*(COS(ALPHA(I)))**4
C
      QR26(I) = (Q11(I)-Q12(I)-2.*Q66(I))*((SIN(ALPHA(I)))**3)*
     1          COS(ALPHA(I))+(Q12(I)-Q22(I)+2.*Q66(I))*SIN(ALPHA(I))*
     2          (COS(ALPHA(I)))**3
C
      QR66(I) = (Q11(I)+Q22(I)-2.*Q12(I)-2.*Q66(I))*((SIN(ALPHA(I)))**2)
     1          *((COS(ALPHA(I)))**2)+Q66(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
C-----------------------------------------------------------------
C     CALCULATE THE TRANSFORMED COMPLIANCE COEFFICIENTS FOR EACH LAMINA
C-----------------------------------------------------------------
      SR11(I) = S11(I)*(COS(ALPHA(I)))**4
     1          +(2.*S12(I)+ S66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+S22(I)*(SIN(ALPHA(I)))**4
C
      SR12(I) = (S11(I)+S22(I)-S66(I))*((SIN(ALPHA(I)))**2)*
     1          ((COS(ALPHA(I)))**2)+S12(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
C
      SR16(I) = (2.*S11(I)-2.*S12(I)-S66(I))*((COS(ALPHA(I)))**3)*
     1          SIN(ALPHA(I))-(2*S22(I)-2*S12(I)-S66(I))*COS(ALPHA(I)
     2          )*(SIN(ALPHA(I)))**3
C
      SR22(I) = S11(I)*(SIN(ALPHA(I)))**4
     1          +(2*S12(I)+S66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+S22(I)*(COS(ALPHA(I)))**4
C
      SR26(I) = (2*S11(I)-2*S12(I)-S66(I))*((SIN(ALPHA(I)))**3)*
     1          COS(ALPHA(I))-(2*S22(I)-2*S12(I)-S66(I))*SIN(ALPHA(I)
     2          )*(COS(ALPHA(I)))**3
C
      SR66(I) = 2*(2*S11(I)+2*S22(I)-4.*S12(I)-S66(I))*((SIN(ALPHA(I))
     1          )**2)*((COS(ALPHA(I)))**2)+S66(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
   11 CONTINUE
C       *****************************************************
C       * RELATE EACH LAMINA THICKNESS TO ITS LOCATION WITHIN *
C       * THE LAMINATE (Z COORDINATE IS 0 AT THE MIDSURFACE)  *
C       *****************************************************
C-----------------------------------------------------------------
C                      INPUT LAMINA THICKNESSES
C-----------------------------------------------------------------
      DO 998 J = 1,NLAY
      THK(J) = .0625
  998 CONTINUE
```

```
C-----------------------------------------------------------------
      TOTTH = 0.0
      DO 25 J = 1,NLAY
      TOTTH = TOTTH + THK(J)
   25 CONTINUE
C-----------------------------------------------------------------
C             CALCULATE THE TOTAL WEIGHT OF THE BEAM
C-----------------------------------------------------------------
      TOTWT = WIDTH*TOTTH*XLB*RHO
C-----------------------------------------------------------------
      HAFTH = TOTTH/2.
      ZT(NLAY) = HAFTH
      DO 24 J = 1,NLAY
      LAM = NLAY - J + 1
      ZT(NLAY-J) = ZT(LAM) - THK(LAM)
   24 CONTINUE
      DO 40 J = 1,NLAY+1
   40 CONTINUE
C------------------------------------------------------------------
C        CALCULATE BENDING STIFFNESSES FOR THE ENTIRE LAMINATE
C------------------------------------------------------------------
      D11 = 0.0
      D12 = 0.0
      D16 = 0.0
      D22 = 0.0
      D26 = 0.0
      D66 = 0.0
      DO 35 J = 1,NLAY
      L = J + 1
      D11 = D11 + QR11(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D12 = D12 + QR12(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D16 = D16 + QR16(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D22 = D22 + QR22(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D26 = D26 + QR26(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D66 = D66 + QR66(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
   35 CONTINUE
C-----------------------------------------------------------------
C          CALCULATE THE DETERMINANT OF THE STIFFNESS MATRIX
C-----------------------------------------------------------------
      DETER = D11*D22*D66 + D12*D26*D16 + D16*D12*D26
     1        - D16*D16*D22 - D12*D12*D66 - D26*D26*D11
C-----------------------------------------------------------------
C  CALCULATE COEFFICIENTS FOR THE COFACTORS OF THE STIFFNESS MATRIX
C-----------------------------------------------------------------
      D11C = D22*D66 - D26*D26
      D12C = -(D12*D66 - D16*D26)
      D16C = D12*D26 - D16*D22
C-----------------------------------------------------------------
C     CALCULATE THE COEFFICIENTS OF THE INVERSE STIFFNESS MATRIX
C-----------------------------------------------------------------
      D11S = D11C/DETER
      D12S = D12C/DETER
      D16S = D16C/DETER
      WRITE(13,*)' D11S =',D11S
```

```
C-----------------------------------------------------------------
C                        IDENTIFY APPPLIED MOMENTS
C-----------------------------------------------------------------
C            THE LOAD MUST BE EXPRESSED IN THE LAPLACE DOMAIN
C               LOAD(S) = LOAD/SL(LAP) [for a CONSTANT load]
C-----------------------------------------------------------------
      XMAP = -292.5/WIDTH
      XMAP = XMAP/SL(LAP)
      PLOAD = 10./SL(LAP)
C     ----------------------------------------------------------------
C     *************** PERFORM DEFLECTION CALCULATIONS ******************
C     ----------------------------------------------------------------
C     ----------------------------------------------------------------
C     CALCULATE THE CURVATURE AT ANY POINT ALONG THE LENGTH OF THE BEAM
C     ----------------------------------------------------------------
      INM1 = 30
      XIN(1) = 0.0
      DO 805 K = 2,31
      XIN(K) = XIN(K-1)+XLB/INM1
  805 CONTINUE
      DO 804 NI = 1,INM1+1
      CAL(NI) = D11S*PLOAD*(XLB-XIN(NI))/WIDTH
  804 CONTINUE
C     ----------------------------------------------------------------
C     PERFORM A POLYNOMIAL CURVE FIT FOR THE CURVATURE AT
C     ANY POINT ALONG THE LENGTH
C     ----------------------------------------------------------------
      CALL POLYCF(XIN,CAL,31,3,XLAMD,NERR)
C     ----------------------------------------------------------------
C     CALCULATE THE TIP DEFLECTION OF THE BEAM BY INTEGRATING THE
C     CURVATURE TWICE WITH RESPECT TO THE LENGTH COORDINATE
C     ----------------------------------------------------------------
      DEF = 0.0
      DO 111 J = 1,4
      DEF = DEF + XLAMD(J)*(XLB**(J+1))/(J*(J+1))
  111 CONTINUE
C-----------------------------------------------------------------
C  CALCULATE THE ASSOCIATED ELASTIC NATURAL FREQUENCY OF THE BEAM
C-----------------------------------------------------------------
      FREQ = 3.51601*SQRT((WIDTH*386.4)/(D11S*TOTWT*(XLB**4)))
C-----------------------------------------------------------------
C           IDENTIFY THE DEFLECTION AS THE ASSOCIATED ELASTIC
C              SOLUTION MULTIPLIED BY THE LAPLACE PARAMETER
C-----------------------------------------------------------------
      DEF = DEF*SL(LAP)
C-----------------------------------------------------------------
C                 CALCULATE THE STRESSES FOR EACH LAMINA
C-----------------------------------------------------------------
      DO 36 J = 1,NLAY
      L = J + 1
     SIGX(J) = ZT(L-2)*XMAP*(QR11(J)*D11S +QR12(J)*D12S +QR16(J)*D16S)
     SIGY(J) = ZT(L-2)*XMAP*(QR12(J)*D11S +QR22(J)*D12S +QR26(J)*D16S)
     SIGXY(J) = ZT(L-2)*XMAP*(QR16(J)*D11S +QR26(J)*D12S +QR66(J)*D16S)
   36 CONTINUE
```

```
C-----------------------------------------------------------------------
C          CALCULATE THE TIME DEPENDENT STRAINS FOR EACH LAMINA
C-----------------------------------------------------------------------
       DO 37 J = 1,NLAY
       EPSX(J)  = SR11(J)*SIGX(J)+SR12(J)*SIGY(J)+SR16(J)*SIGXY(J)
       EPSY(J)  = SR12(J)*SIGX(J)+SR22(J)*SIGY(J)+SR26(J)*SIGXY(J)
       EPSXY(J)  = SR16(J)*SIGX(J)+SR26(J)*SIGY(J)+SR66(J)*SIGXY(J)
    37 CONTINUE
C---------------------------------------------------------------------
C       IDENTIFY STRAINS FOR EACH LAMINA AS THE ASSOCIATED ELASTIC
C       STRAINS MULTIPLIED BY THE LAPLACE PARAMETER
C---------------------------------------------------------------------
       DO 38 J = 1,NLAY
       EPSX(J)  = EPSX(J)*SL(LAP)
       EPSY(J)  = EPSY(J)*SL(LAP)
       EPSXY(J)  = EPSXY(J)*SL(LAP)
    38 CONTINUE
C---------------------------------------------------------------------
C       CALCULATE DIFFERENCE BETWEEN STRAINS, DEFLECTION AND NATURAL
C          FREQUENCY AT TIME ZERO AND AT THE DISCRETE VALUES
C          OF THE LAPLACE PARAMETER FOR INVERSION BACK INTO
C                            THE TIME DOMAIN
C---------------------------------------------------------------------
C       NOTE THAT ONLY THE STRAINS IN THE X DIRECTION ARE USED
C                           FOR THIS PROBLEM
C---------------------------------------------------------------------
       F1(LAP)  = EPS01 - EPSX(1)
       F2(LAP)  = EPS02 - EPSX(2)
       F3(LAP)  = EPS03 - EPSX(3)
       F4(LAP)  = EPS04 - EPSX(4)
       F5(LAP)  = DEFT0 - DEF
       F6(LAP)  = FREQ0 - FREQ
       WRITE(13,902)LAP,FREQ0,FREQ,F6(LAP)
   902 FORMAT('LAP=',I2,' FREQ0=',E12.5,' FREQ=',E12.5,' F6(LAP)=',E12.5)
  1000 CONTINUE
C---------------------------------------------------------------------
C       PERFORM LAPLACE TRANSFORM INVERSION AND SOLVE FOR THE CONSTANTS
C       NEEDED FOR THE CALCULATIONS OF TIME DEPENDENT STRAINS
C---------------------------------------------------------------------
       DO 1001 LAPW = 1,6
       DO 1001 LAPV = 1,6
       ALPWV = EXP(2*(LAPW-LAPV))
       ALAP(LAPW,LAPV)  = 1/(1 + ALPWV)
       ALAP2(LAPW,LAPV)  = 1/(1 + ALPWV)
       ALAP3(LAPW,LAPV)  = 1/(1 + ALPWV)
       ALAP4(LAPW,LAPV)  = 1/(1 + ALPWV)
       ALAP5(LAPW,LAPV)  = 1/(1 + ALPWV)
       ALAP6(LAPW,LAPV)  = 1/(1 + ALPWV)
  1001 CONTINUE
C---------------------------------------------------------------------
C                      SOLVE SIMULTANEOUS EQUATIONS
C---------------------------------------------------------------------
       CALL MATINV(ALAP,6,F1,H1)
       CALL MATINV(ALAP2,6,F2,H2)
```

```
      CALL MATINV(ALAP3,6,F3,H3)
      CALL MATINV(ALAP4,6,F4,H4)
      CALL MATINV(ALAP5,6,F5,H5)
      CALL MATINV(ALAP6,6,F6,H6)
C-----------------------------------------------------------------------
C      EVALUATE GAMMA CONSTANTS FROM INITIAL STRAIN CONDITIONS
C-----------------------------------------------------------------------
      SUMST1 = 0.0
      SUMST2 = 0.0
      SUMST3 = 0.0
      SUMST4 = 0.0
      SUMST5 = 0.0
      SUMST6 = 0.0
      DO 1002 LAP = 1,6
      SUMST1 = SUMST1 + H1(LAP)
      SUMST2 = SUMST2 + H2(LAP)
      SUMST3 = SUMST3 + H3(LAP)
      SUMST4 = SUMST4 + H4(LAP)
      SUMST5 = SUMST5 + H5(LAP)
      SUMST6 = SUMST6 + H6(LAP)
 1002 CONTINUE
      GAMA1 = EPS01 - SUMST1
      GAMA2 = EPS02 - SUMST2
      GAMA3 = EPS03 - SUMST3
      GAMA4 = EPS04 - SUMST4
      GAMA5 = DEFT0 - SUMST5
      GAMA6 = FREQ0 - SUMST6
C-----------------------------------------------------------------------
C              CALCULATE TIME DEPENDENT STRAINS,DEFLECTION
C                         AND NATURAL FREQUENCY
C-----------------------------------------------------------------------
      LTIME = 200
      TIME = .0001
      DO 1003 I = 1,LTIME
      STS1(I) = 0.0
      STS2(I) = 0.0
      STS3(I) = 0.0
      STS4(I) = 0.0
      DEFLEC = 0.0
      FREQCY = 0.0
      DO 1004 LAP = 1,6
      STS1(I) = STS1(I) + H1(LAP)*EXP(-SL(LAP)*TIME)
      STS2(I) = STS2(I) + H2(LAP)*EXP(-SL(LAP)*TIME)
      STS3(I) = STS3(I) + H3(LAP)*EXP(-SL(LAP)*TIME)
      STS4(I) = STS4(I) + H4(LAP)*EXP(-SL(LAP)*TIME)
      DEFLEC = DEFLEC + H5(LAP)*EXP(-SL(LAP)*TIME)
      FREQCY = FREQCY + H6(LAP)*EXP(-SL(LAP)*TIME)
 1004 CONTINUE
      STS1(I) = GAMA1 + STS1(I)
      STS2(I) = GAMA2 + STS2(I)
      STS3(I) = GAMA3 + STS3(I)
      STS4(I) = GAMA4 + STS4(I)
      DEFLEC = GAMA5 + DEFLEC
      FREQCY = GAMA6 + FREQCY
```

```
      WRITE(12,1006)I,TIME,STS1(I),STS2(I),STS3(I),STS4(I),DEFLEC,FREQCY
      TIME = TIME*1.1
 1003 CONTINUE
C----------------------------------------------------------------------
C             NORMALIZE THE STRAINS WITH RESPECT TO THE MAXIMUM
C                    FOR PURPOSES OF GRAPHING THE RESULTS
C                       (TMAG = MAGNIFICATION FACTOR)
C----------------------------------------------------------------------
      TIME = .0001
      TMAG = 100.
      DO 285 I = 1,LTIME
      STS1(I)=TMAG*STS1(I)/STS1(LTIME)
      STS2(I)=TMAG*STS2(I)/STS2(LTIME)
C*******IF THE STRAIN IS ZERO (mid surface) DO NOT NORMALIZE***********
C             STS3(I) IS THE STRAIN AT THE MIDDLE SURFACE
C*********************************************************************
      STS4(I)=TMAG*STS4(I)/STS4(LTIME)
      WRITE(13,1006)I,TIME,STS1(I),STS2(I),STS3(I),STS4(I)
 1006 FORMAT(I5,7E12.5)
      TIME = TIME*1.1
  285 CONTINUE
      STOP
      END
```

APPENDIX C

COMPUTER PROGRAM "VIST0"

```
C*********************************************************************
C*   PROGRAM TO DETERMINE THE INITIAL (T=0.0) STRAINS AND DEFLECTION   *
C*     OF A VISCOELASTIC COMPOSITE BEAM SUBJECTED TO BENDING LOADS     *
C*********************************************************************
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION ALPHA(24)
      DIMENSION E(24,22)
      DIMENSION XNU(24,21),G(24,12)
      DIMENSION Z(25),ZT(25)
      DIMENSION D(6,6),EPSX(24),EPSY(24),EPSXY(24)
      DIMENSION SIGX(24),THK(24),SIGY(24),SIGXY(24)
      DIMENSION QR11(24),QR22(24),QR12(24),QR66(24),QR26(24),QR16(24)
      DIMENSION SR11(24),SR22(24),SR12(24),SR66(24),SR26(24),SR16(24)
      DIMENSION S11(24),S22(24),S12(24),S66(24)
      DIMENSION Q11(24),Q22(24),Q12(24),Q66(24)
      DIMENSION XIN(31),CAL(31),XLAMD(4)
      OPEN(UNIT=13,FILE='C:\FORTRAN\VIST0.DAT')
C-------------------------------------------------------------------
      IFLAG = 0
C-------------------------------------------------------------------
C                  IDENTIFY LAMINA MATERIAL PROPERTIES
C                        AT TIME T = 0.0 SECONDS
C-------------------------------------------------------------------
      PI = 3.14159
      WRITE(6,*)' INPUT THE COMPOSITE "MATRIX" VOLUME FRACTION'
      READ(5,*)VM
      WRITE(6,*)' INPUT THE COMPOSITE "FIBER" VOLUME FRACTION'
      READ(5,*)VF
      WRITE(6,*)' INPUT THE "FIBER" MODULUS OF ELASTICITY'
      READ(5,*)EF
      WRITE(6,*)' INPUT THE "MATRIX" RELAXATION MODULUS AT t=0'
      READ(5,*)EM
      WRITE(6,*)' INPUT THE "MATRIX" POISSONS RATIO AT t=0'
      READ(5,*)XNUM
      WRITE(6,*)' INPUT THE "FIBER" POISSONS RATIO'
      READ(5,*)XNUF
      WRITE(6,*)' INPUT THE "PLY" DENSITY'
      READ(5,*)RHO
C-------------------------------------------------------------------
C                        ENTER NUMBER OF PLIES
C-------------------------------------------------------------------
      WRITE(6,*)' INPUT THE TOTAL NUMBER OF PLIES'
      READ(5,*)NLAY
C-------------------------------------------------------------------
C                ENTER THE WIDTH AND LENGTH OF THE BEAM
C-------------------------------------------------------------------
      WRITE(6,*)' INPUT THE WIDTH OF THE CANTILEVER BEAM'
      READ(5,*)WIDTH
      WRITE(6,*)' INPUT THE LENGTH OF THE CANTILEVER BEAM'
      READ(5,*)XLB
```

```
C-----------------------------------------------------------------
C      CALCULATE SHEAR MODULUS FOR FIBERS AND MATRIX
C-----------------------------------------------------------------
       GF = EF/(2*(1+XNUF))
       GM = EM/(2*(1+XNUM))
C-----------------------------------------------------------------
C             CALCULATE PLY UNIDIRECTIONAL MATERIAL PROPERTIES
C                    (USING MICROMECHANICS APPROACH)
C-----------------------------------------------------------------
       DO 10 I = 1,NLAY
       E(I,11) = EF*VF + EM*VM
       E(I,22) = EF*EM/(VM*EF + VF*EM)
       GNUM = GF*(PI+4*VF) + GM*(PI-4*VF)
       GDEN = GF*(PI-4*VF) + GM*(PI+4*VF)
       GN = GNUM/GDEN
       G121 = ((4-PI)+PI*GN)/4
       G122 = 4*GN/((4-PI)*GN+PI)
       G(I,12) = (GM/2)*(G121 + G122)
       XNU(I,12) = XNUF*VF + XNUM*VM
       XNU(I,21) = XNU(I,12)*E(I,22)/E(I,11)
   10 CONTINUE
C-----------------------------------------------------------------
C      INPUT THE ANGULAR ORIENTATION OF EACH LAMINA WITH RESPECT TO
C      THE LONGITUDINAL AXIS OF THE LAMINATE
C-----------------------------------------------------------------
       DO 702 IJ = 1,NLAY
       WRITE(6,*)' INPUT THE ORIENTATION (ANGLE) OF PLY NO.',IJ
       READ(5,*)ALPHA(IJ)
  702 CONTINUE
C-----------------------------------------------------------------
C      CALCULATE THE REDUCED STIFFNESS COEFFICIENTS FOR EACH LAMINA
C-----------------------------------------------------------------
       DO 11 I = 1,NLAY
       Q11(I) = E(I,11)/(1.-XNU(I,12)*XNU(I,21))
       Q12(I) = XNU(I,12)*E(I,22)/(1.-XNU(I,12)*XNU(I,21))
       Q22(I) = E(I,22)/(1.-XNU(I,12)*XNU(I,21))
       Q66(I) = G(I,12)
C-----------------------------------------------------------------
C      CALCULATE THE COMPLIANCE MATRIX COEFFICIENTS FOR EACH LAMINA
C-----------------------------------------------------------------
       S11(I) = 1/E(I,11)
       S12(I) = -XNU(I,12)/E(I,11)
       S22(I) = 1/E(I,22)
       S66(I) = 1/G(I,12)
C-----------------------------------------------------------------
C      CALCULATE THE TRANSFORMED REDUCED STIFFNESSES FOR EACH LAMINA
C-----------------------------------------------------------------
       ALPHA(I) = ALPHA(I)*PI/180.
       QR11(I) = Q11(I)*(COS(ALPHA(I)))**4
     1          +2.*(Q12(I)+2.*Q66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+Q22(I)*(SIN(ALPHA(I)))**4
C
       QR12(I) = (Q11(I)+Q22(I)-4.*Q66(I))*((SIN(ALPHA(I)))**2)*
     1          ((COS(ALPHA(I)))**2)+Q12(I)*((SIN(ALPHA(I)))**4+
```

```
     2          (COS(ALPHA(I)))**4)
C
      QR16(I) = (Q11(I)-Q12(I)-2.*Q66(I))*((COS(ALPHA(I)))**3)*
     1          SIN(ALPHA(I))+(Q12(I)-Q22(I)+2.*Q66(I))*COS(ALPHA(I))*
     2          (SIN(ALPHA(I)))**3
C
      QR22(I) = Q11(I)*(SIN(ALPHA(I)))**4
     1          +2.*(Q12(I)+2.*Q66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+Q22(I)*(COS(ALPHA(I)))**4
C
      QR26(I) = (Q11(I)-Q12(I)-2.*Q66(I))*((SIN(ALPHA(I)))**3)*
     1          COS(ALPHA(I))+(Q12(I)-Q22(I)+2.*Q66(I))*SIN(ALPHA(I))*
     2          (COS(ALPHA(I)))**3
C
      QR66(I) = (Q11(I)+Q22(I)-2.*Q12(I)-2.*Q66(I))*((SIN(ALPHA(I)))**2)
     1          *((COS(ALPHA(I)))**2)+Q66(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
C-----------------------------------------------------------------
C     CALCULATE THE TRANSFORMED COMPLIANCE COEFFICIENTS FOR EACH LAMINA
C-----------------------------------------------------------------
      SR11(I) = S11(I)*(COS(ALPHA(I)))**4
     1          +(2.*S12(I)+ S66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+S22(I)*(SIN(ALPHA(I)))**4
C
      SR12(I) = (S11(I)+S22(I)-S66(I))*((SIN(ALPHA(I)))**2)*
     1          ((COS(ALPHA(I)))**2)+S12(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
C
      SR16(I) = (2.*S11(I)-2.*S12(I)-S66(I))*((COS(ALPHA(I)))**3)*
     1          SIN(ALPHA(I))-(2*S22(I)-2*S12(I)-S66(I))*COS(ALPHA(I)
     2          )*(SIN(ALPHA(I)))**3
C
      SR22(I) = S11(I)*(SIN(ALPHA(I)))**4
     1          +(2*S12(I)+S66(I))*((SIN(ALPHA(I)))**2)*
     2          ((COS(ALPHA(I)))**2)+S22(I)*(COS(ALPHA(I)))**4
C
      SR26(I) = (2*S11(I)-2*S12(I)-S66(I))*((SIN(ALPHA(I)))**3)*
     1          COS(ALPHA(I))-(2*S22(I)-2*S12(I)-S66(I))*SIN(ALPHA(I)
     2          )*(COS(ALPHA(I)))**3
C
      SR66(I) = 2*(2*S11(I)+2*S22(I)-4.*S12(I)-S66(I))*((SIN(ALPHA(I))
     1          )**2)*((COS(ALPHA(I)))**2)+S66(I)*((SIN(ALPHA(I)))**4+
     2          (COS(ALPHA(I)))**4)
   11 CONTINUE
C-----------------------------------------------------------------
C        RELATE EACH LAMINA THICKNESS TO ITS LOCATION WITHIN THE
C             LAMINATE (Z COORDINATE IS 0 AT THE MIDSURFACE)
C-----------------------------------------------------------------
C                          INPUT LAMINA THICKNESSES
C-----------------------------------------------------------------
  301 DO 998 J = 1,NLAY
      IF(IFLAG.EQ.0) THEN
      WRITE(6,*)' INPUT THE THICKNESS OF PLY NO.',J
      READ(5,*)THK(J)
```

```
      ENDIF
  998 CONTINUE
      IF(IFLAG.EQ.1)GO TO 302
C-----------------------------------------------------------------------
C                 CALCULATE THE WEIGHT OF THE LAMINATE
C-----------------------------------------------------------------------
      TOTWT = 0.0
      DO 700 NA = 1,NLAY
      TOTWT = TOTWT + WIDTH*THK(NA)*XLB*RHO
  700 CONTINUE
C-----------------------------------------------------------------------
  302 TOTTH = 0.0
      DO 25 J = 1,NLAY
      TOTTH = TOTTH + THK(J)
   25 CONTINUE
      HAFTH = TOTTH/2.
      ZT(NLAY) = HAFTH
      DO 24 J = 1,NLAY
      LAM = NLAY - J + 1
      ZT(NLAY-J) = ZT(LAM) - THK(LAM)
   24 CONTINUE
      IF(IFLAG.EQ.1)GO TO 300
C-----------------------------------------------------------------------
C      CALCULATE BENDING STIFFNESSES FOR THE ENTIRE LAMINATE
C-----------------------------------------------------------------------
      D11 = 0.0
      D12 = 0.0
      D16 = 0.0
      D22 = 0.0
      D26 = 0.0
      D66 = 0.0
      DO 35 J = 1,NLAY
      L = J + 1
      D11 = D11 + QR11(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D12 = D12 + QR12(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D16 = D16 + QR16(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D22 = D22 + QR22(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D26 = D26 + QR26(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
      D66 = D66 + QR66(J)*((ZT(L-1)**3.)-(ZT(L-2)**3.))/3.
   35 CONTINUE
C-----------------------------------------------------------------------
C          CALCULATE THE DETERMINANT OF THE STIFFNESS MATRIX
C-----------------------------------------------------------------------
      DETER = D11*D22*D66 + D12*D26*D16 + D16*D12*D26
     *         - D16*D16*D22 - D12*D12*D66 - D26*D26*D11
C-----------------------------------------------------------------------
C  CALCULATE COEFFICIENTS FOR THE COFACTORS OF THE STIFFNESS MATRIX
C-----------------------------------------------------------------------
      D11C = D22*D66 - D26*D26
      D12C = -(D12*D66 - D16*D26)
      D16C = D12*D26 - D16*D22
```

```
C-----------------------------------------------------------------
C      CALCULATE THE COEFFICIENTS OF THE INVERSE STIFFNESS MATRIX
C-----------------------------------------------------------------
       D11S = D11C/DETER
       D12S = D12C/DETER
       D16S = D16C/DETER
C------------------------------------------------------------------
C               IDENTIFY APPLIED MOMENTS (APPLIED LOAD)
C               (LOAD APPLIED AT TIP [free end] OF BEAM)
C------------------------------------------------------------------
       WRITE(6,*)' INPUT APPLIED LOAD AT THE TIP OF THE CANTILEVER BEAM'
       READ(5,*)PLOAD
       XMAP = -PLOAD*XLB/WIDTH
C      ------------------------------------------------------------------
C      *************** PERFORM DEFLECTION CALCULATIONS ******************
C      ------------------------------------------------------------------
C      ------------------------------------------------------------------
C      CALCULATE THE CURVATURE AT ANY POINT ALONG THE LENGTH OF THE BEAM
C      ------------------------------------------------------------------
       INM1 = 30
       XIN(1) = 0.0
       DO 805 K = 2,31
       XIN(K) = XIN(K-1)+XLB/INM1
   805 CONTINUE
       DO 804 NI = 1,INM1+1
       CAL(NI) = D11S*PLOAD*(XLB-XIN(NI))/WIDTH
   804 CONTINUE
C      ------------------------------------------------------------------
C      PERFORM A POLYNOMIAL CURVE FIT FOR THE CURVATURE AT
C      ANY POINT ALONG THE LENGTH
C      ------------------------------------------------------------------
       CALL POLYCF(XIN,CAL,31,3,XLAMD,NERR)
C      ------------------------------------------------------------------
C      CALCULATE THE TIP DEFLECTION OF THE BEAM BY INTEGRATING THE
C      CURVATURE TWICE WITH RESPECT TO THE LENGTH COORDINATE
C      ------------------------------------------------------------------
       DEF = 0.0
       DO 111 J = 1,4
       DEF = DEF + XLAMD(J)*(XLB**(J+1))/(J*(J+1))
   111 CONTINUE
       WRITE(13,109)DEF
   109 FORMAT(' THE TIP DEFLECTION AT TIME ZERO IS',E12.5)
C-----------------------------------------------------------------
C         CALCULATE THE NATURAL FREQUENCY OF THE CANTILEVER BEAM
C-----------------------------------------------------------------
       FREQ = 3.51601*SQRT((WIDTH*386.4)/(D11S*TOTWT*(XLB**4)))
C-----------------------------------------------------------------
       WRITE(13,*)' D11S =',D11S
       WRITE(13,*)' WIDTH =',WIDTH
       WRITE(13,*)' WEIGHT =',TOTWT
       WRITE(13,*)' LENGTH =',XLB
       WRITE(13,701)FREQ
   701 FORMAT(' THE NATURAL FREQUENCY AT TIME ZERO IS',E12.5,' HERTZ')
```

```
C-------------------------------------------------------------------
C                CALCULATE THE STRESSES FOR EACH LAMINA
C-------------------------------------------------------------------
       IFLAG = 1
       GO TO 301
   300 DO 36 J = 1,NLAY
       L = J + 1 - 2
       SIGX(J) = ZT(L)*XMAP*(QR11(J)*D11S +QR12(J)*D12S +QR16(J)*D16S)
       SIGY(J) = ZT(L)*XMAP*(QR12(J)*D11S +QR22(J)*D12S +QR26(J)*D16S)
       SIGXY(J) = ZT(L)*XMAP*(QR16(J)*D11S +QR26(J)*D12S +QR66(J)*D16S)
    36 CONTINUE
C---------------------------------------------------------------------
C      CALCULATE THE INITIAL (T=0.0) STRAINS FOR EACH LAMINA
C---------------------------------------------------------------------
       WRITE(13,251)
   251 FORMAT('   PLY',7X,'EPSX',11X,'EPSY',11X,'EPSXY'/)
       DO 37 J = 1,NLAY
       EPSX(J) = SR11(J)*SIGX(J)+SR12(J)*SIGY(J)+SR16(J)*SIGXY(J)
       EPSY(J) = SR12(J)*SIGX(J)+SR22(J)*SIGY(J)+SR26(J)*SIGXY(J)
       EPSXY(J) = SR16(J)*SIGX(J)+SR26(J)*SIGY(J)+SR66(J)*SIGXY(J)
       WRITE(13,250)J,EPSX(J),EPSY(J),EPSXY(J)
   250 FORMAT(I4,3E15.6)
    37 CONTINUE
C----------------------------------------------------------------------
       STOP
       END
```

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APPROVAL

ON THE DESIGN OF STRUCTURAL COMPONENTS USING MATERIALS WITH TIME-DEPENDENT PROPERTIES

By Pedro I. Rodriguez

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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