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Time Dependent Turbulence Modeling and Analytical Theories of Turbulence

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Abstract

By simplifying the direct interaction approximation (DIA) for turbulent shear flow, time dependent formulas are derived for the Reynolds stresses which can be included in two equation models. The Green's function is treated phenomenologically, however following Smith and Yakhot (Theor. Comput. Fluid Dyn., 4, 197 (1993)), we insist on the short and long time limits required by DIA. For small strain rates, perturbative evaluation of the correlation function yields a time dependent theory which includes normal stress effects in simple shear flows. From this standpoint, the phenomenological Launder-Reece-Rodi model is obtained by replacing the Green's function by its long time limit. Eddy damping corrections to short time behavior initiate too quickly in this model; in contrast, the present theory exhibits strong suppression of eddy damping at short times. A time dependent theory for large strain rates is proposed in which large scales are governed by rapid distortion theory while small scales are governed by Kolmogorov inertial range dynamics. At short times and large strain rates, the theory closely matches rapid distortion theory, but at long times it relaxes to an eddy damping model.

I. Introduction

D. C. Leslie[1] proposed solving the direct interaction approximation (DIA) equations for shear turbulence[2] by treating the shear terms as a weak perturbation of an isotropic turbulent background state. At lowest order, the corresponding perturbation series yields a linear relation between Reynolds stress and strain rate in which the eddy viscosity depends only on the correlation function and response function of the DIA theory of isotropic turbulence. Because DIA is a time dependent theory, this approach leads naturally to a time dependent theory of shear turbulence. The goal of this paper is to develop these time dependent theories explicitly, incorporating some recent observations of Smith and Yakhot[3] on the short time and long time behavior of turbulence. In contrast, Leslie's main concern was the long time, steady state limit of the theory. Evaluating the second order terms in Leslie's expansion leads to a time dependent generalization of Yoshizawa's[4] nonlinear eddy viscosity representation of turbulence in which the Reynolds stresses are quadratic functions of the mean velocity gradient.[5] Finally, the restriction of Leslie's theory to weak shear is removed by summing the perturbation series to all orders. For simple shear flow in which $\partial U_i / \partial x_j = S \delta_{i1} \delta_{j2}$, the summation is accomplished with the help of rapid distortion theory (RDT).[6]

The present theory of weakly sheared turbulence can be compared to a standard phenomenological time dependent model, the Launder-Reece-Rodi (LRR) Reynolds stress transport model.[7] From the viewpoint of the present theory, the LRR model arises by incorrectly replacing the DIA Green's function by its long time limit. This simplification misrepresents the short time response of turbulence to shear. An important consequence is that eddy damping corrections to short time RDT behavior initiate too quickly in the LRR model. This causes excessive growth of turbulence kinetic energy at short times in highly strained homogeneous shear flow. The present theory exhibits a strong suppression of eddy damping at short times. It therefore also predicts a much wider frequency range over which RDT correctly describes oscillating shear flow: in the present theory, the corrections to RDT are of order ω^{-2} where ω is the oscillation frequency, but are of order ω^{-1} for the LRR model. This may explain the success of RDT based models in computing oscillating flows.[8]

The present proposal for strongly sheared turbulence leads to a "two scale" picture of

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shear turbulence in which large scales are rapidly distorted while small scales are governed by Kolmogorov inertial range dynamics. The theory can be described either as RDT with generalized eddy damping, or as RDT with a modified total strain. The introduction of a phenomenological modified total strain to improve the agreement of RDT with experiments has been advocated in the RDT literature;[6,9] in the present theory, the modified total strain is determined by the Green's function of isotropic turbulence. This theory is intrinsically time dependent; however, it can be simplified by assuming that turbulent states at any strain rate can be long time limits. Then for simple shear $S = \partial U_1 / \partial x_2$ held for infinite time, the theory formally reduces to a relation

$$\tau_{12} = C_\nu(\eta) \frac{K^2}{\varepsilon} S$$

with[10] $\eta = SK/\varepsilon$. There are analogous formulas for the normal stresses. Phenomenological expressions of this type have been proposed,[11] but in the present theory, $C_\nu(\eta)$ is exactly determined by RDT.

II. Formulation of the Theory

A. Simplified DIA Analysis of Shear Turbulence

Leslie's theory of shear turbulence[1] can be derived from the generalized Langevin model for isotropic turbulence[12]

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) + \int_0^t ds\ \eta(k, t, s)\ u_i(\mathbf{k}, s) = f_i(\mathbf{k}, t) \tag{1}$$

in which u_i is the random velocity field, $\eta(k, t, s)$ is a deterministic eddy damping factor, and f is a random force. A Fourier space representation is used and $k^2 = \mathbf{k} \cdot \mathbf{k}$. The properties of $\eta(k, t, s)$ and f are given in detail in Ref. 12. They depend on the two time correlation function of the velocity field, so that the linearity of this equation is only apparent. Eq. (1) is also a generic model in the statistical mechanical theory of transport coefficients.[13] DIA gives the exact correlation function for a suitable model of this type.[12]

Suppose that some external agency, such as shear or buoyancy forces, is present. We will generalize Eq. (1), representing the effect of the external agency by adding a suitable force F_i to the right side:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) + \int_0^t ds\ \eta(k, t, s)\ u_i(\mathbf{k}, s) = f_i(\mathbf{k}, t) + F_i(u_p(\mathbf{k}, t)) \tag{2}$$

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This general model, which can perhaps be attributed to Leslie, is a simplification for each force F of a corresponding complete DIA theory. It simplifies the dynamics by ignoring any effects of the external agency on either the eddy damping or the random force; in particular, the eddy damping always remains isotropic. Except for some tentative proposals of Tchen[14] and the suggestions of Cambon et al,[15] such effects have been little investigated. Ignoring them amounts to treating shear turbulence, for example, as the outcome of straining by large scales acting against isotropic eddy damping. Although this picture is oversimplified, it is a plausible starting point and should capture some of the physics of shear flow. Modifying the damping in the region of strong effects of F may overcome some of the limitations of the model.[16] In any case, a more complete investigation based on a full DIA analysis will be considerably more difficult.

With these assumptions, the model equation for shear turbulence is

$$\frac{D}{Dt} u_i(\mathbf{k}, t) + \int_0^t ds\ \eta(k, t, s)\ u_i(\mathbf{k}, s) = f_i(\mathbf{k}, t) + S_{ip}(\mathbf{k}, t) u_p(\mathbf{k}, t) \tag{3}$$

where

$$S_{im} = -A_{im} + 2k^{-2} k_i k_p A_{pm} + \delta_{im} k_s A_{sr} \partial/\partial k_r$$

$$A_{im} = \partial U_i / \partial x_m$$

Note that the viscous term of Eq. (2) has been ignored in Eq. (3). The shear terms result[17] from Reynolds averaging the Navier Stokes Equations with a mean velocity field $U_i = A_{ij} x_j$. Reynolds averaging also introduces the convective derivative in Eq. (3) which replaces the time derivative in Eq. (2). Since convection by the mean flow does not affect eddy damping or straining, this effect will be ignored in what follows. Closely related models have been proposed and investigated by Cambon et al[15] in the context of EDQNM especially for rotational effects. Since our goal is to derive single point models rather than to study spectral dynamics, Leslie's much simpler formulation seems adequate.

Eq. (3) can be rewritten in terms of a Green's tensor or response tensor

$$G_{ij}(\mathbf{k}, t, s) = G(k, t, s) P_{ij}(\mathbf{k})$$

where

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$$

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as

$$u_i(\mathbf{k}, t) = \int_0^t ds\ G(k, t, s)\ [f_i(\mathbf{k}, s) + P_{iq}(\mathbf{k})S_{qp}(\mathbf{k}, s)u_q(\mathbf{k}, s)] \tag{4}$$

where G satisfies the integrodifferential equation

$$\partial G(k, t, s)/\partial t + \int_s^t dr\ \eta(k, t, r)\ G(k, r, s) = 0 \tag{5}$$

and the conditions

$$G(k, s, s) = 1$$

$$G(k, t, s) = 0 \text{ for } t < s \tag{6}$$

In what follows, the condition Eq. (6) on the Green's function will always be understood. Moreover, in order to avoid additional notation, the symbol G will be used to denote different Green's functions dependent on different numbers of arguments. Eq. (3) will be applied to derive Reynolds stress models, however it should be possible to deduce K and ε transport equations from it as well.

B. Relation to RDT

Eq. (3) with both the random force and the eddy damping terms neglected is just RDT.[6] An approximate condition that RDT apply is therefore that

$$\int_0^t \eta(k, t, s)ds\ << \ (A_{pq}A_{pq})^{1/2} \tag{7}$$

Because η models eddy damping due to nonlinear interaction, this condition agrees with the usual idea that RDT applies when the shear dominates nonlinearity. At short times, the left side of Eq. (7) is $O(t)$; therefore, the short time response is always governed by RDT in this theory. At finite times, Eq. (7) states a *scale dependent* criterion for the applicability of RDT which will be discussed later.

In the RDT limit with $\eta(k, t, s) \equiv 0$, Eqs. (5),(6) become simply $G(k, t, s) \equiv 1$. Thus, a second criterion for the applicability of RDT is

$$|G(k, t, s) - 1| << 1 \tag{8}$$

In what follows, eddy damping will be described by G rather than by η, and Eq. (4) will be used as the basis of the theory.

C. Stationary Green's Functions

The time stationary case

$$G(k,t,s) = G(k,t-s) \tag{9}$$

should be characterized by universal inertial range forms with Kolmogorov similarity

$$G(k,t-s) = G(\xi), \ \xi = \varepsilon^{1/3}k^{2/3}(t-s) \tag{10}$$

No attempt to calculate the time dependent function $G(k,t-s)$ theoretically has yet succeeded. DIA itself is inconsistent[18] with the Kolmogorov scaling of Eq. (10); the Lagrangian modification[19] of DIA which restores Kolmogorov scaling does not give satisfactory predictions for the long time behavior of time correlations.[16] At this time, it is therefore necessary to postulate functional forms for $G(\xi)$. However, Smith and Yakhot observed[3] that the short and long time limits of G alone have important consequences. It follows from Eq. (5) that at short time separations,

$$G(k,t-s) = 1 + O((t-s)^2) \text{ for } (t-s) \sim 0 \tag{11}$$

and it is generally believed that at long time separations, the function $G(\xi)$ decays exponentially,

$$G(k,t-s) \sim \exp(-C_D\xi) \text{ for } (t-s) \sim \infty \tag{12}$$

with C_D a universal constant: in the Yakhot-Orszag theory,[20] $C_D = .49$. This limit corresponds to eddy damping. We will follow Smith and Yakhot[3] by leaving the functional form of $G(\xi)$ unspecified, but insisting on the limits in Eqs. (11),(12).

An important observation is that the long time limit, Eq. (12), does not satisfy the short time limit Eq. (11). In fact, reference to Eq. (5) shows that the long time limit is defined by the singular damping function

$$\eta(k,t,s) = C_D\varepsilon^{1/3}k^{2/3}\delta(t-s) \tag{13}$$

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which "Markovianizes" Eq. (3) as in Kraichnan's test field model.[12] It will be shown later that the LRR stress transport model also assumes Markovian eddy damping.

A consequence of the short and long time limits follows from the definition of ξ in Eq. (10): at any fixed nonzero $(t - s)$, ξ is small for large scales and large for small scales. Eqs. (8),(10) imply that *sufficiently large scales are governed by the short time limit, RDT, while sufficiently small scales are governed by the long time limit, inertial range eddy damping.* This observation suggests a "two-scale" theory of shear turbulence.

D. Nonstationary Green's Functions

For completely general conditions, the time dependence of $G(k, t, s)$ can only be found from DIA. In the present simplified theory, the time dependence must be postulated instead, recognizing that some form of universality is indispensable in turbulence modeling. This will restrict the applicability of the theory, but such restrictions are inevitable in any case: there are time dependent problems accessible to DIA, such as the relaxation of turbulence with strong k space anisotropy, or the generation of a Kolmogorov spectrum from an arbitrary initial spectrum, which cannot be usefully described at the single point level.

Let the inertial range be characterized by its time dependent dissipation rate $\varepsilon(t)$ and inverse integral scale $k_0(t)$. In the Yakhot-Orszag theory,[20] k_0 is defined so that

$$K(t) = \int_{k_0(t)}^{\infty} E(k, t) dk$$

where K denotes the turbulence kinetic energy; for the Kolmogorov spectrum written as

$$E = C_K \varepsilon^{2/3} k^{-5/3} \text{ for } k \geq k_0 \tag{14}$$

where C_K is the Kolmogorov constant, K and k_0 are related by

$$K(t) = \frac{3}{2} C_K \varepsilon(t)^{2/3} k_0(t)^{-2/3} \tag{15}$$

Either pair of functions $\varepsilon(t)$ and $k_0(t)$ or $\varepsilon(t)$ and $K(t)$ defines a time dependent inertial range: the second pair will be assumed to be known from the solution of a two equation model.

Define the frequencies

$$\theta(k,t) = \varepsilon^{1/3}(t)k^{2/3} \text{ for } k \geq k_0(t)$$

$$\Theta(t) = \varepsilon(t)/K(t)$$

A universal time dependent damping function can be postulated by assuming that a stationary Green's function $G(\xi)$ satisfying the limits Eqs. (11),(12) is known, and replacing the similarity variable $\xi = \theta(t-s)$ of Eq. (10) by the generalization $\int_s^t \theta \, dr$, so that

$$G(k,t,s) = G(\int_s^t \theta(k,r)dr) \tag{16}$$

This postulate has two consistency properties: it reduces to the stationary form when the inertial range is time independent, and it is exact for the singular case Eq. (13) in which $G(\xi) = e^{-C_D\xi}$.

A simpler formulation, closer in spirit to single point modeling, is to treat damping as scale independent by setting $G(k,t,s) = G(t,s)$ only. In this case, the damping is due to the action of the inertial range as a whole. It will be convenient to call this type of model a *global damping model*. The appropriate similarity variable in the stationary case is $(t-s)\Theta$. The short time limit is

$$G(t-s) = 1 + O((t-s)^2) \text{ for } (t-s) \sim 0 \tag{17}$$

and the long time limit is

$$G(t-s) \sim e^{-C_R(\varepsilon/K)(t-s)} \text{ for } (t-s) \sim \infty \tag{18}$$

where C_R is another universal constant: in the Yakhot-Orszag theory,[20] $C_R \sim 1.58$. Assuming that $G(t-s)$ satisfying Eqs. (17),(18) is known, we can define by analogy to Eq. (16),

$$G(t,s) = G(\int_s^t \Theta(r)dr) \tag{19}$$

Dropping the k dependence in Eq. (5) shows, by analogy to Eq. (13) that the long time limit corresponds to the singular damping function

$$\eta(t,s) = \Theta(s)\delta(t-s) \tag{20}$$

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Eq. (19) is exact in this case, for which $G(t-s) = \exp(-C_R(t-s)\Theta)$.

III. Time Dependent Eddy Viscosity

To derive the time dependent eddy viscosity, we follow Leslie[1] and expand Eq. (4) in powers of the mean strain rate about an isotropic background state $u^{(0)}$:

$$u = u^{(0)} + u^{(1)} + \cdots \tag{21}$$

We have assumed that the force f is independent of the mean strain rate; therefore, the effect of f is absorbed entirely in the background state, and $u^{(1)}$ is given by

$$u_i^{(1)}(\mathbf{k},t) = \int_0^t ds\ G(k,t,s)\ P_{im}(\mathbf{k})\ S_{mn}(\mathbf{k},s)\ u_n^{(0)}(\mathbf{k},s) \tag{22}$$

If, corresponding to Eq. (21) the single time correlation function is expanded as

$$Q = Q^{(0)} + Q^{(1)} + \cdots \tag{23}$$

where

$$Q_{ij}^{(1)}(\mathbf{k},t) = < u_i^{(1)}(\mathbf{k},t)\ u_j^{(0)}(-\mathbf{k},t) + u_i^{(0)}(\mathbf{k},t)u_j^{(1)}(-\mathbf{k},t) > /\delta(\mathbf{k})$$

and the higher order correlations are defined similarly, then Eq. (22) implies

$$
\begin{aligned}
Q_{im}^{(1)}(\mathbf{k},t) = &\int_0^t ds\ G(k,t,s)(-A_{ir} + 2k_i k_p k^{-2} A_{pr})P_{rm}Q^{(0)}(k,t,s) \\
&+ (im) + G(k,t,s)k_r\ A_{rn}\ \frac{\partial}{\partial k_n}\ Q_{im}^{(0)}(k,t,s) \\
&- Q_{im}^{(0)}(k,t,s)k_r\ A_{rn}\ \frac{\partial}{\partial k_n}G(k,t,s)
\end{aligned}
\tag{24}
$$

where (im) denotes index interchange in the immediately preceding term. In view of the isotropy of the background field, $Q_{ij}^{(0)}(\mathbf{k},t,s) = Q^{(0)}(k,t,s)P_{ij}(\mathbf{k})$. The occurence of two time correlation functions in the formula for the single time correlation function is characteristic of DIA. The time stationary form of this equation was stated by Leslie.[1] We follow Smith and Yakhot[3] and assume the nonstationary fluctuation dissipation relation

$$Q^{(0)}(k,t,s) = Q^{(0)}(k,s)\Big[G(k,t,s) + G(k,s,t)\Big]$$

A decomposition of the Reynolds stress follows from Eq. (23):

$$\tau_{ij} = \tau_{ij}^{(0)} + \tau_{ij}^{(1)} + \cdots$$

where

$$\tau_{ij}^{(n)}(t) = \int d\mathbf{k}\, Q_{ij}^{(n)}(\mathbf{k}, t) \tag{25}$$

Note that the sign convention established here is

$$\tau_{ij} = + <u_i u_j>$$

Evaluation of the angular integrals in Eqs. (24),(25) leads to

$$\tau_{ij}^{(1)}(t) = -\int_0^t ds \int_0^\infty dk \left\{ \frac{4}{15} G(k,t,s)^2 \right. \tag{26}$$
$$\left. -\frac{2}{15} G(k,t,s)k \frac{d}{dk} G(k,t,s) \right\} E(k,s)\, S_{ij}(s)$$

where $E(k,s)$ is the energy spectrum at time s and

$$S_{ij} = A_{ij} + A_{ji}$$

We adopt the viewpoint of the Yakhot-Orszag theory[20] and evaluate Eq. (26) over the inertial range $k \geq k_0$ only using the similarity form Eq. (16) for the Green's function and the Kolmogorov spectrum, Eq. (14). The result will have the general form

$$\tau_{ij}^{(1)} = \int_0^t ds\, \Gamma(t,s) S_{ij}(s)$$

in which the integral kernel Γ itself depends in a complicated manner on the evolution of the inertial range parameters K and ε for times between 0 and t. In the stationary case, the result can be rewritten as

$$\tau_{ij}^{(1)}(t) = -C_K \frac{K^2}{\varepsilon} \int_0^{\varepsilon t/K} dr\, S_{ij}(rK/\varepsilon) \times \tag{27}$$
$$r \int_{(\frac{3}{2}C_K r)^{3/2}}^\infty dk\, \kappa^{-5/3} \left\{ \frac{4}{15} G(\kappa^{2/3}) - \frac{4}{45} \kappa^{2/3} G(\kappa^{2/3}) G'(\kappa^{2/3}) \right\}$$

Recall that G here denotes the inertial range similarity form Eq. (10). Given a functional form for G, the second integral in Eq. (27) is a universal function of the dimensionless time variable r.

As Smith and Yakhot emphasize,[3] the universal short and long time limits for G imply universal short and long time limits for $\tau_{ij}(t)$. Namely, substituting the short time limit for G given in Eq. (11) in Eq. (26) leads to

$$\tau_{ij} = -\frac{4}{15} K t \, S_{ij} \qquad (28)$$

in agreement with Crow's[21] RDT calculation. To derive the long time limit, use the stationary form Eq. (10) for $G(k, t - s)$ and define constants C_1 and C_2 by

$$\int_0^\infty G(k, \sigma)^2 \, d\sigma = C_1/\theta$$

$$\int_0^\infty G(k, \sigma) \, k \, dG(k, \sigma)/dk \, d\sigma = C_2/\theta$$

Then assuming constant strain rate and substituting in Eq. (26), using Eq. (15) to eliminate k_0 in favor of K,

$$\tau_{ij} = -C_\nu \, \frac{K^2}{\varepsilon} \, S_{ij}$$

where

$$C_\nu = (\frac{4}{C_1} - \frac{2}{C_2})/45 C_K$$

There are some other useful special cases of Eq. (26). By setting $G \equiv 1$, we obtain RDT expanded to first order in the mean strain rate. The possibility of a "viscoelastic" representation of turbulence, as in Eqs. (26),(27) was suggested by Crow[21] and others. Crow's theory arises in the present formalism by assuming viscous damping

$$G(k, t, s) = e^{-\nu k^2(t - s)}$$

instead of eddy damping.

Global damping models greatly simplify these formulas while retaining the idea of eddy damping. The global damping analog of the general nonstationary model Eq. (26) is

$$\tau_{ij}^{(1)} = -\frac{4}{15} \int_0^t ds \, G(t, s)^2 \, K(s) \, S_{ij}(s) \qquad (29)$$

An important special case of this formula is obtained by substituting the exponential form for G of Eqs. (18),(19); although this form does not satisfy the short time constraint Eq.

(17), it will connect the present theory with stress transport models. In this case, Eq. (29) becomes

$$\tau_{ij}^{(1)}(t) \;=\; -\frac{4}{15}\int_0^t ds\,\Big[exp\int_s^t -2C_R\Theta(r)\,dr\Big]\,K(s)\,S_{ij}(s)$$

Equivalently,

$$\dot{\tau}_{ij}^{(1)} \;=\; -2C_R\frac{\varepsilon}{K}\tau_{ij}^{(1)} - \frac{4}{15}KS_{ij} \qquad (30)$$

or for constant strain in simple shear in which $A_{ij} = S\delta_{i1}\delta_{j2}$

$$\tau_{12} \;=\; -\frac{2}{15C_R}\,\frac{K^2}{\varepsilon}\Big(1 - exp(-2C_R\,\frac{\varepsilon}{K}t)\Big)S$$

In the Yakhot-Orszag theory[20], $2/15C_R = C_\nu \sim .08$ exactly equals the usual eddy viscosity constant.

Comparison with the LRR model[7] requires that both convection and diffusion be ignored: these are inhomogeneous effects extraneous to the present analysis. Writing τ_{ij}^D for the deviatoric, or anisotropic part of the stress, the simplified LRR model is

$$\begin{aligned}
\frac{\partial \tau_{ij}^D}{\partial t} =\; & -C_1\frac{\varepsilon}{K}\tau_{ij}^D - \frac{4}{15}KS_{ij} \\
& + C_2[\tau_{ip}^D\frac{\partial U_j}{\partial x_p} + \tau_{jp}^D\frac{\partial U_i}{\partial x_p} - \frac{2}{3}\delta_{ij}\tau_{qp}^D\frac{\partial U_q}{\partial x_p}] \\
& + C_3[\tau_{ip}^D\frac{\partial U_p}{\partial x_j} + \tau_{jp}^D\frac{\partial U_p}{\partial x_i} - \frac{2}{3}\delta_{ij}\tau_{qp}^D\frac{\partial U_p}{\partial x_q}]
\end{aligned} \qquad (31)$$

Solving this equation by a perturbation series in the strain rate analogous to Eq. (21),

$$\tau_{ij}^D \;=\; \tau_{ij}^{(1)} + \tau_{ij}^{(2)} + \cdots$$

we find that $\tau_{ij}^{(1)}$ satisfies Eq. (30) with $2C_R = C_1$. This establishes a simple connection between the present theory and LRR, namely that to lowest order in this perturbation theory, the LRR model is a global damping model defined by the Green's function

$$G(t,s) \;=\; \exp\int_s^t -C_1\Theta(r)\,dr \qquad (32)$$

which does not satisfy the short time limit Eq. (17). Instead, this choice of Green's function corresponds to the long time limit Eq. (18), to the singular eddy damping of Eq. (20), and to Markovian damping in Eq. (3).

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The LRR model satisfies the Crow constraint on short time behavior, Eq. (28). The short time expansion $G(t-s) \sim 1 + O(t-s)$ of the LRR Green's function Eq. (32) implies that corrections to the Crow constraint are of order t^2; Eq. (17) implies a correction of order t^3. Summarizing, the short time corrections to the Crow constraint are

$$\tau \sim t + O(t^2) \quad \text{LRR}$$
$$\tau \sim t + O(t^3) \quad \text{present} \tag{33}$$
$$\tau \sim t + O(t^5) \quad \text{RDT}^9$$

These distinctions are important in oscillating shear flow. Consider oscillating simple shear flow in which $S = \partial U_1 / \partial x_2$ is the only nonzero mean velocity component, and let $\tau = \tau_{12}$ be the shear stress. Suppose that S is oscillatory:

$$S = d\alpha/dt = ai\omega e^{i\omega t}$$

In rapidly oscillating flow, it is reasonable to ignore oscillations in K and ε and to replace them by their time averages. Then $\Theta = \varepsilon/K$ is constant. Damping is then described by the stationary Green's function $G((t-s)\Theta)$. Then τ is given by

$$\tau(t) = -\frac{4}{15} aiK\omega \int_0^t ds \, G(\Theta(t-s))^2 e^{i\omega s}$$

The steady state solution when ω is large satisfies[22]

$$\tau \sim -4aK/15 \left[A_0 + \frac{A_1}{i\omega} + \frac{A_2}{i^2\omega^2} + \cdots \right] e^{i\omega t}$$

where

$$A_0 = G(0) = 1$$
$$A_1 = 2G(0)G'(0)\Theta$$
$$A_2 = [2G(0)G''(0) + 2G'(0)^2] \Theta^2$$

indicating as expected that in the limit $\omega \to \infty$, the response is elastic and is governed by RDT. In the LRR model, $G'(0) \neq 0$; therefore the corrections to RDT are of order ω^{-1}. But if G satisfies Eq. (17), then $A_1 = 0$ and the correction to the RDT solution depends on ω^{-2} instead of on ω^{-1}. Moreover, the phase lag between stress and strain is of order ω^{-3}. This may explain the success of RDT in solving high frequency oscillating

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flows.[8] It appears that the LRR model will overpredict the effects of eddy damping at high frequencies.

The corrections to RDT at short times summarized by Eq. (33) are also important in transient homogeneous shear flow. Eq. (33) indicates that the short time corrections in both LRR and the present theory are associated with the initiation of eddy damping. It is known[23] that at high strain rates, the LRR model predicts a much too rapid growth of K. Energy growth is due to the onset of turbulence production by eddy damping. In view of Eq. (33), eddy damping corrections are of order t^2 in LRR, but are of order t^3 in the present theory. This strong short time suppression of eddy damping suggests that RDT will apply in the present theory for longer times than it does in the LRR model. This may improve the agreement with DNS studies[24] of highly strained turbulence.

IV. Second Order Analysis

Calculation of Leslie's expansion to second order is lengthy but routine. The result has the form

$$
\begin{aligned}
Q_{ij}^{(2)}(\mathbf{k},t) = \sum_{1\leq N\leq 6} I^{(N)}[& a^{(N)}A_{ip}(s)A_{jp}(r) + b^{(N)}A_{ip}(s)A_{pj}(r) \\
& + c^{(N)}A_{pi}(s)A_{jp}(r) + d^{(N)}A_{pi}(s)A_{pj}(r) \\
& + e^{(N)}\delta_{ij}A_{pq}(s)A_{qp}(r) + f^{(N)}\delta_{ij}A_{pq}(s)A_{pq}(r)] + (ij)
\end{aligned}
\tag{33}
$$

in which the $I^{(N)}$ are integral operators

$$
\begin{aligned}
I^{(1)} &= \int_0^t ds\, G(k,t,s) \int_0^s dr\, G(k,s,r)G(k,t,r)Q^{(0)}(k,r) \\
I^{(2)} &= \int_0^t ds\, G(k,t,s) \int_0^s dr\, G(k,s,r)k\frac{d}{dk}[G(k,t,r)Q^{(0)}(k,r)] \\
I^{(3)} &= \int_0^t ds\, G(k,t,s) \int_0^s dr\, G(k,s,r)k^2\frac{d^2}{dk^2}[G(k,t,r)Q^{(0)}(k,r)] \\
I^{(4)} &= \int_0^t ds\, G(k,t,s) \int_0^t dr\, G(k,t,r)G(k,\mid s,r \mid)Q^{(0)}(k,r) \\
I^{(5)} &= \int_0^t ds\, G(k,t,s) \int_0^t dr\, G(k,t,r)k\frac{d}{dk}[G(k,\mid s,r \mid)Q^{(0)}(k,r)] \\
I^{(6)} &= \int_0^t ds\, G(k,t,s) \int_0^t dr\, G(k,t,r)k^2\frac{d^2}{dk^2}[G(k,\mid s,r \mid)Q^{(0)}(k,r)]
\end{aligned}
\tag{34}
$$

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$a^{(N)},...f^{(N)}$ are the following geometric constants:

	(1)	(2)	(3)	(4)	(5)	(6)
$105a^{(N)}$	27	-1	-2	$\frac{19}{2}$	$-\frac{1}{2}$	-1
$105b^{(N)}$	20	6	-2	6	3	-1
$105c^{(N)}$	-15	-15	-2	$\frac{5}{2}$	$-\frac{15}{2}$	-1
$105d^{(N)}$	20	-8	-2	-22	-4	-1
$105e^{(N)}$	10	24	6	3	12	3
$105f^{(N)}$	-4	24	6	10	12	3

and

$$G(k, \mid s, r \mid) = G(k, s, r) + G(k, r, s)$$

As in all derivations of this type[4,5,9], these constants arise from integrating even order products $k_i k_j, ...$ over spheres $k = $ constant. A useful constraint on the calculation is that it must reduce to the rapid distortion results of Maxey[9] when $G \equiv 1$. Single point models are derived exactly as in Sect. II and will not be described explicitly.

In a global damping model, the integrals in Eq. (34) satisfy the relations

$$I^{(2)} = -3I^{(1)}$$

$$I^{(3)} = 12I^{(1)}$$

$$I^{(5)} = -3I^{(4)}$$

$$I^{(6)} = 12I^{(4)}$$

and Eq. (33) reduces, after performing the wavenumber integrals, to

$$\tau_{ij}^{(2)}(t) = \int_0^t ds\, G(t,s) \int_0^s dr\, G(s,r)G(t,r)K(r) \times$$
$$\{ + \frac{4}{105}[S_{ip}(s)A_{jp}(r) + S_{jp}(s)A_{ip}(r)] \tag{35}$$
$$- \frac{24}{105}[S_{ip}(s)A_{pj}(r) + S_{jp}(s)A_{pi}(r)]$$
$$+ \frac{32}{105}\delta_{ij}[A_{pq}(s)A_{qp}(r) + A_{pq}(s)A_{pq}(r)]\}$$

To compare this calculation with LRR, evaluate the second order solution of Eq. (31):

$$\tau_{ij}^{(2)}(t) = \int_0^t ds\, G(t,s) \int_0^s dr\, G(s,r)K(r) \times$$
$$+ C_2[S_{ip}(s)A_{jp}(r) + S_{jp}(s)A_{ip}(r)]$$
$$+ C_3[S_{ip}(s)A_{pj}(r) + S_{jp}(s)A_{pi}(r)]$$

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where G is the Green's function for the LRR model, Eq. (32). The occurence of an additional Green's function in Eq. (35), which arises from the two-time correlation, means that at second order, the present theory does not reduce to LRR. However, as in the discussion of shear stress, the important difference between the theories is the correction to short time behavior: in both the LRR model and the global damping model, the normal stresses are of order t^2 at short times. In the LRR model, the corrections are of order t^3, whereas in a global damping model in which G satisfies the short time limit Eq. (17), the corrections are of order t^4. Thus, the present theory resembles RDT longer in transient homogeneous shear flow and at lower frequencies in oscillating shear flow than the LRR model.

V. Models Valid for Large Strain Rates

The perturbative derivation of these models limits their applicability to moderately strained flows. A theory valid at arbitrary strains can be derived by summing Leslie's expansion[1] to all orders. The summation is simplest in the time stationary case which will be considered first. In an obvious operator notation, Eq. (22) can be written as

$$u^{(1)} = \mathbf{GS}u^{(0)}$$

where $\mathbf{G}$ denotes time convolution by the (stationary) Green's function,

$$(\mathbf{G}a)(t) = (G * a)(t) = \int_0^t G(t - s)a(s) \, ds$$

(only the time arguments have been shown explicitly), and $\mathbf{S}$ denotes the action of the strain dependent terms. In view of Eq. (3),

$$u^{(n)} = \mathbf{GS}u^{(n-1)} = (\mathbf{GS})^n u^{(0)}$$

where operator products are understood; therefore,

$$u = [\mathbf{I} + \mathbf{GS} + (\mathbf{GS})^2 + \cdots]u^{(0)} \tag{36}$$

and the problem is to find a useful representation for the Neumann series on the right side. When $G \equiv 1$, $\mathbf{G}$ is simply time integration, which can be written

$$(\mathbf{G}a)(t) = (H * a)(t) = \int_0^t a(s) \, ds$$

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where H denotes the usual unit step function. In this case, the sum in Eq. (36) defines RDT, for which the sum can be given explicitly for some important special mean velocity gradients A_{ij}.

Consider a global damping model $G(k, t, s) = G(t, s)$, so that in each term of the series in Eq. (36), the k-derivative in **S** does not act on G. RDT can be written as the special case of Eq. (4),

$$u_i(\mathbf{k}, t) = \int_0^t ds \ H(t - s) \ P_{iq}(\mathbf{k}) \mathbf{S}_{qp}(\mathbf{k}, s) u_p(\mathbf{k}, s) \tag{37}$$

Following Ref. 9, write the solution of RDT for simple shear

$$A_{im}(t) = S(t) \delta_{i1} \delta_{m2}$$

as

$$u_i(\mathbf{k}, t) = M_{ip}(\mathbf{m}(\mathbf{k}, \alpha(t)), \alpha(t)) u_p(\mathbf{k}, 0) \tag{38}$$

where $\alpha(t)$ is the total strain

$$\alpha(t) = \int_0^t S(s) \ ds = (H * S)(t) \tag{39}$$

and **m** is defined in Ref. 9. Define the modified velocity gradient A^*_{im} by

$$A^*_{im} = X * A_{im} \tag{40}$$

where componentwise convolution is understood and the function X is chosen so that

$$H * X = G \tag{41}$$

therefore

$$H * A^*_{im} = G * A_{im} \tag{42}$$

Comparing Eqs. (37), (40), and (42), it is evident that the solution of Eq. (37) with the modified velocity gradient A^*_{im} is the solution of Eq. (4) with an arbitrary Green's function G.

The solution of Eq. (41) is

$$X = G' + \delta \tag{43}$$

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where G' is the derivative of the stationary Green's function $G(t-s)$ with respect to its argument. For simple shear, Eq. (40) reduces to

$$S^* = X * S \tag{44}$$

Therefore the solution of Eq. (4) for simple shear is the modification of Eq. (38),

$$u_i(\mathbf{k}, t) = M_{ip}(\mathbf{m}(\mathbf{k}, \alpha^*(t)), \alpha^*(t))u_p(\mathbf{k}, 0) \tag{45}$$

where, in view of Eqs. (41),(44), α^* is defined by

$$\alpha^* = H * S^* = G * S$$

Explicitly,

$$\alpha^*(t) = \int_0^t G((t-s)\Theta)S(s) \, ds \tag{46}$$

Eqs. (45) and (46) solve the problem of summing Leslie's perturbation theory in the special case of simple shear. α^* may be called the *modified total strain.*

The introduction of a phenomenological modified total strain has often been suggested in the RDT literature[6,8,9] as a way to improve the agreement between RDT and data from flows which are not evidently rapidly distorted. The short and long time properties of Eq. (46) are interesting from this viewpoint. At short times, $G \sim 1$ and Eq. (46) becomes

$$\alpha^* \sim \alpha \sim S(0)t \text{ for } t \sim 0 \tag{47}$$

indicating that the short time limit of this theory is RDT. At long times and constant strain rate S,

$$\alpha^* \sim \int_0^\infty e^{-C_R\Theta s}S \, ds = \frac{1}{C_R}\frac{SK}{\varepsilon} \text{ for } t \sim \infty \tag{48}$$

In shear flows nearly in a production equals dissipation steady state, $SK/\varepsilon \sim 3.0$; for flows which evolve to this state, the modified total strain saturates after growing linearly for short times. In the RDT solution for constant strain rate, the total strain grows linearly for all times. The phenomenological modifications of RDT suppress this growth by forcing saturation of the modified total strain. Here, this saturation is a consequence of Eq. (46).

Some obvious generalizations of this theory are to nonstationary problems and to general wavenumber dependent damping models. Arguing by exact analogy to the stationary

case, to develop a nonstationary theory, it would be necessary to interpret the operator **G** as a Volterra operator

$$(\mathbf{G}a)(t) = \int_0^t G(t,s)a(s) \ ds$$

and to define the modified velocity gradient as the solution of a Volterra integral equation. In view of the phenomenological character of the nonstationary Green's function, it seems preferable just to generalize Eq. (46) directly as

$$\alpha^*(t) = \int_0^t G(t,s)S(s) \ ds$$

When the stationary Green's function is k-dependent, the summation fails because the **k** derivatives in the strain operator **S** also act on $G(k, t-s)$. The direct generalization of Eq. (46) for simple shear flow,

$$\alpha^*(k,t) = \int_0^t G(k, t-s)S(s) \ ds \tag{49}$$

therefore does not lead to an exact summation of Eq. (36). But inserting the wavenumber dependent modified total strain of Eq. (49) in Eq. (45),

$$u_i(\mathbf{k},t) = M_{ip}(\mathbf{m}, \mathbf{k}, \alpha^*(k,t)), \alpha^*(k,t))u_p(\mathbf{k},0) \tag{50}$$

does at least sum terms of all orders in the strain rate in Eq. (36); it is therefore a plausible approximate summation of this series. Eq. (50) brings about a generalization of the short and long time limits of Eqs. (47) and (48): since $G(k, t-s) \sim 1$ for sufficiently large scales, whatever the value of $(t-s)$, the RDT solution for large strain rates can apply *even at finite times* to sufficiently large scales. This idea is also stated in a similar context by Cambon et al.[15] However, for sufficiently small scales, G rapidly assumes its asymptotic eddy damping form. Eq. (50) states a "two-scale" theory of shear turbulence in which large scales can be governed by RDT while small scales exhibit the eddy damping characteristic of a steady state Kolmogorov inertial range.

V. Approximate Theory of Highly Strained Flows

There have been recent proposals[11] to modify the standard eddy viscosity

$$\nu_T = C_\nu K^2/\varepsilon, \ C_\nu \sim .09 \tag{51}$$

to improve its behavior for large strain rates. These proposals all introduce functions $C_\nu(\eta)$ in Eq. (51), where in simple shear flow with $A_{ij} = S\delta_{i1}\delta_{j2}$, η is the dimensionless strain rate $\eta = SK/\varepsilon$. This idea can be attributed to Yakhot et al,[10] who replace the constant $C_{\epsilon 1}$ of the two equation model by a function $C_{\epsilon 1}(\eta)$. Applied to the nonlinear eddy viscosity model of Yoshizawa,[4] this approach suggests

$$
\begin{aligned}
\tau_{ij} = {} & \frac{2}{3}K\delta_{ij} - C_\nu(\eta)\frac{K^2}{\varepsilon}S_{ij} \\
& + C_{\tau 1}(\eta)\frac{K^3}{\varepsilon^2}[A_{ip}A_{jp} - \frac{1}{3}\delta_{ij}A_{qp}A_{qp}] \\
& + C_{\tau 2}(\eta)\frac{K^3}{\varepsilon^2}[A_{ip}A_{pj} + A_{jp}A_{pi} - \frac{2}{3}\delta_{ij}A_{qp}A_{pq}] \\
& + C_{\tau 3}(\eta)\frac{K^3}{\varepsilon^2}[A_{pi}A_{pj} - \frac{1}{3}\delta_{ij}A_{qp}A_{qp}]
\end{aligned}
\tag{52}
$$

This equation, with constant $C_\nu, C_{\tau 1}, C_{\tau 2}$, and $C_{\tau 3}$ cannot be applied to flow regions, like the near wall, in which η is large: it predicts that the stress ratios τ_{ij}/K all increase with η, completely contrary to the data, and even predicts that some of the normal stresses become negative. These problems can be overcome if the functions of η in Eq. (52) are chosen suitably. An early, and especially interesting model of this form was proposed by Pope[25] who solved the "algebraic" form of the LRR model due to Rodi[26] for two dimensional mean flow.

A model of this type can be derived for simple shear flow under the additional hypothesis that Eq. (48) defines a long time limit for any value of η. It must be stressed that this is an additional assumption; it is not a consequence of this theory. Analytically, this assumption states that α^* in Eq. (45) can be replaced by its formal long time limit, η/C_R from Eq. (48), so that

$$
u_i(\mathbf{k}, t) = M_{ip}(\mathbf{m}(\mathbf{k}, \eta/C_R), \eta/C_R)u_p(\mathbf{k}, 0)
\tag{53}
$$

Then

$$
\tau = -C_\nu(\eta)\frac{K^2}{\varepsilon}S
\tag{54}
$$

where $\tau = \tau_{12}$ and the function $C_\nu(\eta)$ is found from RDT as follows. By forming moments and integrating over wavenumbers, Maxey[9] presents τ/K graphically as a function of $\alpha = St$, say

$$
\tau/K = F(\alpha)
$$

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Replace α by η/C_R as in Eq. (53); the result is

$$\tau = KF(\eta/C_R) = -C_\nu(\eta)\,\frac{K^2}{\varepsilon}S$$

where

$$C_\nu(\eta) = -\frac{F(\eta/C_R)}{\eta} \qquad (55)$$

and the function F is known from RDT. Note[9] that since $F \to 0$ when $\eta \to \infty$

$$\eta C_\nu(\eta) \to 0, \qquad \eta \to \infty$$

in agreement with the observation that strong shear suppresses the shear stress so that $\tau/K \to 0$ when η is very large. Strain dependent coefficients $C_{\tau 1}(\eta), C_{\tau 3}(\eta)$ for Eq. (52) are also easily obtained from Maxey's RDT results.

This theory helps explain a curious feature of homogeneous shear flow data: the ratio τ/K is about the same both in fully developed homogeneous shear flow in which $\eta \sim 4.4$ and in simple shear flows in energy equilibrium in which $\eta \sim 3.0$. Therefore, if the usual formula Eq. (51) is written as

$$\tau/K = -C_\nu \eta$$

and is calibrated for equilibrium shear flows, it will predict a viscosity which is too large in homogeneous shear flow. The data is summarized in Table I and compared with the predictions of Eqs. (54)-(55). The theory predicts that τ/K has approximately the same value in both flows because in both flows η/C_R is near the maximum, at about $\alpha = 2$, of $F(\alpha)$ according to Maxey's RDT calculation.[9]

Table II compares τ/K in near wall channel flow with the theoretical predictions. Making this comparison invokes the suggestion of Lee et al[24] that near wall turbulent states are similar to highly strained homogeneous shear flow. As in the homogeneous shear flow comparison, there is qualitative agreement with the trends, but the theory underpredicts the reduction in τ/K as η increases. In both Tables, results for the theory with transport corrections are also listed. These arise as follows: transport effects (convection by the mean flow and turbulent diffusion) are inhomogeneous effects which have been ignored in this theory. Rodi's[26] algebraic transport correction is useful because it is model independent; a rough way to incorporate it in this model is to reduce the viscosity by the factor $C_1/(C_1 +$

$P/\varepsilon - 1$) where C_1 is the constant of the LRR model Eq. (31) and P is production. We have set $C_1 = 1.6$ in applying this correction. It improves the agreement with the data in some cases and greatly overcorrects in others; however, the comparisons are not meaningful unless some correction for transport is made.

A referee has suggested comparison with Pope's explicit solution[25] of the algebraic LRR model. Further comparisons with this model are given in Sect. VI. The comparison shows that the algebraic LRR model also reduces the ratio τ/K as η increases. The constants of Ref. 25 have been used to make the comparisons. The quantitative agreement could be improved by adjustments of these constants.

The quantitative limitations of the theory can be attributed to its basic assumption that the large η states can be considered long time limits. It is more likely that highly strained states are transient; although near wall flow can be steady in time, this steady state is maintained by the continual diffusion of highly strained turbulence away from the near wall production region into to bulk of the flow. From this (Lagrangian) viewpoint, turbulence is highly strained only for a finite time.[16] The simplicity of models like Eq. (52) makes them attractive, but they all assume the steady state character of highly strained states. Therefore, they should not be accepted uncritically, although their value in "regularizing" the stresses in large η flow regions will make them useful in calculations.

In Sects. IV and V, simple shear flow has been emphasized because an explicit RDT solution exists for this flow. In principle, the constructions of these sections can be generalized to any mean velocity gradient, but they will be explicit only when a corresponding RDT solution is known. Thus, a theory of this type applicable to near wall calculations in square duct flow will require solving RDT for a mean velocity gradient with the structure

$$\begin{bmatrix} \partial U_1/\partial x_1 & \partial U_1/\partial x_2 & 0 \\ \partial U_2/\partial x_1 & \partial U_2/\partial x_2 & 0 \\ \partial U_3/\partial x_1 & \partial U_3/\partial x_2 & 0 \end{bmatrix} \quad \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} = 0$$

It should not be assumed that superposing results for simple shear flows will provide a good approximation, or that the results will depend on a single parameter like η. The development of a more general theory, in which a rotational analog of the strain parameter η enters[23] is an interesting possibility.

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VI. Normal stresses and the one-component limit

The predictions of these theories for homogeneous shear flow in the limit of very large η are compared in Table III. It has been shown[24] that RDT predicts the initial evolution of these flows very well. The tensor b_{ij} is defined as usual by $b_{ij} = -\tau_{ij}/2K + 1/3\delta_{ij}$. The line NLEV refers to the nonlinear eddy viscosity model, Eq. (52) with constant coefficients $C_\nu, C_{\tau 1}, C_{\tau 2}, C_{\tau 3}$. The unphysical results are shown to motivate the introduction of η-dependent coefficients; all derivations of this model [4,5] have computed perturbatively assuming that the strain rate is small. The results for the explicit algebraic LRR model of Pope are shown using the notation of Ref. 25; b_2 and b_3 depend on the choice of model constants. This model predicts a qualitatively correct shear stress ratio b_{12}, however the normal stress ratios are model dependent. By construction, Eq. (53) recovers the RDT normal stresses, the one component limit.[24] It is more important that the present time dependent theories, Eqs. (45) and (50) also predict this limit at short times. This fact shows once more the importance of the short time corrections to RDT: even if the LRR model could be calibrated for agreement with the one component state in the limit of large η, eddy damping would quickly drive the solution away from this state; the strong suppression of eddy damping at short times in the present theory will maintain the RDT limit longer.

The difference between the present theory and LRR can be understood by comparing Eqs. (3) and (31): although the production term containing A_{im} is treated identically, the "rapid pressure strain" term $k_i k_p A_{pm} k^{-2}$ is treated exactly in the present theory by RDT but is modeled in LRR by the terms bilinear in stress and strain rate. Renormalization group analysis[27] showed that the LRR model is a rational lowest order approximation for the rapid pressure strain term; however, it is not valid for very large strain rates.

VII. Conclusions

The present theory should be compared to the LRR stress transport model and to RDT. Both LRR and RDT arise from a special choice of the Green's function, corresponding respectively to the long time and short time limits of the present theory. The present theory reduces to RDT at short times and at large strain rates, and to an eddy damping model qualitatively similar to LRR at long times.

TABLE I

Shear Stress Ratios in Homogeneous Shear Flow and in Equilibrium Simple Shear Flows

	$-\tau/K$			
	$\eta \sim 3.0$	$\eta \sim 4.4$		
Experiments	.33	.33	&&	
Theory	.32	.32		&
AlgebraicLRR	.37	.38	&&&	
Theory with transport corrections	.32	.23	&	

TABLE II

Stress Ratios in Near-Wall Turbulence

η	$-\tau/K(DNS)$[26]	$-\tau/K$(Theory)	AlgebraicLRR	$-\tau/K$(Corrected Theory)
6.6	.178	.274	.380	.246
8.4	.159	.237	.361	.195
9.6	.149	.216	.347	.170
10.9	.138	.183	.332	.139
12.5	.127	.161	.314	.117
14.0	.116	.146	.295	.105

TABLE III

Theoretical predictions in the large η limit

	b_{12}	b_{11}	b_{22}	b_{33}
NLEV	$O(\eta^2)$	$O(\eta^3)$	$O(\eta^3)$	$O(\eta^3)$
LRR	$O(\eta^{-1})$	$\frac{4}{15}\frac{3b_3-b_2}{3b_3^2-b_2^2}$	$-\frac{4}{15}\frac{3b_3+b_2}{3b_3^2-b_2^2}$	$\frac{8}{45}\frac{3b_2}{3b_3^2-b_2^2}$
Eq.(53)	$O(\eta^{-1})$	$2/3$	$-1/3$	$-1/3$
RDT	$O(\eta^{-1})$	$2/3$	$-1/3$	$-1/3$

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By simplifying the direct interaction approximation (DIA) for turbulent shear flow, time dependent formulas are derived for the Reynolds stresses which can be included in two equation models. The Green's function is treated phenomenologically, however following Smith and Yakhot, we insist on the short and long time limits required by DIA. For small strain rates, perturbative evaluation of the correlation function yields a time dependent theory which includes normal stress effects in simple shear flows. From this standpoint, the phenomenological Launder-Reece-Rodi model is obtained by replacing the Green's function by its long time limit. Eddy damping corrections to short time behavior initiate too quickly in this model; in contrast, the present theory exhibits strong suppression of eddy damping at short times. A time dependent theory for large strain rates is proposed in which large scales are governed by rapid distortion theory while small scales are governed by Kolmogorov inertial range dynamics. At short times and large strain rates, the theory closely matches rapid distortion theory, but at long times it relaxes to an eddy damping model.

14. SUBJECT TERMS

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