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WAVELET MULTIRESOLUTION ANALYSES ADAPTED FOR THE FAST SOLUTION OF BOUNDARY VALUE ORDINARY DIFFERENTIAL EQUATIONS *

Björn Jawerth
University of South Carolina
Columbia, SC

Wim Sweldens
Katholieke Universteit Leuven, Belgium
University of South Carolina

SUMMARY

We present ideas on how to use wavelets in the solution of boundary value ordinary differential equations. Rather than using classical wavelets, we adapt their construction so that they become (bi)orthogonal with respect to the inner product defined by the operator. The stiffness matrix in a Galerkin method then becomes diagonal and can thus be trivially inverted. We show how one can construct an $\mathcal{O}(N)$ algorithm for various constant and variable coefficient operators.

INTRODUCTION

The purpose of this paper is to use wavelets in the solution of certain linear ordinary differential equations of the form

$$L\,u(x) \;=\; f(x) \quad \text{for} \quad x \in [0,1], \quad \text{where} \quad L \;=\; \sum_{j=0}^{m} a_j(x)\, D^j,$$

and with appropriate boundary conditions on $u(x)$ for $x = 0, 1$.

Currently there exist two major solution techniques. First, if the coefficients $a_j(x)$ of the operator are constants, then the Fourier transform is well suited for solving these equations. The underlying reason is that the complex exponentials are eigenfunctions of a constant coefficient operator and they form an orthogonal system. As a result the operator becomes diagonal in the

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Fourier basis and can thus trivially be inverted. The numerical algorithm then boils down to calculating the discrete Fourier transform of the right hand side, dividing each coefficient by its corresponding entry in a diagonal matrix and finally taking the inverse Fourier transform to obtain the solution. This can be done quickly using the fast Fourier transform which has a complexity of $N \log N$, where N is the number of unknowns in the discretization.

If the coefficients are not constant one typically uses finite element or finite difference methods to discretize the problem. We focus here on finite element methods. Define the *operator inner product* associated with an operator L by

$$\langle\langle u, v \rangle\rangle \;=\; \langle Lu, v \rangle .$$

A weak solution u can be found with a Petrov-Galerkin method, i.e. consider two spaces $\mathcal{S}$ and $\mathcal{S}^*$ and look for a solution $u \in \mathcal{S}$ such that

$$\langle\langle u, v \rangle\rangle \;=\; \langle f, v \rangle ,$$

for all v in $\mathcal{S}^*$. If $\mathcal{S}$ and $\mathcal{S}^*$ are finite dimensional spaces with the same dimension, this leads to a linear system of equations. The matrix of this system, also referred to as the *stiffness matrix*, has as elements the operator inner products of the basis functions of $\mathcal{S}$ and $\mathcal{S}^*$.

Traditionally one uses very local finite elements such that the stiffness matrix has a banded structure. The linear system can then be solved efficiently with an iterative method. These classical finite elements however have the disadvantage that the stiffness matrix becomes ill conditioned as the problem size grows. This slows down the convergence speed of the iterative algorithm dramatically. It is well understood by now that this can be solved with multiresolution techniques such as multigrid or hierarchical basis functions [1, 2]. Multiresolution finite element bases can provide preconditioners which result in a uniformly bounded condition number, see e.g. [3, 4, 5]. The convergence of the linear system is then independent of the problem size.

The research presented here is motived by the question of how good wavelets are for the solution of ordinary differential equations. We know that there are basically four main properties of wavelets; namely, they provide a multiresolution basis for a wide variety of function spaces, they are local in both space and frequency, they satisfy (bi)orthogonality conditions and fast transform algorithms are available. Because of these properties, wavelets have already proven to be a valuable substitute for the Fourier transform in many applications.

One possible idea, as proposed by several researchers, is to use wavelets as basis functions in a Galerkin method. This has proven to work and results in a linear system that is sparse because of the compact support of the wavelets, and that, after preconditioning, has a condition number independent of problem size because of the multiresolution structure. However, in this setting the wavelets do not provide significantly better results than more general multiresolution techniques (cfr. supra) and in fact one of their major properties, namely their (bi)orthogonality, is not exploited at all.

Three questions are addressed in this research. The first, how can one make use of the (bi)orthogonality property of the wavelets? The second, which operators can be diagonalized by wavelets? The last, are fast algorithms available and what is their complexity?

PRELIMINARIES

Notation and definitions

Much of the notation will be presented as we go along. Here we just note that the inner product of two square integrable functions $f, g \in L^2(IR)$ is defined by

$$\langle f, g \rangle \;=\; \int_{-\infty}^{+\infty} f(x)\,\overline{g(x)}\,dx,$$

and that the Fourier transform of a function f is defined as

$$\hat{f}(\omega) \;=\; \int_{-\infty}^{+\infty} f(x)\,e^{-i\omega x}\,dx.$$

We say that a function w is an *L-spline* if

$$L^*Lw \;=\; 0 \quad \text{and} \quad w \in C^{2m-2},$$

where L^* is the adjoint of L, a linear differential operator of order m. This definition leads to the classical piecewise polynomial splines in case $L = D^m$.

Multiresolution analysis

We give a brief review of wavelets and multiresolution analysis. For more information one can consult [6, 7, 8, 9]. A *multiresolution analysis* of $L^2(IR)$ is defined as a set of closed subspaces V_j, with $j \in \mathbb{Z}$, that exhibit the following properties:

1. $V_j \subset V_{j+1}$,
2. $v(x) \in V_j \Leftrightarrow v(2x) \in V_{j+1}$ and $v(x) \in V_0 \Leftrightarrow v(x+1) \in V_0$,
3. $\bigcup_{j=-\infty}^{+\infty} V_j$ is dense in $L^2(IR)$ and $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$,
4. A *scaling function* $\phi(x) \in V_0$ exists such that the set of functions $\{\phi_{j,l}(x) \mid l \in \mathbb{Z}\}$, with $\phi_{j,l}(x) \;=\; \sqrt{2^j}\,\phi(2^j x - l)$, is a Riesz basis of V_j.

As a result there is a sequence $\{h_k \mid k \in \mathbb{Z}\}$ such that the scaling function satisfies a *refinement equation*

$$\phi(x) \;=\; 2\sum_k h_k\,\phi(2x-k). \tag{1}$$

Define W_j now as a complementary space of V_j in V_{j+1}, such that $V_{j+1} = V_j \oplus W_j$ ($\oplus$ stands for direct sum) and, consequently,

$$\bigoplus_j W_j = L^2(IR).$$

Note that this definition of W_j as a complementary space is non unique.

A function $\psi(x)$ is a *wavelet* if the set of functions $\{\psi(x-l) \mid l \in ZZ\}$ is a Riesz basis of W_0. The set of wavelet functions $\{\psi_{j,l}(x) \mid l,j \in ZZ\}$ is then a Riesz basis of $L^2(IR)$. Since the wavelet is an element of V_1, it too satisfies a refinement relation,

$$\psi(x) = 2\sum_k g_k \phi(2x-k). \tag{2}$$

There are dual functions $\tilde{\phi}_{j,l}(x) = \sqrt{2^j}\tilde{\phi}(2^j x - l)$ and $\tilde{\psi}_{j,l}(x) = \sqrt{2^j}\tilde{\psi}(2^j x - l)$ that exist so that the projection operators P_j and Q_j onto V_j and W_j, respectively, are given by

$$P_j f(x) = \sum_l \langle f, \tilde{\phi}_{j,l} \rangle \phi_{j,l}(x) \quad \text{and} \quad Q_j f(x) = \sum_l \langle f, \tilde{\psi}_{j,l} \rangle \psi_{j,l}(x).$$

The basis functions and dual functions are biorthogonal,

$$\langle \phi_{j,l}, \tilde{\phi}_{j,l} \rangle = \delta_{l-l'} \quad \text{and} \quad \langle \psi_{j,l}, \tilde{\psi}_{j',l'} \rangle = \delta_{j-j'}\delta_{l-l'}. \tag{3}$$

If the basis functions are orthogonal, they coincide with the dual functions and the projections are orthogonal.

The dual scaling function and wavelet satisfy

$$\tilde{\phi}(x) = 2\sum_k \tilde{h}_k \tilde{\phi}(2x-k), \qquad \tilde{\psi}(x) = 2\sum_k \tilde{g}_k \tilde{\phi}(2x-k), \tag{4}$$

and

$$\tilde{\phi}(2x-k) = \sum_l h_{k-2l}\, \tilde{\phi}(x-l) + \sum_l g_{k-2l}\, \tilde{\psi}(x-l). \tag{5}$$

Taking the Fourier transform of the refinement equations (1) and (2) yields

$$\hat{\phi}(\omega) = h(\omega/2)\,\hat{\phi}(\omega/2) \quad \text{with} \quad h(\omega) = \sum_k h_k\, e^{-ik\omega}$$

and

$$\hat{\psi}(\omega) = g(\omega/2)\,\hat{\psi}(\omega/2), \quad \text{with} \quad g(\omega) = \sum_k g_k\, e^{-ik\omega}.$$

Here $h(\omega)$ and $g(\omega)$ are 2π-periodic functions that correspond to discrete filters. Similar definitions and equations hold for the dual functions. A necessary condition for biorthogonality is then

$$\forall \omega \in IR : \; \tilde{m}(\omega)\overline{m^t(\omega)} = 1,$$

where

$$m(\omega) = \begin{bmatrix} h(\omega) & h(\omega+\pi) \\ g(\omega) & g(\omega+\pi) \end{bmatrix}$$

and similarly for $\tilde{m}(\omega)$. The existence of the dual filters is guaranteed by the following lemma:

Lemma 1 *The space generated by the set of functions* $\{\psi_{j,l} \mid l \in \mathbb{Z}\}$ *complements* V_j *in* V_{j+1} *if and only if* $\delta(\omega) = det\, m(\omega)$ *does not vanish.*

The following statements are now equivalent :

- The dual wavelet has M vanishing moments.
- Any polynomial with degree less than M can be written as a linear combination of the functions $\phi_{j,l}(x)$ with $l \in \mathbb{Z}$.
- If $f \in C^M$, then the error of the approximation $\|f - P_j f\|$ decays as $\mathcal{O}(h^M)$ with $h = 2^{-j}$.

These statements are also equivalent with the Strang-Fix condition [10].

The fast wavelet transform

Since V_j is equal to $V_{j-1} \oplus W_{j-1}$, a function $v_j \in V_j$ can be written uniquely as the sum of a function $v_{j-1} \in V_{j-1}$ and a function $w_{j-1} \in W_{j-1}$:

$$\begin{aligned} v_j(x) &= \sum_k \nu_{j,k}\, \phi_{j,k}(x) \;=\; v_{j-1}(x) + w_{j-1}(x) \\ &= \sum_l \nu_{j-1,l}\, \phi_{j-1,l}(x) + \sum_l \mu_{j-1,l}\, \psi_{j-1,l}(x). \end{aligned}$$

There is a one-to-one relationship between the coefficients in the different representations. The decomposition formulae can be found using (4):

$$\nu_{j-1,l} \;=\; \sqrt{2} \sum_k \tilde{h}_{k-2l}\, \nu_{j,k}, \quad \text{and} \quad \mu_{j-1,l} = \sqrt{2} \sum_k \tilde{g}_{k-2l}\, \nu_{j,k}.$$

The reconstruction step involves calculating the $\nu_{j,k}$ from the $\nu_{j-1,l}$ and the $\mu_{j-1,l}$. Using (5) we have

$$\nu_{j,k} = \sqrt{2} \sum_l h_{k-2l}\, \nu_{j-1,l} + \sqrt{2} \sum_l g_{k-2l}\, \mu_{j-1,l}.$$

When applied recursively, these formulae define a transformation, the *fast wavelet transform* [8, 11]. The decomposition step consists of applying a low-pass ($\tilde{h}$) and a band-pass ($\tilde{g}$) filter followed by downsampling (i.e. retaining only the even index samples). The reconstruction consists of upsampling (i.e. adding a zero between every two samples) followed by filtering and addition. Note that the filter coefficients of the fast wavelet transform are given by the coefficients of the refinement equations.

There are many constructions of wavelets. Here we shall only consider compactly supported wavelets as in [12, 13]. In this case the filters used in the fast wavelet transform are finite impulse response filters and a fast accurate implementation is assured.

General idea

We shall assume that L is self-adjoint and positive definite and, in particular, we can write

$$L = V^* V,$$

where V^* is the adjoint of V. We call V the *square root operator* of L. Suppose that $\{\Psi_{j,l}\}$ and $\{\Psi^*_{j,l}\}$, for an appropriate range of indices, are bases for $\mathcal{S}$ and $\mathcal{S}^*$ respectively. The entries of the stiffness matrix are then given by

$$\langle\langle \Psi_{j,l}, \Psi^*_{j',l'} \rangle\rangle \;=\; \langle L\Psi_{j,l}, \Psi^*_{j',l'} \rangle \;=\; \langle V\Psi_{j,l}, V\Psi^*_{j',l'} \rangle .$$

Now, the idea is to let

$$\Psi_{j,l} \;=\; V^{-1}\psi_{j,l} \quad \text{and} \quad \Psi^*_{j,l} \;=\; V^{-1}\tilde{\psi}_{j,l} \,,$$

where ψ and $\tilde{\psi}$ are the wavelets of a classical multiresolution analysis. Because of the biorthogonality (3), the stiffness matrix becomes a diagonal matrix which can trivially be inverted. This avoids the use of an iterative algorithm. We will call the Ψ and Ψ^* functions the *operator wavelets* and the ψ functions the *original wavelets*. The operator wavelets are biorthogonal with respect to the operator inner product, a property we refer to as *operator biorthogonal*.

This idea can be powerful, but there are a few problems. First of all one has to check whether the operator wavelets still provide an multiresolution analysis where the successive approximations to a general function converge sufficiently fast (cfr the Strang-Fix condition). Secondly one has to construct a fast wavelet transform for this operator multiresolution analysis. We want operator wavelets to be compactly supported and to be able to construct compactly supported operator scaling functions $\Phi_{j,l}$. We will see that the latter is not as simple as just applying V^{-1} to the original scaling functions.

The analysis is relatively straightforward for simple constant coefficient operators such as the Laplace and polyharmonic operator. For more general constant coefficient operators, we will show that one needs to modify the construction of the original wavelets for the operator wavelets to satisfy all the desired properties. We will discuss the Helmholz operator as a typical example. At the end of the paper we shall consider a variable coefficient operator.

A similar idea was described in [14, 15]. However there only the operator wavelets of different levels are operator orthogonal and not the ones from the same level. As a result, one does not obtain a full diagonalization, but rather a decoupling of equations corresponding to different levels.

Our idea is different from the technique presented in [16]. There wavelets are used to efficiently compute the inverse of the matrix that comes from a finite difference discretization. It is also shown that the wavelets provide a diagonal preconditioner which yields uniformly bounded condition numbers.

In [17, 18] antiderivates of wavelets are used in a Galerkin method. This parallels our construction in the case of the Laplace or polyharmonic operator.

The one dimensional Laplace operator and its square root are

$$L = -D^2 \quad \text{and} \quad V = D.$$

The associated *operator inner product* is therefore $\langle\langle u, v \rangle\rangle = \langle u', v' \rangle$. Since the action of V^{-1} is simply taking the antiderivative here, we define the operator wavelets as

$$\Psi(x) = \int_{-\infty}^{x} \psi(t)\, dt, \quad \text{and} \quad \Psi^*(x) = \int_{-\infty}^{x} \tilde{\psi}(t)\, dt.$$

The operator wavelets are compactly supported because the integral of the original wavelets has to vanish. Also translation and dilation invariance is preserved, so we define

$$\Psi_{j,l}(x) = \Psi(2^j x - l) \quad \text{and} \quad \Psi^*_{j,l}(x) = \Psi^*(2^j x - l).$$

It is then easy to see that

$$\langle\langle \Psi^*_{j,l}, \Psi_{j',l'} \rangle\rangle = 2^j \delta_{j-j'} \delta_{l-l'} \quad \text{for} \quad j, j', l, l' \in \mathbb{Z}.$$

This means that the stiffness matrix is diagonal with powers of 2 on its diagonal.

We now need to find an operator scaling function Φ. The antiderivative of the original scaling function is not compactly supported and hence not suited. We instead construct the operator scaling function Φ by taking the convolution of the original scaling function with the indicator function on $[0, 1]$,

$$\Phi = \phi * \chi_{[0,1]},$$

and similarly for the dual functions. We will show that these functions indeed generate a multiresolution analysis. To this end define

$$V_j = \text{clos span}\, \{\Phi_{j,l} \mid l \in \mathbb{Z}\} \quad \text{and} \quad W_j = \text{clos span}\, \{\Psi_{j,l} \mid l \in \mathbb{Z}\}.$$

We show that the V_j spaces are nested and that W_j complements V_j in V_{j+1}.

In the Fourier domain we have

$$\hat{\Phi}(\omega) = \frac{1 - e^{-i\omega}}{i\omega} \hat{\phi}(\omega) \quad \text{and} \quad \hat{\Psi}(\omega) = \frac{1}{i\omega} \hat{\psi}(\omega).$$

A simple calculation shows that the operator scaling function satisfies a refinement equation

$$\hat{\Phi}(\omega) = \hat{\Phi}(\omega/2)\, H(\omega/2) \quad \text{with} \quad H(\omega) = \frac{1 + e^{-i\omega}}{2} h(\omega).$$

Consequently, the V_j spaces are nested. If we can find a function G such that

$$\hat{\Psi}(\omega) = \hat{\Phi}(\omega/2)\, G(\omega/2),$$

then this implies that W_j is a subset of V_{j+1}. It is easy to see that this holds with

$$G(\omega) = \frac{1}{2(1-e^{-i\omega})} g(\omega).$$

This function is well defined because $g(0) = 0$.

The space W_j complements V_j in V_{j+1} if

$$\Delta(\omega) = \det \begin{bmatrix} H(\omega) & H(\omega+\pi) \\ G(\omega) & G(\omega+\pi) \end{bmatrix}$$

does not vanish, see lemma 1. In fact, we readily see that $\Delta(\omega) = \delta(\omega)/4$, and this cannot vanish since ϕ and ψ generate a multiresolution analysis. The construction of the dual functions Φ^* and Ψ^* from $\tilde{\phi}$ and $\tilde{\psi}$ is competely similar. The coefficients of the trigonometric functions H, H^*, G and G^* now define a fast wavelet transform.

Note that there is no reason why the operator scaling functions should be operator biorthogonal and in fact one can prove that this never happens. Note also that if true, this property would make the use of wavelets superfluous.

Algorithm

We will describe the algorithm in the case of periodic boundary conditions. This implies that the basis functions on the interval $[0,1]$ are just the periodization of the basis functions on the real line.

Let $\mathcal{S} = V^n$ and consider the basis $\{\Phi_{n,l} \mid 0 \leqslant l < 2^n\}$. Define vectors b and x such that

$$b_l = \langle f, \Phi^*_{n,l} \rangle, \quad \text{and} \quad u = \sum_{l=0}^{2^n-1} x_l \Phi_{n,l}.$$

The Galerkin method with this basis then yields a system

$$A x = b \quad \text{with} \quad A_{k,l} = \langle\langle \Phi_{n,l}, \Phi_{n,k} \rangle\rangle.$$

As we mentioned earlier, the matrix A cannot be diagonal. Also its condition number grows as $\mathcal{O}(2^{2n})$. Consider now the decomposition

$$V_n = V_0 \oplus W_0 \oplus \cdots \oplus W_{n-1},$$

and the corresponding wavelet basis. The space V_0 has dimension one and contains constant functions. We now switch to a one index notation such that the sets

$$\{1, \Psi_{j,l} \mid 0 \leqslant j < n,\ 0 \leqslant l < 2^j\} \quad \text{and} \quad \{\Psi_k \mid 0 \leqslant k < 2^n\}$$

coincide. Define the vectors $\tilde{b}$ and $\tilde{x}$ such that

$$\tilde{b}_l = \langle f, \Psi^*_l \rangle \quad \text{and} \quad u = \sum_{l=0}^{2^n-1} \tilde{x}_l \Psi_l.$$

We know that there exists matrices T and T^* such that

$$\tilde{b} = T^* b \quad \text{and} \quad x = T \tilde{x}$$

The matrix T^* corresponds to the fast wavelet transform decomposition with filters H^* and G^* and T corresponds to reconstruction with filters H and G. The complexity of the matrix vector multiplication is $\mathcal{O}(N)$, $N = 2^n$. In the wavelet basis the system becomes

$$\tilde{A}\,\tilde{x} \;=\; \tilde{b} \quad \text{with} \quad \tilde{A} \;=\; T^* A T \quad \text{and} \quad \tilde{A}_{k,l} \;=\; \langle\langle\, \Psi_{n,l}, \Psi_{n,k} \,\rangle\rangle\,.$$

Since $\tilde{A}$ is diagonal, it can be trivially inverted and the solution is then given by

$$x \;=\; T\,\tilde{A}^{-1}\,T^*\,b.$$

This means that one has to calculate the wavelet decomposition of the right hand side, divide each coefficient by its corresponding diagonal element and reconstruct to find the solution. The complexity is $\mathcal{O}(N)$.

The constant basis function of V_0 has a zero as diagonal element and its coefficient is thus undetermined. Note that this leads to an inconsistency if the integral of f does not vanish.

Boundary conditions

Our general idea to deal with boundary conditions is to let the operator wavelets satisfy the homogeneous boundary conditions and to let the component in the V_0 space satisfy the imposed boundary conditions. This requires the use of special boundary wavelets as described in [19]. With only a slight change of basis one can then incorporate Dirichlet, Neumann, mixed and periodic boundary conditions. The details of this construction go beyond the scope of this paper. We will describe the construction in some specific cases.

Example

In this section we shall take a look at a simple example, namely the basis we get by starting from the Haar multiresolution analysis, where

$$\phi \;=\; \chi_{[0,1]} \quad \text{and} \quad \psi(x) \;=\; \phi(2x) - \phi(2x-1).$$

Define the *hat function* as

$$\Lambda \;=\; \chi_{[0,1]} * \chi_{[0,1]}, \quad \text{such that} \quad \Phi \;=\; \Lambda \quad \text{and} \quad \Psi(x) \;=\; \Lambda(2x).$$

The original wavelets are orthogonal and as a consequence the basis functions and dual functions coincide.

The operator scaling functions can represent linears which means they satisfy the Stang-Fix condition with $M = 2$ and the convergence is of order h^2. One can prove that higher order wavelets with more vanishing moments (M) will in general not yield faster convergence because the solution u is not smooth enough. The underlying reason is that the solution u belongs to the Sobolev space W_2. One can get faster convergence only by imposing extra regularity conditions on the right hand side. So in a way this basis seems to be the most natural one to work with. Note that these piecewise linear basis functions are local solutions of the homogeneous equation such that the

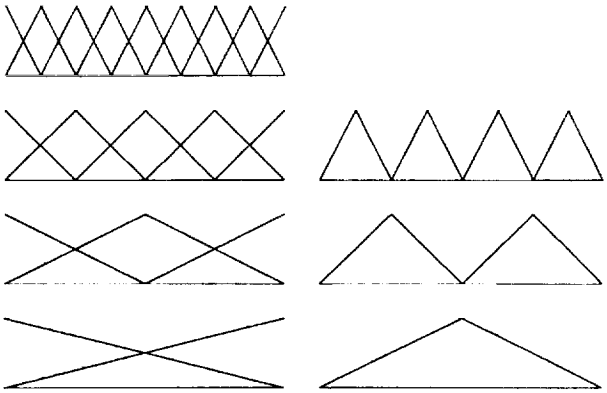

Figure 1: Basis for Dirichlet problem.

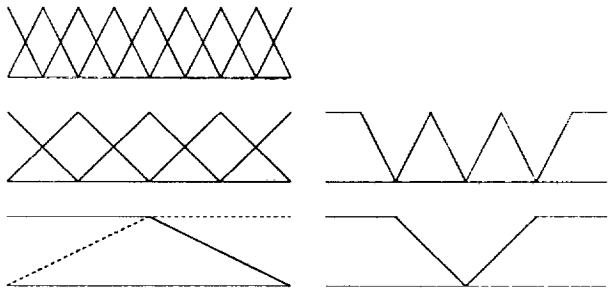

Figure 2: Basis for Neumann problem.

operator scaling functions and wavelets are V-splines. This basis also coincides with Yserentant's hierarchical basis.

Figure 1 shows the basis functions in the case of Dirichlet boundary conditions and $n = 3$. The left part are the bases for the spaces V_0 up to V_3 while the right part are the bases for W_0 up to W_2, which provide the diagonalization. The coefficients of the two functions in the V_0 space are determined by the boundary conditions. The fast wavelet transform differs from the periodic algorithm here in the sense that different coefficients are used for the wavelets at the boundary. Note the "half hat" functions here. The basis in case of the Neumann problem is shown in figure 2. The boundary conditions are handled by the two functions in the V_1 space. Again the coefficient of the constant is undetermined. The algorithm leads to an inconsistency in case the integral of f is not equal to $u'(1) - u'(0)$. Note that in both cases the operator wavelets satisfy the homogeneous boundary conditions.

MORE GENERAL CONSTANT COEFFICIENT OPERATORS

The polyharmonic operator

The polyharmonic equation is defined as

$$-u^{(2m)} = f,$$

and the square root operator is now $V = D^m$. The operator scaling function Φ is now m times the convolution of the original scaling function ϕ with the box function and the operator wavelet Ψ is m times the antiderivative of the original wavelet ψ. In order to get a compactly supported wavelet, the original wavelet now needs to have at least m vanishing moments, a property which can be satisfied by all known wavelet families. The construction and algorithm are then completely similar

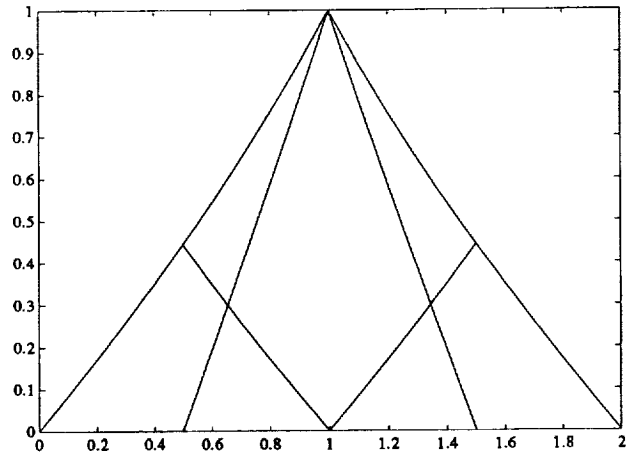

Figure 3: The refinement relation for the piecewise exponentials.

to the case of the Laplace operator.

The Helmholz operator

The general definition of the one dimensional Helmholz operator is:

$$L = -D^2 + k^2 \quad \text{such that} \quad V = D + k.$$

Here we shall assume that $k = 1$ which can always be obtained from a simple transformation. Observe that $V = D + I = e^{-x} D e^{x}$ and thus $V^{-1} = e^{-x} D^{-1} e^{x}$. One easily verifies that applying V^{-1} to a wavelet will not necessarily yield a compactly supported function since $e^{x}\psi_{j,l}$ in general does not have a vanishing integral. Therefore we let $\Psi_{j,l} = V^{-1} e^{-x} \psi_{j,l} = e^{-x} D^{-1} \psi_{j,l}$. If $\psi_{j,l}$ has a vanishing integral, then $\Psi_{j,l}$ is compactly supported.

In order to diagonalize the stiffness matrix, the original wavelets now need to be orthogonal with respect to a weighted inner product with weight function e^{-2x} because

$$\langle\langle\, \Psi_{j,l}, \Psi^{*}_{j',l'} \,\rangle\rangle \;=\; \int_{-\infty}^{+\infty} e^{-2x}\, \psi_{j,l}(x)\, \tilde{\psi}_{j',l'}(x)\, dx.$$

Finding such wavelets is a hard problem to solve in general. Inspired by the Haar basis, we construct a solution where the orthogonality of the wavelets on each level immediately follows from their disjoint support, by letting $\operatorname{supp} \psi_{j,l} = [2^{-j} l, 2^{-j}(l+1)]$. To get orthogonality between the different levels, we need that V_j is orthogonal to $W_{j'}$ for $j' \geqslant j$ or

$$\int_{-\infty}^{+\infty} e^{-2x}\, \phi_{j,l}(x)\, \tilde{\psi}_{j',l'}(x)\, dx \;=\; 0 \quad \text{for} \quad j' \geqslant j.$$

We now let the scaling function coincide with e^{2x} on the support of the finer scale wavelets,

$$\phi_{j,l} \;=\; e^{2x}\, \chi_{j,l},$$

where $\chi_{j,l}$ is the indicator function on the interval $[2^{-j}l, 2^{-j}(l+1)]$, normalized such that the integral of the scaling functions is a constant. As in the Haar case we choose the wavelets as

$$\psi_{j,l} \;=\; \phi_{j+1,2l} - \phi_{j+1,2l+1},$$

so that they have a vanishing integral. The orthogonality between levels now follows from the fact that the scaling functions coincide with e^{2x} on the support of the finer scale wavelets, and from the vanishing integral of the wavelets

$$\int_{-\infty}^{+\infty} e^{-2x}\,\phi_{j,l}(x)\,\tilde{\psi}_{j',l'}(x)\,dx \;=\; \int_{-\infty}^{+\infty} \chi_{j,l}(x)\,\tilde{\psi}_{j',l'}(x)\,dx \;=\; \int_{-\infty}^{+\infty} \tilde{\psi}_{j',l'}(x)\,dx \;=\; 0.$$

One can see that the operator wavelets are now piecewise hyperbolic functions (piecewise combinations of e^x and e^{-x}). The scaling functions are chosen as

$$\Phi_{j,l} \;=\; e^{-x}D^{-1}(\phi_{j,l} - \phi_{j,l+1}) \;\text{ so that }\; \Psi_{j,l} \;=\; \Phi_{j+1,2l}.$$

With the right normalization, one gets

$$\Phi_{j,l}(x) \;=\; \begin{cases} \dfrac{\sinh(x - l2^{-j})}{\sinh(2^{-j})} & \text{for } x \in [l2^{-j}, (l+1)2^{-j}] \\ \dfrac{\sinh((l+2)2^{-j} - x)}{\sinh(2^{-j})} & \text{for } x \in [(l+1)2^{-j}, (l+2)2^{-j}] \\ 0 & \text{elsewhere.} \end{cases}$$

The operator scaling functions on one level are translates of each other but the ones on different levels are no longer dilates of each other. They are supported on exactly the same sets as the ones in figure 1 and they roughly look similar. The operator scaling functions satisfy a refinement relation

$$\Phi_{j,l} \;=\; \sum_{k=0}^{2} H_k^j\,\Phi_{j+1,2l+k},$$

with

$$H_0^j \;=\; H_2^j \;=\; \sinh(2^{-j-1})/\sinh(2^{-j}) \quad\text{and}\quad H_1^j \;=\; 1.$$

Figure 3 shows the refinement relation for the scaling functions. The 3 finer scale functions are not the dilates of the coarse scale one but they still add up to it.

The Helmholz operator in this basis of hyperbolic wavelets again is diagonal and the algorithm is completely similar to the Laplace case. The only difference in implementation is that the filter coefficients H_k^j used in the fast wavelet transform now depend on the level.

Note that these functions again are V-splines and, in a way, are the most natural to work with. Also note that

$$\lim_{j\to\infty} \phi_{j,0}(2^{-j}x) \;=\; \Lambda(x).$$

Despite the fact that the Strang-Fix conditions are not satisfied, one can prove that the convergence is still of order h^2.

So we can conclude that a wavelet transform can diagonalize constant coefficient operators similar to the Fourier transform. The resulting algorithm is a little faster ($\mathcal{O}(N)$ instead of

$\mathcal{O}(N \log N)$). This gain in speed is a consequence of the subsampling of the coarser levels in the wavelet transform (the ones that correspond to the low frequency components of the solution) which is not present in the Fourier transform. Also boundary conditions are taken care of more easily than in the Fourier case.

VARIABLE COEFFICIENTS

Naturally, the next question is how to use wavelets for variable coefficient operators. The underlying reason why wavelets can diagonalize constant coefficient operators is their locality in the frequency domain. We want to understand if we can exploit the localization in space to diagonalize variable coefficient operators. The answer is (perhaps quite surprisingly) yes and this really justifies the use of wavelets for differential equations. No other technique (to our knowledge) has been able to accomplish this.

We take a closer look at the following operator

$$L \;=\; -D\,p^2(x)D,$$

where p is sufficiently smooth and positive. The square root is now $V = p\,D$ and $V^{-1} = D^{-1}\,1/p$. The rest of the analysis is very similar to the case of the Helmholz operator. Applying V^{-1} directly to a wavelet does not yield a compactly supported function. We therefore take $\Psi_{j,l} \;=\; V^{-1}\,p\,\psi_{j,l}$ which implies that the wavelets need to be (bi)orthogonal with respect to a weighted inner product with p^2 as weight function. We use the same trick as for the Helmholz equation to construct such functions. This means that we let the scaling functions $\phi_{j,l}$ coincide with $1/p^2$ on the dyadic interval $[2^{-j}l, 2^{-j}(l+1)]$ and normalize them such that they have a constant integral. We then take the wavelets $\psi_{j,l}$ to be equal to $\phi_{j+1,2l} - \phi_{j+1,2l+1}$ so they have a vanishing integral and the operator wavelets are compactly supported. The operator wavelets are now piecewise functions that locally look like $AP + B$ where P is the antiderivative of $1/p^2$ and again are V-splines. Their support also coincides with the support of the functions of figure 1, and since p is smooth they will converge to hat functions as the level goes to infinity. The operator wavelets are neither dilates nor translates of one function, since their behavior locally depends on p. This is not a problem because they still generate a multiresolution analysis and satisfy refinement relations. The coefficients in the fast wavelet transform are now different everywhere and they depend in a very simple way on the Haar wavelet transform of $1/p^2$. The entries of the diagonal stiffness matrix can be calculated from the wavelet transform of $1/p^2$. The algorithm is completely similar to previous cases and is of order N. Boundary conditions are as easy to handle as in the case of the Laplace operator. Note that the operator scaling functions do not satisfy the Strang-Fix conditions. It is however again possible to prove that the method has a convergence of order h^2. As mentioned earlier, higher convergence orders can not be obtained in general.

NUMERICAL EXAMPLE

We solve the equation

$$-D\,e^{x^2}\,D\,u(x) \;=\; e^{x^2}\left(\sin(x)(3x^2-2) + \cos(x)(2x-2x^3)\right)/x^3, \quad \text{with} \quad u(0)=1 \quad \text{and} \quad u(1)=\sin(1),$$

l	L_∞ error
1	1.22e-02
2	3.37e-03
3	8.66e-04
4	2.18e-04
5	5.45e-05
6	1.36e-05
7	3.41e-06
8	8.52e-07
9	2.13e-07

such that the exact solution is given by $u(x) = \sin(x)/x$. The L_∞ error of the numerically computed solution is a function of the number of levels (l) shown in the above table . Each time the number of levels is increased the error is divided almost exactly by a factor of 4, which agrees with the $\mathcal{O}(h^2)$ convergence.

CONCLUSION

In this paper we showed how wavelets can be adapted to be useful in the solution of differential equations. Like the Fourier transform, wavelets can diagonalize constant coefficient operators. The resulting algorithm is slightly faster. The main result however is that even non-constant coefficient operators can be diagonalized with the right choice of basis which evidently yields a much faster algorithm than more classical iterative methods.

This technique can also be applied to the solution of implicit time stepping discretizations of equations of the form $\partial u/\partial t = Lu + f$ even when L is non-linear. Future research includes the study of non self adjoint operators where a splitting $L = V\tilde{V}^*$ is needed and the study of the possible generalization of these ideas to partial differential equations.

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