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A Digital Combining-Weight Estimation Algorithm for Broadband Sources With the Array Feed Compensation System

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An algorithm for estimating the optimum combining weights for the Ka-band (33.7-GHz) array feed compensation system has been developed and analyzed. The input signal is assumed to be broadband radiation of thermal origin, generated by a distant radio source. Currently, seven video converters operating in conjunction with the real-time correlator are used to obtain these weight estimates. The algorithm described here requires only simple operations that can be implemented on a PC-based combining system, greatly reducing the amount of hardware. Therefore, system reliability and portability will be improved.

I. Introduction

At the present time, there is considerable interest in operating the DSN at increasingly higher carrier frequencies in order to realize the inherent advantages associated with shorter wavelengths, namely, greater antenna gains, increased useful bandwidth, and reduced sensitivity to plasma effects. Consequently, there is an effort underway to demonstrate the feasibility of using Ka-band (33.7-GHz) carrier frequencies for deep space telemetry. However, there are also disadvantages associated with the use of higher carrier frequencies, such as greater sensitivity to weather effects, increased requirements on pointing accuracy, and reduced antenna gains due to imperfections in the antenna's reflecting surfaces. Such imperfections become particularly noticeable on large receiving antennas,

where gravitational distortions, wind-induced vibrations, and collimation problems can seriously degrade antenna performance. Some of these losses can be recovered with a properly designed compensation system employing an array of receiving horns in the focal plane of the antenna; a conceptual design of such a combining system is shown in Fig. 1. Complete descriptions and analyses of a real-time array feed compensation system designed for deep space telemetry can be found in the literature [1–3]. Recently, a seven-element array feed compensation system has been installed at DSS 13 for the purpose of demonstrating combining concepts in real time.

Perhaps the most serious problem encountered during the Ka-band array feed compensation effort was the lack of reliable coherent sources in the antenna's far field. Since

this is not an operational frequency band, no spacecraft has yet been built employing Ka-band carrier frequencies (Mars Observer did carry a low-power Ka-band beacon, but this spacecraft ceased to function before reaching its target). However, since only spatial coherence is required to demonstrate the array feed combining concept, it is possible to carry out weighted combining operations using natural radio sources, such as quasars and planets. Since these are always in the antenna's far field, the only additional requirement is that the sources remain unresolved. This article is devoted to the derivation of the optimum combining weights for maximizing he signal-to-noise ratio (SNR) of the combined signal and the estimation of the optimum combining weights in real time from natural radio signals observed in the presence of additive noise.

II. The Received Signal

A functional block diagram of the combining system is shown in Fig. 2. The array horns receive a broadband signal of thermal origin from a distant point source, each with different amplitude and phase introduced by the antenna distortions. An independent noise waveform is added to each signal in every channel, the result of receiver noise plus background radiation received from all directions in space. After passing through narrowband filters of center frequency ω, the received signal can be represented in the kth channel as

$$r_k(t) = s_k(t) + n_k(t) \qquad k = 1, 2, \ldots, K \qquad (1a)$$

where

$$s_k(t) = \sqrt{2}S_k \{a_c(t) \cos(\omega t + \theta_k)$$
$$+ a_s(t) \sin(\omega t + \theta_k)\} \qquad (1b)$$

$$n_k(t) = \sqrt{2}\{n_{ck}(t) \cos(\omega t) + n_{sk}(t) \sin(\omega t)\} \qquad (1c)$$

Note that the signal components in the various channels differ from each other only in their amplitude and phase, having been generated by the same point source. Thus, the random processes $s_k(t)$ are completely correlated. However, the noise processes $n_k(t)$ are assumed to be uncorrelated in all channels, as these are composed of thermal noise generated within the receivers and background radiation arriving from all directions in space.

The received waveforms are downconverted to baseband in-phase and quadrature signals $r_{Ik}(t)$ and $r_{Qk}(t)$

by premultiplying with local oscillator signals of the form $\sqrt{2}\cos(\omega t)$ and $\sqrt{2}\sin(\omega t)$, and low-pass filtering:

$$r_{Ik}(t) = s_{Ik}(t) + n_{Ik}(t) \qquad (2a)$$

$$r_{Qk}(t) = s_{Qk}(t) + n_{Qk}(t) \qquad (2b)$$

$$s_{Ik}(t) = S_k \{a_c(t) \cos(\theta_k) - a_s(t) \sin(\theta_k)\} \qquad (2c)$$

$$s_{Qk}(t) = S_k \{a_c(t) \sin(\theta_k) + a_s(t) \cos(\theta_k)\} \qquad (2d)$$

$$n_{Ik}(t) = n_{ck}(t) - n_{sk}(t) \qquad (2e)$$

$$n_{Qk}(t) = n_{ck}(t) + n_{sk}(t) \qquad (2f)$$

After sampling the baseband waveforms, the resulting in-phase and quadrature samples may be treated as complex samples $\tilde{r}_k(i)$, defined as

$$\tilde{r}_k(i) = \tilde{s}_k(i) + \tilde{n}_k(i) \qquad (3a)$$

where

$$\tilde{s}_k(i) = s_{Ik}(i) + js_{Qk}(i) \qquad (3b)$$

$$\tilde{n}_k(i) = n_{Ik}(i) + jn_{Qk}(i) \qquad (3c)$$

In other words, we shall use complex arithmetic to operate on these samples.

Defining the complex coefficient $\tilde{S}_k = S_k e^{j\theta_k}$, Eq. (3b) can also be written as

$$\tilde{s}_k(i) = \tilde{S}_k \{a_c(i) + ja_S(i)\} = \tilde{S}_k \tilde{a}(i) \qquad (4)$$

which shows that the complex channel scaling factors separate from the temporal variations.

The real and imaginary parts of the complex noise sequence are independent random sequences, each with variance σ_k^2. The components of the signal sequence, $a_c(i)$ and $a_s(i)$, are also independent random sequences, being of thermal origin. However, since we are interested in extracting the complex magnitude $\tilde{S}_k$, it is reasonable to let each signal component have variance $1/2$, so as not to introduce additional scaling. In the following, the complex signal coefficients and the noise variances will be assumed to be constants independent of time.

III. Combining Weights to Maximize SNR

The goal of the combining operation is to maximize the SNR of the combined sequence. The approach is to multiply each sequence by a complex combining weight with the property that the sum of the weighted sequences achieves the greatest SNR.

Let the kth combining weight be denoted by $\tilde{w}_k$. Multiplying each received sequence by the corresponding complex weight and summing yields the combined sequence $z(i)$:

$$\tilde{z}(i) = \sum_{k=1}^{K} \tilde{r}_k(i)\,\tilde{w}_k(i) \tag{5}$$

With $\tilde{s}_c(i) = \sum_{k=1}^{K} \tilde{s}_k(i)\,\tilde{w}_k(i)$ denoting the combined signal, the SNR of the combined sequence is defined as

$$\rho_z = \frac{E\,|\tilde{s}_c(i)|^2}{\mathrm{var}\,\{\tilde{z}(i)\}} = \frac{\left|\sum_{k=1}^{K} \tilde{S}_k \tilde{w}_k\right|^2}{\sum_{k=1}^{K} 2\,|\tilde{w}_k|^2\,\sigma_k^2} \tag{6}$$

This is the quantity we wish to maximize by judicious selection of the combining weights.

The optimum combining weights are obtained by means of the Schwarz inequality. Writing the combined sequence as

$$\left|\sum_{k=1}^{K} \tilde{S}_k \tilde{w}_k\right|^2 = \left|\sum_{k=1}^{K} \left(\sqrt{2}\tilde{w}_k\sigma_k\right)\left(\frac{\tilde{S}_k}{\sqrt{2}\sigma_k}\right)\right|^2 \tag{7}$$

and applying the Schwarz inequality, yields

$$\left|\sum_{k=1}^{K} \left(\sqrt{2}\tilde{w}_k\sigma_k\right)\left(\frac{\tilde{S}_k}{\sqrt{2}\sigma_k}\right)\right|^2 \leq \tag{8}$$

$$\left(\sum_{k=1}^{K} 2\,|\tilde{w}_k|^2\,\sigma_k^2\right)\left(\sum_{k=1}^{K} \frac{|\tilde{S}_k|^2}{2\sigma_k^2}\right)$$

Dividing both sides by the first term on the right-hand side of Eq. (8), we obtain

$$\frac{\sum_{k=1}^{K} \left(\sqrt{2}\tilde{w}_k\sigma_k\right)\left(\frac{\tilde{S}_k}{\sqrt{2}\sigma_k}\right)}{\sum_{k=1}^{K} 2\,|\tilde{w}_k|^2\,\sigma_k^2} \leq \sum_{k=1}^{K} \frac{|\tilde{S}_k|^2}{2\sigma_k^2} \triangleq \rho_0 \tag{9}$$

which shows that for any choice of combining weights, the achievable SNR is bounded from above by the sum of the channel SNRs. Except for an arbitrary complex factor, equality is achieved when we let $\sqrt{2}\tilde{w}_k\sigma_k = \tilde{S}_k^*/\sqrt{2}\sigma_k$, whereby the optimum combining weights are determined in terms of the signal and noise parameters as

$$\tilde{w}_k = \frac{\tilde{S}_k^*}{2\sigma_k^2} \tag{10}$$

The combined SNR is maximized when each sequence is multiplied by the conjugate of the signal scaling factor and divided by the variance of the additive noise in that channel. Therefore, these quantities have to be estimated in real time to determine the correct combining weights.

IV. Parameter Estimates

The estimator described here is based on the observation that the temporal variation of the signal components is identical in every channel. This implies that the expected value of the product of a received sequence with the complex conjugate of a sequence from any other channel is equal to the product of the complex coefficients. That is,

$$E\,[\tilde{r}_\ell^*(i)\tilde{r}_m(i)] = E\left[\left(\tilde{S}_\ell\tilde{a}(i) + \tilde{n}_\ell(i)\right)^*\left(\tilde{S}_m\tilde{a}(i) + \tilde{n}_m(i)\right)\right]$$

$$= \tilde{S}_\ell^*\tilde{S}_m \tag{11}$$

The last equality follows from the definition $E|\tilde{a}(i)|^2 = 1$, and the assumption that noise sequences are not correlated with other noise sequences or with the signal sequence. If the received sequences are ergodic, then ensemble and time averages are identical, suggesting the following estimates for the coefficient products:

$$\widehat{\tilde{S}_\ell^*\tilde{S}_m}_{\ell \neq m} = \frac{1}{L}\sum_{i=1}^{L} \tilde{r}_\ell^*(i)\tilde{r}_m(i)$$

$$\ell, m = 1, 2, \ldots, K; \quad \ell \neq m \tag{12}$$

As the number of terms grows without bound, the estimation error approaches zero. With the total number of channels equal to K, index the channels according to the SNR, so that channel number 1 contains the greatest SNR, channel number 2 the second-greatest, and so on (in case of equalities, an arbitrary choice can be made). Thus, we may view channel 1 as the "reference channel," as it provides the highest fidelity signal. Consider the estimates of the products $\tilde{S}_1^* \tilde{S}_m$:

$$\widehat{\tilde{S}_1^* \tilde{S}_m} = \frac{1}{L} \sum_{i=1}^{L} \tilde{r}_1^*(i) \tilde{r}_m(i) \quad m = 2, 3, \ldots, K \qquad (13)$$

If $\tilde{S}_1^*$ were known, then we could estimate $\tilde{S}_m$, $m = 2, 3, \ldots, K$, by means of the formula

$$\widehat{\tilde{S}_m} = \frac{\widehat{\tilde{S}_1^* \tilde{S}_m}}{\widehat{\tilde{S}_1}} \qquad (14)$$

Even if only $|\tilde{S}_1|$ were known (that is, if no phase information could be obtained), we could still determine $\tilde{S}_m e^{-j\theta_1}$, where θ_1 is the argument of $\tilde{S}_1$. Since in our application the combining weights may be multiplied by a complex constant without affecting the combined SNR, the phase of the reference channel need not be estimated. Thus, we turn our attention to the estimation of the signal magnitude in the central reference channel.

An estimate of $|\tilde{S}_1|$ can be obtained in the following manner, using well-established experimental techniques. With the antenna pointing "on-source," we obtain an estimate of the total power P_{t1}, by averaging L independent sample powers:

$$\widehat{P_{t1}} = \frac{1}{L} \sum_{i=1}^{L} \left| \tilde{S}_1 \tilde{a}(i) + \tilde{n}_1(i) \right|^2 \qquad (15a)$$

Next, we point the antenna "off-source" and obtain an estimate of the noise power using subsequent samples:

$$\widehat{P_{n1}} = \frac{1}{L} \sum_{i=L+1}^{2L} |\tilde{n}_1(i)|^2 \qquad (15b)$$

Since

$$E\left[\widehat{P_{t1}}\right] = \left|\tilde{S}_1\right|^2 + 2\sigma_1^2 \qquad (16a)$$

and

$$E\left[\widehat{P_{n1}}\right] = 2\sigma_1^2 \qquad (16b)$$

it follows that

$$\left|\tilde{S}_1\right|^2 = \begin{cases} \widehat{P_{t1}} - \widehat{P_{n1}}; & \widehat{P_{t1}} \geq \widehat{P_{n1}} \\ \text{not defined}; & \widehat{P_{n1}} > \widehat{P_{t1}} \end{cases} \qquad (17)$$

is a reasonable estimate of $|\tilde{S}_1|^2$ when the average signal power is large compared to the random variations. When the signal power is not sufficiently great, it is possible for the estimate to become negative, which is meaningless for a power estimate and, therefore, should not be used.

The estimate of $\left|\tilde{S}_1\right|$ follows from Eq. (17) as

$$\widehat{\left|\tilde{S}_1\right|} = \left(\widehat{\left|\tilde{S}_1\right|^2}\right)^{1/2} \qquad (18)$$

Using this estimate, the signal coefficients for the outer channels may be obtained from Eqs. (12) and (14), except for a common complex coefficient, as follows:

$$\widehat{\tilde{S}_m e^{-j\theta_1}} = \frac{\widehat{\tilde{S}_1^* \tilde{S}_m}}{\widehat{\left|\tilde{S}_1\right|}} \qquad (19)$$

The complex coefficient $e^{-j\theta_1}$ has no effect on the combiner performance, since the combining weights will include this factor, effectively setting the average phase of the combined signal to zero.

V. Statistics of the Weight Estimates

The combining weights that maximize the SNR of the combined samples are defined in Eq. (10), repeated here for convenience:

$$\widehat{w_k} = \frac{\widehat{\tilde{S}_k^*}}{2\sigma_k^2} \qquad (20)$$

Each complex weight may be multiplied by a complex constant without affecting combiner performance. Thus,

for our purposes, it is sufficient to estimate the "rotated" weights

$$\widehat{\tilde{w}_k e^{-j\theta_1}} = \frac{\widehat{\tilde{S}_k^* e^{-j\theta}}}{2\widehat{\sigma_k^2}} = \frac{\widehat{\tilde{S}_1 \tilde{S}_k^*}}{2|\widehat{\tilde{S}_1}|\widehat{\sigma_k^2}} \qquad (21)$$

instead, without explicitly determining the phase of the central channel, θ_1.

The denominator of each weight is simply an estimate of the noise variance in that channel. The noise variance in the kth channel, $2\sigma_k^2$, is estimated by means of simultaneous off-source power measurements in all K channels (both real and imaginary noise components have variance σ_k^2, hence the factor of 2 in the variance of the complex noise). If M samples are used, the estimate of the variance in the kth channel is of the form

$$\widehat{\sigma_k^2} = \frac{1}{2M} \sum_{i=1}^{M} |\tilde{n}_k(i)|^2$$

$$\qquad (22)$$

$$= \frac{1}{2M} \left\{ \sum_{i=1}^{M} n_R^2(i) + \sum_{i=1}^{M} n_I^2(i) \right\}$$

Clearly, the expected value of this estimate is the desired quantity,

$$E\widehat{(\sigma_k^2)} = \sigma_k^2 \qquad (23)$$

hence the estimator is unbiased. Assuming the underlying processes are Gaussian, the estimate is a scaled central chi-square random variable with $2M$ degrees of freedom and variance

$$\text{var}\widehat{(\sigma_k^2)} = \frac{1}{4M^2} \left(4M\sigma_k^4 \right) = \frac{\sigma_k^4}{M} \qquad (24)$$

In typical applications, the component of the noise variance due to the receiver remains constant, while the component due to the background changes slowly with elevation and azimuth, but may be considered constant for many weight estimates. The frequency with which off-source measurements must be performed depends on the specific details of the experiment.

The numerator of each rotated weight estimate can be obtained using Eq. (19). In addition to an estimate of the

product $\tilde{S}_1 \tilde{S}_k^*$ (whose expectation is proportional to the correlation between the first and kth channels), this approach also requires an estimate of the magnitude of the complex scale factor in the central channel. This quantity can be updated with each new weight estimate, using the last noise power measurement, or a separate on-source measurement can be made periodically for the central channel as well. Both cases will be considered. First, assume estimates are made with each update, using the same number of samples as for the weights (L). The estimate of the signal power in the central channel then becomes

$$\widehat{|\tilde{S}_1|^2} = \frac{1}{L} \sum_{i=1}^{L} \left| \tilde{S}_1 \tilde{a}(i) + \tilde{n}_1(i) \right|^2 - \frac{1}{M} \sum_{i=L+1}^{L+M} |\tilde{n}_1(i)|^2 \quad (25)$$

with expectation

$$E\widehat{|\tilde{S}_1|^2} = |\tilde{S}_1|^2 \qquad (26)$$

and variance

$$\text{var}_L \left(\widehat{|\tilde{S}_1|^2} \right) = \frac{2}{L} \left(2\sigma_1^2 + |\tilde{S}_1|^2 \right)^2 + \frac{8}{M}\sigma_1^4 \qquad (27)$$

Again, we observe from Eq. (26) that an unbiased estimate is obtained. If a separate power measurement is carried out for the central channel using N independent samples, then the signal power estimate becomes

$$\widehat{|\tilde{S}_1|^2} = \frac{1}{N} \sum_{i=1}^{N} \left| \tilde{S}_1 \tilde{a}(i) + \tilde{n}_1(i) \right|^2 - \frac{1}{M} \sum_{i=N+1}^{N+M} |\tilde{n}_1(i)|^2 \quad (28)$$

This estimate is also unbiased; hence, Eq. (26) still holds. However, the variance of the estimate decreases if $N > L$, as shown by comparing the following expression with Eq. (27):

$$\text{var}_N \left(\widehat{|\tilde{S}_1|^2} \right) = \frac{2}{N} \left(2\sigma_1^2 + |\tilde{S}_1|^2 \right)^2 + \frac{8}{M}\sigma_1^4 \qquad (29)$$

The estimate of the signal power can, therefore, be made arbitrarily good by making both N and M sufficiently large, provided the natural time scales of the relevant process variations are not exceeded.

Next consider the statistics of the coefficient product estimates, $\tilde{S}_j \tilde{S}_k^*$. As shown in Eq. (11), the expectation of the product of the received samples between any two channels is the desired coefficient product, implying the unbiased estimator structure defined in Eq. (13).

The variance of this estimate can be obtained by first deriving the second moment of the coefficient product, subtracting the square of the expectation, and dividing the result by the total number of independent sample products

averaged to obtain the estimate. If L sample products are averaged, this can be expressed as

$$\text{var}\left(\tilde{S}_j \tilde{S}_k^*\right) = \frac{1}{L}\left\{ E\left|\tilde{S}_j \tilde{S}_k^*\right|^2 - \left|E\left(\tilde{S}_j \tilde{S}_k^*\right)\right|^2 \right\} \quad (30)$$

Consider the second moment first. Writing the received samples as in Eqs. (3a) and (4), the second moment of the sample products between two distinct channels j and k becomes

$$E|\tilde{S}_j \tilde{S}_k^*|^2 = E\left\{ \left(\tilde{S}_j \tilde{S}_k^*|\tilde{a}|^2 + \tilde{S}_j \tilde{a}\tilde{n}_k^* + \tilde{S}_k^* \tilde{a}^* \tilde{n}_j + \tilde{n}_j \tilde{n}_k^*\right)\left(\tilde{S}_j^* \tilde{S}_k|\tilde{a}|^2 + \tilde{S}_j^* \tilde{a}^* \tilde{n}_k + \tilde{S}_k \tilde{a}\tilde{n}_j^* + \tilde{n}_j^* \tilde{n}_k\right) \right\}$$

$$= E\left\{ |\tilde{S}_j|^2|\tilde{S}_k|^2|\tilde{a}|^4 + |\tilde{S}_j|^2|\tilde{a}|^2|\tilde{n}_k|^2 + |\tilde{S}_k|^2|\tilde{a}|^2|\tilde{n}_j|^2 + |\tilde{n}_j|^2|\tilde{n}_k|^2 \right\}$$

$$+ \text{ (products whose expectation goes to zero)}$$

$$= |\tilde{S}_j|^2|\tilde{S}_k|^2\overline{|a|^4} + |\tilde{S}_j|^2\overline{|\tilde{a}|^2}\ \overline{|\tilde{n}_k|^2} + |\tilde{S}_k|^2\overline{|a|^2}\ \overline{|\tilde{n}_j|^2} + \overline{|\tilde{n}_j|^2}\ \overline{|\tilde{n}_k|^2} \quad (31)$$

where the overbar denotes expectation. Letting $\overline{|\tilde{n}_j|^2} = 2\sigma_j^2$, $\overline{|\tilde{a}|^2} = 1$, and with $\overline{|\tilde{a}|^4} = 2$, the second moment becomes

$$E|\tilde{S}_j \tilde{S}_k^*|^2 = 2|\tilde{S}_j|^2|\tilde{S}_k|^2 + 2|\tilde{S}_j|^2\sigma_k^2 + 2|\tilde{S}_k|^2\sigma_j^2 + \sigma_k^2\sigma_j^2$$

$$(32)$$

Subtracting the square of the expected value and dividing by L yields an expression for the variance:

$$\text{var}\left(\tilde{S}_j \tilde{S}_k^*\right) = |\tilde{S}_j|^2|\tilde{S}_k|^2$$

$$+ 2\left(|\tilde{S}_j|^2\sigma_k^2 + |\tilde{S}_k|^2\sigma_j^2\right) + 4\sigma_j^2\sigma_k^2 \quad (33)$$

If the estimation errors are sufficiently small, each estimate may be written as its true value plus a small random deviation. Assuming this to be the case, we have

$$\widehat{\tilde{S}_1^* \tilde{S}_k} = \tilde{S}_1^* \tilde{S}_k(1 + \alpha) \quad (34a)$$

$$\widehat{|\tilde{S}_1|^2} = |\tilde{S}_1|^2(1 + \beta) \quad (34b)$$

$$\widehat{\sigma_k^2} = \sigma_k^2(1 + \gamma) \quad (34c)$$

Using Eqs. (24), (27), and (33), it follows that each error term is a zero-mean random variable with variance

$$\text{var}(\alpha) = \frac{2}{L}$$

$$\times \left\{ 1 + \frac{4\sigma_1^2\sigma_k^2}{|\tilde{S}_1|^2|\tilde{S}_k|^2} + 2\left(\frac{\sigma_k^2}{|\tilde{S}_k|^2} + \frac{\sigma_1^2}{|\tilde{S}_1|^2}\right) \right\} \quad (35a)$$

$$\text{var}(\beta) = \frac{1}{|\tilde{S}_1|^4}\left\{ \frac{2}{Q}\left(2\sigma_1^2 + |\tilde{S}_1|^2\right)^2 + \frac{8}{M}\sigma_1^4 \right\};$$

$$Q = L \text{ or } N \quad (35b)$$

$$\text{var } (\gamma) = \frac{1}{M} \qquad (35c)$$

The estimate of the rotated weight can now be written as

$$\widehat{\tilde{w}_k e^{-j\theta_1}} = \frac{\tilde{S}_1 \tilde{S}_k^*}{2\sigma_k^2 |\tilde{S}_1|} \frac{(1+\alpha)}{\sqrt{(1+\beta)(1+\gamma)}}$$

$$\approx \tilde{C}_k \frac{(1+\alpha)}{\left(1+\frac{\beta}{2}\right)(1+\gamma)} \qquad (36)$$

where $\tilde{C}_k \triangleq \tilde{S}_1 \tilde{S}_k^* / 2\sigma_k^2 |\tilde{S}_1|$. Expanding the denominator, multiplying through, and keeping terms up to second order yields

$$\widehat{\tilde{w}_k e^{-j\theta_1}} = \tilde{C}_k \left[1 - \frac{\beta}{2}\left(1 - \frac{\beta}{2} + \alpha - \gamma\right) \right.$$

$$\left. + \alpha - \gamma (1 + \alpha - \gamma) \right] \qquad (37)$$

The expected value of this expression is

$$E\left(\widehat{\tilde{w}_k e^{-j\theta_1}}\right) \cong \tilde{C}_k \left(1 + \frac{\overline{\beta^2}}{4} + \overline{\gamma^2}\right)$$

$$= \tilde{C}_k \left[1 + \text{var}\left(\frac{\beta}{2}\right) + \text{var }(\gamma)\right] \qquad (38)$$

which shows that the estimate is biased, but becomes asymptotically unbiased as the number of samples grows without bound. Writing the rotated weight estimate as

$$\widehat{\tilde{w}_k e^{-j\theta_1}} = \tilde{C}_k (1 + \zeta) \qquad (39)$$

where $\zeta = \alpha - \frac{\beta}{2} - \gamma - \alpha\gamma - \alpha\frac{\beta}{2} + \gamma\frac{\beta}{2} + \frac{\beta^2}{4} + \gamma^2$, it follows that

$$\text{var }\left(\widehat{\tilde{w}_k e^{-j\theta_1}}\right) = |\tilde{C}_k|^2 \text{ var }(\zeta) \qquad (40)$$

Here ζ is a random variable with second-order statistics

$$E(\zeta) = \frac{\overline{\beta^2}}{4} + \overline{\gamma^2} \qquad (41)$$

and

$$\text{var }(\zeta) = E\left[\alpha - \frac{\beta}{2} - \gamma - \alpha\frac{\beta}{2} - \alpha\gamma + \gamma\frac{\beta}{2}\right]^2$$

$$= \overline{\alpha^2} + \frac{\overline{\beta^2}}{4} + \overline{\gamma^2} + \overline{\gamma^2}\frac{\overline{\beta^2}}{4} + \overline{\alpha^2}\ \overline{\gamma^2} + \frac{\overline{\beta^2}}{4}\overline{\gamma^2}$$

$$+ \text{ (products whose expectation is zero)} \qquad (42)$$

Since $E(\alpha) = E(\beta) = E(\gamma) = 0$, and α, β, and γ are statistically independent, it follows that all terms in Eq. (42) containing α, β, or γ to the first power go to zero. Thus, the normalized variance of the rotated weight may be expressed in symmetric form as

$$|\tilde{C}_k|^{-2} \text{ var } \left(\widehat{\tilde{w}_k e^{-j\theta_1}}\right) =$$

$$\text{var }(\alpha) + \text{ var }\left(\frac{\beta}{2}\right) + \text{ var }(\gamma)$$

$$+ \text{ var }(\alpha) \text{ var }\left(\frac{\beta}{2}\right) + \text{ var }(\gamma) \text{ var }\left(\frac{\beta}{2}\right)$$

$$+ \text{ var }(\alpha) \text{ var }(\gamma) \qquad (43)$$

The statistics of these rotated weight estimates for specific parameter values are examined in the next section.

VI. Numerical Results

The expressions for the normalized bias and variance of the rotated weights developed in Eqs. (41) and (42) are functions of the parameters M, N, and L, as well as of the signal powers and noise variances in the combiner channels. The noise variance is largely determined by the physical characteristics of the front-end amplifiers, and includes contributions from background radiation as well. The signal power in each channel depends on the strength of the source, the antenna aperture, and the amount of distortion suffered by the main reflector. However, the remaining parameters can be selected to achieve a desired level of estimator performance.

In a typical combining experiment, the antenna is first pointed off source, and the noise power in each complex channel is estimated according to Eq. (22). Denote the number of samples used to make this estimate M. Next, the antenna is pointed on source, and the signal in the central channel is measured by means of Eq. (25) or Eq. (28), using either L or N samples, where L is the number of samples per update. Of course, the signal levels in the other channels could also be determined, but that information is not used by this algorithm; instead, the rotated complex weights are estimated according to Eq. (19). Theoretically, some improvement could be obtained if independent signal power measurements were used to improve the magnitude estimates, but that issue will not be addressed here.

After the coefficients are determined, combining weight updates are obtained every L samples for use in the real-time combining operation. The inequality $(M, N) >> L$ is usually valid, since both signal and noise power levels tend to remain constant over a great many coefficient updates. However, we shall also consider the case $N = L$, corresponding to a situation where the signal power in the central channel changes fast enough to warrant its measurement with each update. This could occur if wind-induced dynamics or time-varying pointing errors were present.

Representative values were chosen for the signal power levels, normalized to the noise variance, which was assigned a value of 1 power unit. With a channel noise temperature of roughly 100 K, a strong Ka-band radio source (such as Venus or Jupiter) may produce a 10-K rise in the central channel when observed with the 34-m antenna at DSS 13. At low and high elevations, gravitationally induced distortions of the main reflector generally deflect as much as 10 percent of the total signal power to some of the outer channels. Thus, the values $\sigma_k^2 = 1, |\tilde{S}_1|^2 = 0.1$ and $|\tilde{S}_k|^2 = 0.01, k \neq 1$ will be assumed for the numerical examples that follow, meaning that the signal power in the central (reference) channel is one-tenth of the noise power, while typical signal powers in the outer channels are 10 times smaller than those in the central channel.

The normalized bias defined in Eq. (41) is shown in Fig. 3 as a function of N, with M as a parameter. For any N, the minimum achievable bias is a monotone decreasing function of M that approaches an asymptotic limit from above. Minimum values of M and N are clearly specified for any desired bias level: for example, if we wish to maintain a normalized bias of 0.001, then M must be greater than about 10^5 and N must be at least 3×10^5; at a sampling rate of 2×10^5 samples/sec, this would take a mere

$2 \times 1.5 = 3$ sec, which is short compared to typical time scales encountered in practice. If we let M exceed 10^6, then the above value of N will suffice. Suppose we select this value of N in order to meet the bias requirements, and examine the normalized variance defined in Eq. (43).

The normalized variance of the rotated weight estimates is shown in Fig. 4 as a function of L. It is clear that a given value of N specifies a limit for the smallest attainable variance and, hence, limits the performance of the weight estimator. For the specified value of N, this limit is about 4×10^{-4}, even as M and L grow without bound. The dependence on N disappears if the signal power is recomputed with each update, using the previous L samples, corresponding to the case $N = L$. Now any desired level of performance can be achieved provided that both M and L are sufficiently great, as shown in Fig. 5. Given M and L, the corresponding normalized bias can be determined from Fig. 3 by substituting L for N. Note that for very small values of L, the variance increases rapidly when $N = L$, due to the greater error in the signal power estimate as a result of insufficient observations.

VII. Summary and Conclusions

A digital combining-weight estimation algorithm for use with broadband sources has been described and analyzed. Although the algorithm provides a biased estimate of the combining weights, the bias can be reduced to any desired level by observing enough samples. The normalized variance of the weight estimates can be similarly reduced; however, care must be taken to obtain accurate signal and noise power estimates, as these quantities ultimately limit the performance of the estimator at all update rates. With this algorithm, only three primitive estimates are needed to obtain each combining weight estimate: the noise variance in the kth channel, the signal power in the central channel, and the complex correlation coefficient between the central and kth channels. The first two are simple power measurements requiring magnitude squaring and accumulation, while the third is a complex multiply-and-accumulate operation; both operations can be carried out with current PC-based digital hardware, provided the sampling rates do not exceed roughly 250,000 samples/sec. Consequently, this approach is useful for reducing estimator complexity while greatly increasing the portability of the entire array feed compensation system, enabling the demonstration of combining gain on a variety of DSN antennas.

References

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[2] V. A. Vilnrotter, E. R. Rodemich, and S. J. Dolinar, Jr., "Real-Time Combining of Residual Carrier Array Signals Using ML Weight Estimates," *IEEE Transactions on Communications*, vol. 40, no. 3, pp. 604–615, March 1992.

[3] S. Zohar and V. Vilnrotter, *Using a Focal-Plane Array to Estimate Antenna Pointing Errors*, JPL Publication 91-45, Jet Propulsion Laboratory, Pasadena, California, October 25, 1991.
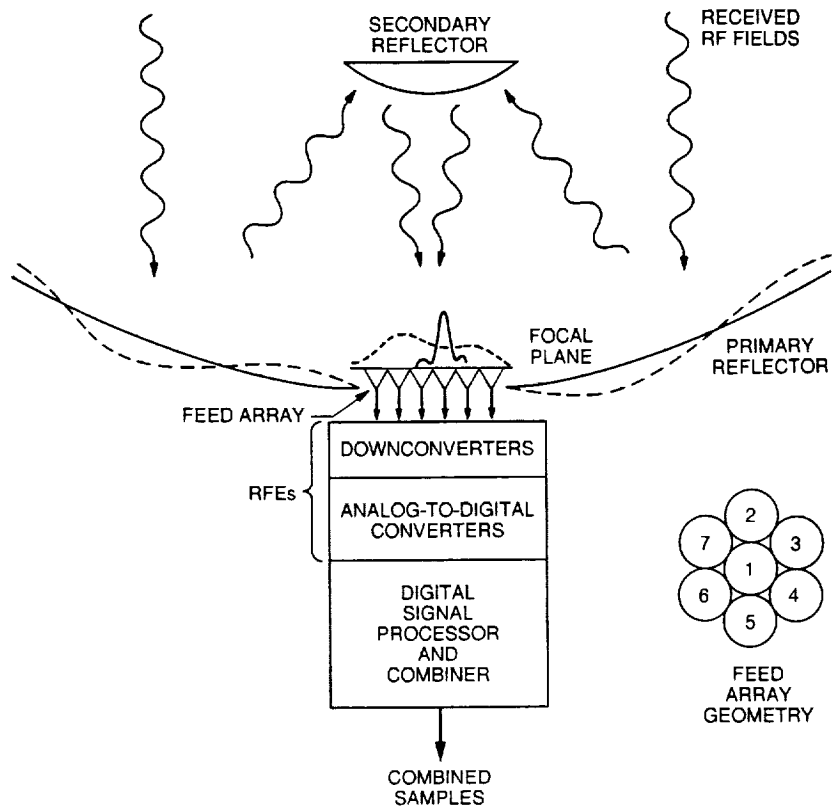
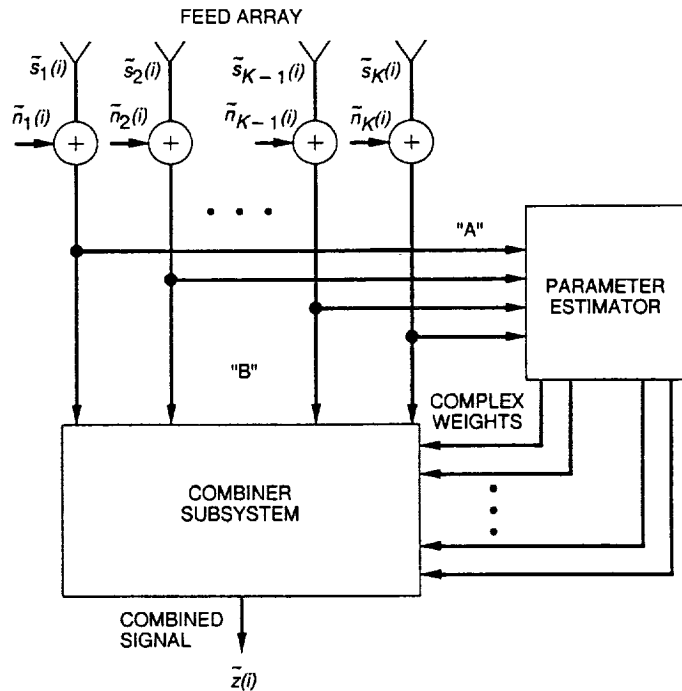Fig. 1. Real-time antenna-compensation system conceptual design.

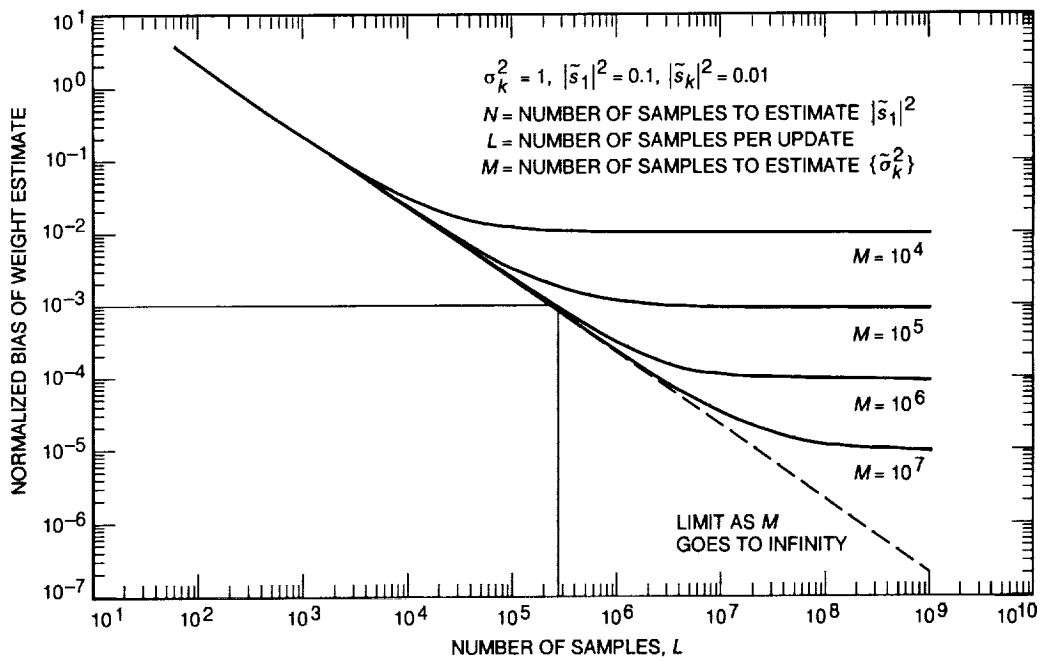**Fig. 2. Signal-combining system (baseband model).**

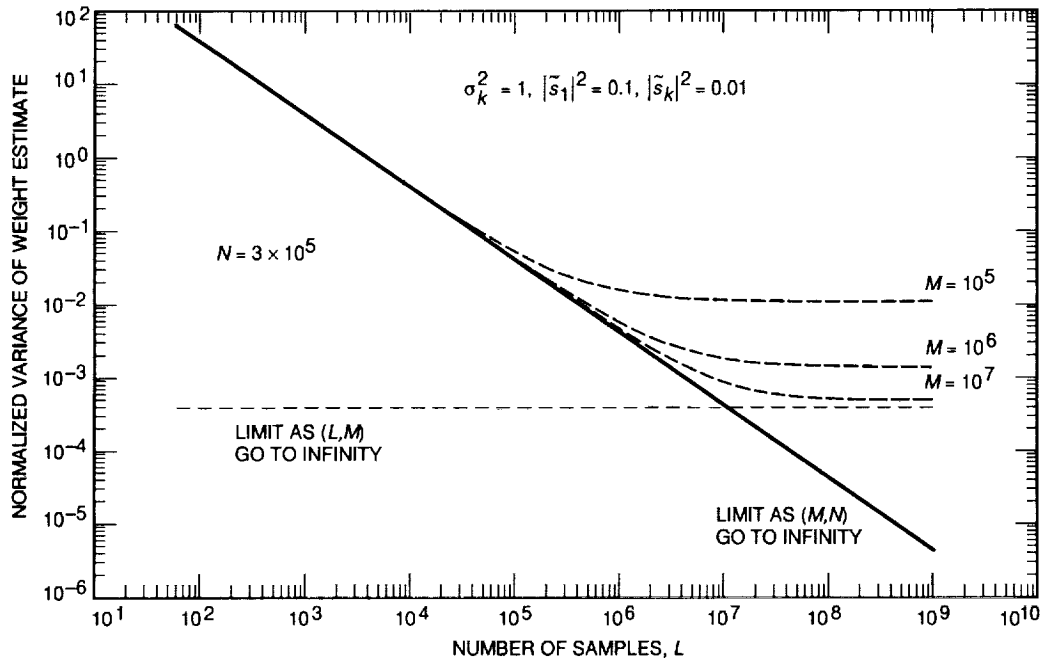**Fig. 3. Normalized bias versus number of samples (N).**
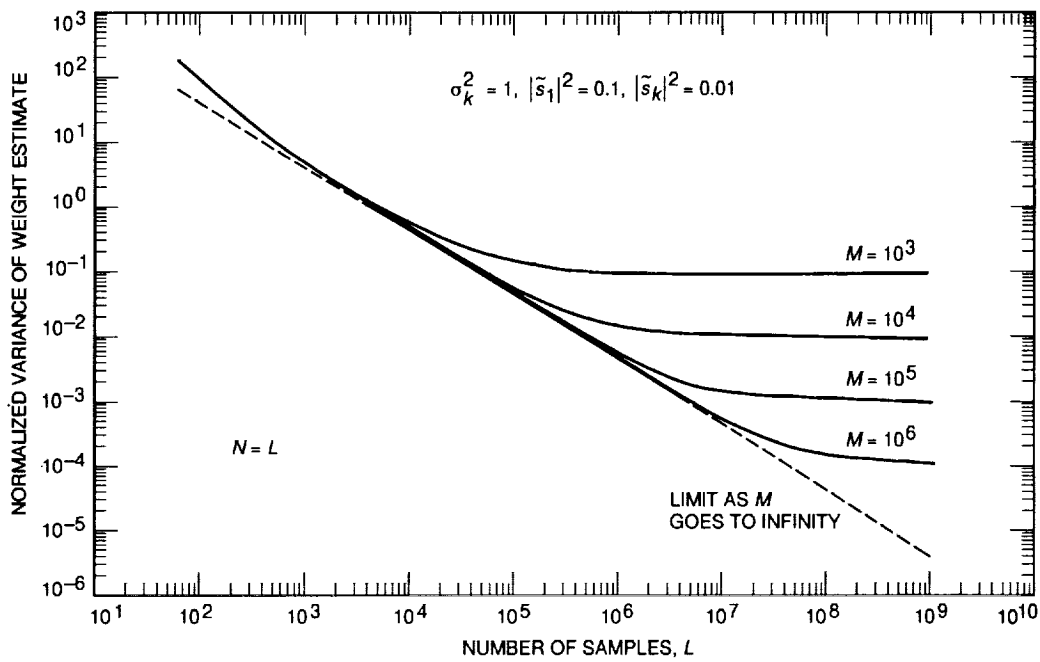
Fig. 4. Normalized variance versus number of samples (L).

Fig. 5. Normalized variance versus number of samples (N = L).