A Complex Symbol Signal-to-Noise Ratio Estimator and Its Performance

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This article presents an algorithm for estimating the signal-to-noise ratio (SNR) of signals that contain data on a downconverted suppressed carrier or the first harmonic of a square-wave subcarrier. This algorithm can be used to determine the performance of the full-spectrum combiner for the Galileo S-band (2.2- to 2.3-GHz) mission by measuring the input and output symbol SNR. A performance analysis of the algorithm shows that the estimator can estimate the complex symbol SNR using 10,000 symbols at a true symbol SNR of -5 dB with a mean of -4.9985 dB and a standard deviation of 0.2454 dB, and these analytical results are checked by simulations of 100 runs with a mean of -5.06 dB and a standard deviation of 0.2506 dB.

I. Introduction

Current plans are for the performance of the full-spectrum recorder (FSR) and full-spectrum combiner (FSC) for the Galileo S-band (2.2- to 2.3-GHz) mission to be measured by running symbol-error-rate (SER) curves on the demodulated data. To do so, one needs to generate test signals at various points of the system, and the test signals must be consistent from one point to the other, which may be difficult. The measurement can only be done at the output of the demodulator, which means that if the demodulator is malfunctioning, other modules (e.g., the FSR or FSC) cannot be tested. Also, since test signals are needed, online testing is impossible.

To overcome the disadvantages of measuring the performance from the SER curves, one would like to directly measure the symbol signal-to-noise ratio (SNR) at various points of the system. This means that the symbol SNR has to be estimated through complex symbols. By complex symbols, we mean the real symbols at an offset frequency with in-phase (I) and quadrature (Q) components. This article presents a complex symbol SNR estimator that estimates the symbol SNR of data with I and Q components of a carrier and/or the first harmonic of a square-wave subcarrier. This SNR estimator can be used to measure the performance of the FSC as well as that of the complex symbol combiner for the Galileo S-band mission.

The technique of the complex symbol SNR estimator is similar to that of the split-symbol moments estimator [1,2], except with complex symbols rather than real ones. The idea behind splitting a real symbol into two halves and correlating them (originally suggested by Larry Howard [2]), is to get the signal power by correlating two halves of a symbol, where the signal parts of the two halves of

the same symbol are correlated but the noises on the two halves of the symbol are uncorrelated.

The same idea is applied here to the complex symbols, with the assumption that the offset frequency is known. The idea is to correlate the first half of a symbol with the complex conjugate of the second half of the same symbol, and then to take the real part to get the signal power multiplied by a factor that is a function of the offset frequency. This factor can be divided out if the offset frequency is known. The same step is repeated for N symbols, and the results are averaged to get a better estimate of the signal power.

The signal power plus the noise power can be obtained by averaging the magnitudes squared of the sums of the samples in a complex symbol over N symbols. The estimate of the SNR ratio is then readily obtained. In the next section, a brief description of the complex symbol SNR estimator is given.

Some practical problems are not considered here. For instance, the frequency offset may not be known and may not be a constant. Also, if the symbol synchronization is off, say by a sample, the performance of the estimator will be affected, and in the future, the effect needs to be quantified.

II. Brief Description of the Estimator

For given samples of I and Q components of baseband data with an offset frequency ω_o, we wish to estimate the symbol SNR. The I and Q components of the lth sample in the kth symbol are expressed as follows:

$$y_{I_{lk}} = \frac{m}{N_s} d_k \cos{(\omega_o l T_s + \phi_o)} + n_{I_{lk}} \qquad (1)$$

$$y_{Q_{lk}} = \frac{m}{N_s} d_k \sin{(\omega_o l T_s + \phi_o)} + n_{Q_{lk}} \qquad (2)$$

where

$$\begin{aligned} l &= 0, ..., N_s - 1 \\ k &= 1, ..., N \end{aligned}$$

Here N_s is the number of samples per symbol and is assumed to be an even integer, T_s is the sample period, and $d_k = \pm 1$ is the kth symbol. The noise samples in the I and Q channels are assumed to be independent with zero mean and equal variance σ_n^2 / N_s. The true symbol SNR is

$$\text{SNR} = \frac{m^2}{2\sigma_n^2} \qquad (3)$$

Presented here is an algorithm to estimate this symbol SNR. First, add all samples in the first half of a symbol for I and Q separately,

$$Y_{\alpha I_k} = \sum_{l=0}^{N_s/2 - 1} y_{I_{lk}} \qquad (4)$$

$$Y_{\alpha Q_k} = \sum_{l=0}^{N_s/2 - 1} y_{Q_{lk}} \qquad (5)$$

Repeat the first step for the second half of the symbol

$$Y_{\beta I_k} = \sum_{l=N_s/2}^{N_s - 1} y_{I_{lk}} \qquad (6)$$

$$Y_{\beta Q_k} = \sum_{l=N_s/2}^{N_s - 1} y_{Q_{lk}} \qquad (7)$$

Multiply the I component of the first half of a symbol by that of the second half; do the same for the Q components; and add the results. This is equivalent to taking the real part of the product of the first half and the complex conjugate of the second half of a symbol. Repeat the same procedure for N symbols, and average the results to obtain the parameter m_p

$$m_p = \frac{1}{N} \sum_{k=1}^{N} (Y_{\alpha I_k} Y_{\beta I_k} + Y_{\alpha Q_k} Y_{\beta Q_k})$$

$$= \frac{1}{N} \sum_{k=1}^{N} \Re \left\{ Y_{\alpha_k} Y_{\beta_k}^* \right\} \qquad (8)$$

where

$$Y_{\alpha_k} = Y_{\alpha I_k} + j Y_{\alpha Q_k} \qquad (9)$$

$$Y_{\beta_k} = Y_{\beta I_k} + j Y_{\beta Q_k} \qquad (10)$$

Take the magnitude squared of the complex sum of the two symbol halves, repeat for N symbols, and average the results to obtain m_{ss}

$$m_{ss} = \frac{1}{N} \sum_{k=1}^{N} |Y_{\alpha_k} + Y_{\beta_k}|^2 \qquad (11)$$

Mathematical expressions for the complex symbol halves Y_{α_k} and Y_{β_k} and the parameters m_p and m_{ss} are given in Appendix A in terms of the underlying model parameters of Eqs. (1) and (2). Finally, use m_p and m_{ss} to obtain the estimated symbol SNR

$$\mathrm{SNR}^* =$$

$$\frac{4m_p \, \mathrm{sinc}^2 \, (\omega_o T_s/2) \, / \, [\mathrm{sinc}^2 \, (\omega_o T_{sy}/4) \cos \, (\omega_o T_{sy}/2)]}{m_{ss} - m_p \, (2 + 2/\cos \, (\omega_o T_{sy}/2))}$$

$$(12)$$

where ω_o is the offset angular frequency, and T_{sy} is the symbol duration,

$$T_{sy} = N_s T_s$$

The estimate, SNR^*, defined in Eq. (12) is shown in the next section to have an expected value equal to the true symbol SNR defined in Eq. (3), plus some bias that decreases with the number of symbols averaged. The variance of SNR^* is also derived in the next section as a function of the number of symbols averaged and the true symbol SNR.

III. Mean and Variance of Symbol SNR Estimates

In Eq. (12), the estimate of the complex symbol SNR, SNR^*, is a function of the random variables m_p and m_{ss}. Let g denote that function

$$\mathrm{SNR}^* = g(m_p, m_{ss}) \qquad (13)$$

Assume that $g(m_p, m_{ss})$ is "smooth" in the vicinity of the point $(\overline{m}_p, \overline{m}_{ss})$, where $\overline{m}_p$ and $\overline{m}_{ss}$ are the means of m_p and m_{ss}, respectively. The expectation of the estimate, SNR^*, can then be approximated by [3, p. 212]

$$E\left\{\mathrm{SNR}^*\right\} \simeq g\left(\overline{m}_p, \overline{m}_{ss}\right) + \frac{1}{2}$$

$$\times \left(\sigma_{m_p}^2 \frac{\partial^2 g}{\partial m_p^2} + 2 \, \mathrm{cov} \, (m_p, m_{ss}) \frac{\partial^2 g}{\partial m_p \partial m_{ss}} + \sigma_{m_{ss}}^2 \frac{\partial^2 g}{\partial m_{ss}^2}\right)$$

$$(14)$$

where $\sigma_{m_p}^2$ and $\sigma_{m_{ss}}^2$ are the variances of m_p and m_{ss}, respectively, and $\mathrm{cov} \, (m_p, m_{ss})$ is their covariance. The variance of the estimate, SNR^*, can be approximated by [3, p. 212]

$$\sigma_{\mathrm{SNR}^*}^2 \simeq \left(\frac{\partial g}{\partial m_p}\right)^2 \sigma_{m_p}^2 + \left(\frac{\partial g}{\partial m_{ss}}\right)^2 \sigma_{m_{ss}}^2$$

$$+ \, 2\frac{\partial g}{\partial m_p} \frac{\partial g}{\partial m_{ss}} \, \mathrm{cov} \, (m_p, m_{ss}) \qquad (15)$$

In Eqs. (14) and (15), all the partial derivatives and the covariance are evaluated at $(\overline{m}_p, \overline{m}_{ss})$. Expressions for the mean and variance of m_p, the mean and variance of m_{ss}, the covariance of m_p and m_{ss}, and the partial derivatives are derived in Appendices B through E, respectively. Substituting all the terms from the appendices into Eqs. (14) and (15), the mean and variance of SNR^* are obtained,

$$E\left\{\mathrm{SNR}^*\right\} = \mathrm{SNR} + \frac{1}{N \cos^2 (z/2)}$$

$$\times \left[\frac{1}{C_1^2} + \mathrm{SNR}\left(\frac{5}{4} - \frac{1}{4}\cos (z)\right)\right.$$

$$\left. + \, \mathrm{SNR}^2 \frac{C_1^2}{4} (1 - \cos (z))\right] \qquad (16)$$

and

$$\sigma_{\mathrm{SNR}^*}^2 = \frac{1}{N \cos^2(z/2)} \left[\frac{2}{C_1^4} + \mathrm{SNR}\frac{4}{C_1^2} + \mathrm{SNR}^2\right.$$

$$\left. \times \left(\frac{7}{4} - \frac{3}{4}\cos (z)\right) \mathrm{SNR}^3 \frac{C_1^2}{4} (1 - \cos (z))\right]$$

$$(17)$$

where

$$z = \omega_o T_{sy}$$

and

$$C_1 = \frac{\text{sinc}\ (\omega_o T_{sy}/4)}{\text{sinc}\ (\omega_o T_s/2)}$$

IV. Analysis Verification

Two methods are used to verify the analytical results of the mean and the variance in the complex symbol SNR estimation:

(1) Compare the numerical results of the analysis to simulation results.

(2) Set the offset frequency to zero and compare the performance of the complex symbol SNR estimator and that of the real symbol SNR estimator.

A. Simulation Results

Simulation results for the mean and variance of SNR* are shown in Figs. (1) and (2), and are compared to the theoretical results of Eqs. (16) and (17). The analytical and simulation results are also compared in Table 1. In all cases, the offset frequency, $f_o = \omega_o/(2\pi)$, is set at 20 Hz. The simulation results are based on 100 runs each, and the simulated means and variances are obtained by averaging over the 100 runs.

B. Special Case

A special case is when the offset frequency $\omega_o = 0$. In this case, the performance of the complex symbol SNR estimator should reduce to that of the real symbol SNR estimator given in [1] and [5], with N samples of pure noise and N samples of the signal plus noise.

For this special case, the mean and variance of the complex symbol SNR estimates reduce to

$$E\{\text{SNR}^*\} = \text{SNR} + \frac{1}{N}(1 + \text{SNR}) \qquad (18)$$

$$\sigma^2_{\text{SNR}^*} = \frac{1}{N}(2 + 4\text{SNR} + \text{SNR}^2) \qquad (19)$$

The analysis in [1], which assumed a constant signal level for all the samples, is easily extended to cover the case of a signal in only half the samples. The real symbol SNR and the real estimate, $E\{\text{SNR}^*\}$, in Eqs. (17a) and (17b) of [1] are simply replaced by values $\overline{\text{SNR}}$ and $\overline{\text{SNR}^*}$, respectively, representing averages over the $2N$ real samples. These average values are one-half the corresponding complex SNR and complex estimate SNR* defined here. With these correspondences, the mean of the real estimate in [1] reduces to

$$E\{\text{SNR}^*\} = \text{SNR} + \frac{1}{N-1}(1 + \text{SNR}) \qquad (20)$$

and the variance of the real estimate is

$$\sigma^2_{\text{SNR}^*} = \left(\frac{N}{N-1}\right)^2 \frac{1}{N-2}$$

$$\times \left[(\text{SNR}^2 + 2\text{SNR} + 1) + \frac{N-1}{N}(2\text{SNR} + 1)\right]$$

$$(21)$$

For a large N, Eq. (20) is equivalent to

$$E\{\text{SNR}^*\} = \text{SNR} + \frac{1}{N}(1 + \text{SNR}) \qquad (22)$$

and Eq. (21) approximates to

$$\sigma^2_{\text{SNR}^*} = \frac{1}{N}(\text{SNR}^2 + 4\text{SNR} + 2) \qquad (23)$$

Comparing Eq. (18) with Eq. (22), and Eq. (19) with Eq. (23), it is clear that the special case of the complex symbol SNR estimator has the same performance as the real symbol SNR with N signal-plus-noise samples and N pure noise samples.

V. Conclusions

A complex symbol signal-to-noise ratio estimator is presented in this article. This estimator modifies the split-symbol moments estimator for estimating the real symbol SNR [2] in order to accommodate complex symbols, with the assumption that the offset frequency is known. This estimator can be used to measure the performance of the full spectrum combiner as well as the complex symbol combiner for the Galileo S-band mission.

Acknowledgments

Table 1. Simulation results of the complex symbol SNR estimation.

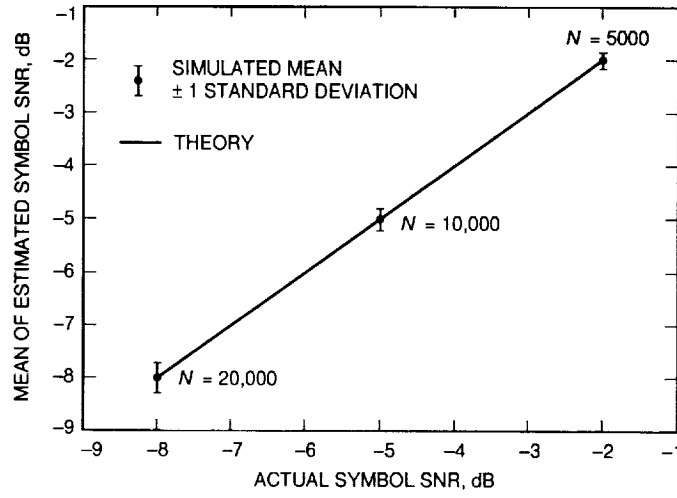| SNR, dB	Mean of SNR*, dB		σ of SNR*, dB		N	N_s	Runs
	Theory	Simulation	Theory	Simulation			
-2	-1.9968	-2.0039	0.2956	0.2834	2,500	10	100
-2	-1.9984	-2.0136	0.2112	0.1937	5,000	10	100
-5	-4.9985	-5.0589	0.2454	0.2506	10,000	10	100
-8	-7.9986	-8.0437	0.3056	0.2787	20,000	10	100

Fig. 1. Means and standard deviations of the symbol SNR estimates.
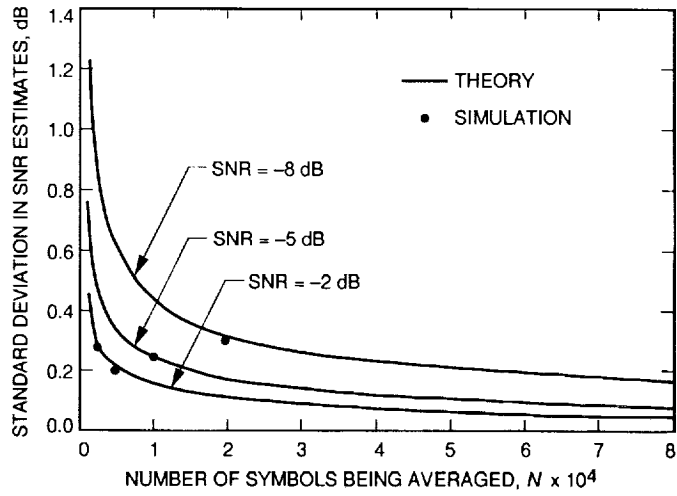

Fig. 2. Standard deviations of the complex symbol SNR estimates.

Appendix A

Sums of the Halves of a Symbol and Definitions of m_p and m_{ss}

The sum of the first half of the kth symbol is

$$
\begin{aligned}
Y_{\alpha_k} &= \sum_{l=0}^{N_s/2-1} \frac{m}{N_s} d_k e^{j(\omega_o T_s l + \phi_o)} + n_{\alpha_k} \\
&= \frac{m}{N_s} d_k e^{j\phi_o} \frac{1 - e^{j\omega_o T_s N_s/2}}{1 - e^{j\omega_o T_s}} + n_{\alpha_k} \\
&= \frac{1}{2} m d_k e^{j(1/4\omega_o Tsy \,-\, 1/2\omega_o T_s \,+\, \phi_o)} \frac{\mathrm{sinc}\,(\omega_o T_{sy}/4)}{\mathrm{sinc}\,(\omega_o T_s/2)} \\
&\quad + n_{\alpha_k}
\end{aligned}
\tag{A-1}
$$

and the sum of the second half of the kth symbol is

$$
\begin{aligned}
Y_{\beta_k} &= \sum_{l=N_s/2}^{N_s-1} \frac{m}{N_s} d_k e^{j(\omega_o T_s l + \phi_o)} + n_{\beta_k} \\
&= \frac{m}{N_s} d_k e^{j(\omega_o T_s N_s/2 + \phi_o)} \frac{1 - e^{j(\omega_o T_s N_s/2)}}{1 - e^{j\omega_o T_s}} + n_{\beta_k} \\
&= \frac{1}{2} m d_k e^{j(3/4\omega_o Tsy \,-\, 1/2\omega_o T_s \,+\, \phi_o)} \frac{\mathrm{sinc}\,(\omega_o T_{sy}/4)}{\mathrm{sinc}\,(\omega_o T_s/2)} \\
&\quad + n_{\beta_k}
\end{aligned}
\tag{A-2}
$$

where the noise terms, n_{α_k} and n_{β_k}, are complex Gaussian noises, with each of the components (real and imaginary) having a variance of $\sigma_n^2/2$. In terms of the in-phase and quadrature noise samples in Eqs. (1) and (2), the complex noise terms, n_{α_k} and n_{β_k}, are

$$
n_{\alpha_k} = \sum_{l=0}^{N_s/2-1} (n_{I_{lk}} + j n_{Q_{lk}})
\tag{A-3}
$$

and

$$
n n_{\beta_k} = \sum_{l=N_s/2}^{N_s-1} (n_{I_{lk}} + j n_{Q_{lk}})
\tag{A-4}
$$

where $j = \sqrt{-1}$.

Let s_{α_k} and s_{β_k} denote the signal parts of Y_{α_k} and Y_{β_k}, respectively. Then Y_{α_k} and Y_{β_k} can be expressed as

$$
Y_{\alpha_k} = s_{\alpha_k} + n_{\alpha_k}
\tag{A-5}
$$

and

$$
Y_{\beta_k} = s_{\beta_k} + n_{\beta_k}
\tag{A-6}
$$

The product of the sum of the first half of the kth symbol and the conjugate of the sum of the second half of the same symbol is

$$
\begin{aligned}
Y_{\alpha_k} Y_{\beta_k}^* &= s_{\alpha_k} s_{\beta_k}^* + s_{\alpha_k} n_{\beta_k}^* + n_{\alpha_k} s_{\beta_k}^* + n_{\alpha_k} n_{\beta_k}^* \\
&= \frac{1}{4} m^2 e^{j(-\omega_o T_{sy}/2)} \frac{\mathrm{sinc}^2(\omega_o T_{sy}/4)}{\mathrm{sinc}^2(\omega_o T_s/2)} \\
&\quad + s_{\alpha_k} n_{\beta_k}^* + n_{\alpha_k} s_{\beta_k}^* + n_{\alpha_k} n_{\beta_k}^*
\end{aligned}
\tag{A-7}
$$

Taking the real part of the product, one obtains

$$
\begin{aligned}
\Re\{Y_{\alpha_k} Y_{\beta_k}^*\} &= \frac{1}{4} m^2 \cos\left(\frac{\omega_o T_{sy}}{2}\right) \frac{\mathrm{sinc}^2\,(\omega_o T_{sy}/4)}{\mathrm{sinc}^2(\omega_o T_s/2)} \\
&\quad + \Re\{s_{\alpha_k} n_{\beta_k}^* + n_{\alpha_k} s_{\beta_k}^* + n_{\alpha_k} n_{\beta_k}^*\}
\end{aligned}
\tag{A-8}
$$

Define m_p as

$$
m_p = \sum_{k=1}^{N} \Re\{Y_{\alpha_k} Y_{\beta_k}^*\}
\tag{A-9}
$$

The sum of the whole kth symbol is

$$Y_k = \sum_{l=0}^{N_s-1} \frac{m}{N_s} d_k e^{j(\omega_o T_s l + \phi_o)} + n_k$$

$$= \frac{m}{N_s} d_k e^{j\phi_o} \frac{1 - e^{j\omega_o T_s N_s}}{1 - e^{j\omega_o T_s}} + n_k$$

$$= m d_k e^{j\left(1/2\omega_o T_{sy} - 1/2\omega_o T_s + \phi_o\right)} \frac{\text{sinc}\left(\omega_o T_{sy}/2\right)}{\text{sinc}\left(\omega_o T_s/2\right)} + n_k$$

$$= s_k + n_k \tag{A-10}$$

where s_k is the signal part of the whole symbol and n_k is a complex Gaussian noise with a variance of σ_n^2 in each component (real and imaginary), and

$$s_k = s_{\alpha_k} + s_{\beta_k}$$

$$n_k = n_{\alpha_k} + n_{\beta_k}$$

Note that the signal and noise parts of a whole complex symbol, s_k and n_k, can also be expressed in terms of their in-phase and quadrature components, that is,

$$s_k = s_{I_k} + j s_{Q_k} \tag{A-11}$$

$$n_k = n_{I_k} + j n_{Q_k} \tag{A-12}$$

where s_{I_k}, s_{Q_k} and n_{I_k}, n_{Q_k} denote the I and Q components of the signal and the noise in the kth symbol. Furthermore, Y_k can be expressed as

$$Y_k = s_{I_k} + j s_{Q_k} + n_{I_k} + j n_{Q_k} \tag{A-13}$$

The sum squared of the kth symbol is

$$|Y_k|^2 = m^2 \frac{\text{sinc}^2\left(\omega_o T_{sy}/2\right)}{\text{sinc}^2\left(\omega_o T_s/2\right)} + |n_k|^2 + 2\Re\{n_k s_k^*\}$$

$$= m^2 C_2^2 + |n_k|^2 + 2\Re\{n_k s_k^*\} \tag{A-14}$$

where

$$C_2^2 = \frac{\text{sinc}^2\left(\omega_o T_{sy}/2\right)}{\text{sinc}^2(\omega_o T_s/2)}$$

Define m_{ss} as

$$m_{ss} = \sum_{k=1}^{N} |Y_k|^2 \tag{A-15}$$

Appendix B

Evaluation of the Mean and Variance of m_p

All the expectations in the following are conditional on knowing the symbol synchronization.

$$E\{m_p\} = \frac{1}{N} \sum_{k=1}^{N} E\{U_k\}$$

$$= \frac{1}{4} m^2 \frac{\text{sinc}^2 \left(\omega_o T_{sy}/4\right)}{\text{sinc}^2 \left(\omega_o T_s/2\right)} \cos\left(\frac{\omega_o T_{sy}}{2}\right)$$

$$+ \frac{1}{N} \sum_{k=1}^{N} E\{\Re\{s_{\alpha_k} n_{\beta_k}^* + n_{\alpha_k} s_{\beta_k}^* + n_{\alpha_k} n_{\beta_k}^*\}\}$$

$$= \frac{1}{4} m^2 \frac{\text{sinc}^2 \left(\omega_o T_{sy}/4\right)}{\text{sinc}^2 \left(\omega_o T_s/2\right)} \cos\left(\frac{\omega_o T_{sy}}{2}\right) \qquad \text{(B-1)}$$

where

$$U_k = \Re\{Y_{\alpha_k} Y_{\beta_k}^*\}$$

$$= \Re\{(s_{\alpha_k} + n_{\alpha_k})(s_{\beta_k}^* + n_{\beta_k}^*)\}$$

and s_{α_k} and s_{β_k} denote the signal part of Y_{α_k} and Y_{β_k}, respectively, as in Eqs. (A-5) and (A-6). The last step in Eq. (B-1) follows from the independence of the noises with zero mean in the two halves of each symbol. Note that $T_{sy} = N_s T_s$.

Because the $\{U_k\}$ are independent, the variance of m_p can be computed as [4, p. 352]

$$\sigma_{m_p}^2 = \frac{1}{N^2} \sum_{k=1}^{N} \sigma_{U_k}^2 \qquad \text{(B-2)}$$

Now let subscripts I and Q denote the in-phase and quadrature components of the signals. The second moment of U_k can be expressed as

$$E\{U_k^2\} = \frac{\sigma_n^2}{2} E\left\{s_{\beta I_k}^2 + s_{\alpha I_k}^2 + s_{\beta Q_k}^2 + s_{\alpha Q_k}^2\right\}$$

$$+ \frac{\sigma_n^4}{2} + E\{(s_{\alpha I_k} s_{\beta I_k} + s_{\alpha Q_k} s_{\beta Q_k})^2\}$$

$$= \frac{m^2}{2} \frac{\text{sinc}^2(\omega_o T_{sy}/4)}{\text{sinc}^2(\omega_o T_s/2)} \frac{\sigma_n^2}{2} + \frac{\sigma_n^4}{2}$$

$$+ E\{(s_{\alpha I_k} s_{\beta I_k} + s_{\alpha Q_k} s_{\beta Q_k})^2\} \qquad \text{(B-3)}$$

Combining Eqs. (B-1), (B-3), and (B-2) yields

$$\sigma_{m_p}^2 = \frac{1}{N} \left(\frac{1}{4} m^2 \frac{\text{sinc}^2 \left(\omega_o T_{sy}/4\right)}{\text{sinc}^2 \left(\omega_o T_s/2\right)} \sigma_n^2 + \frac{\sigma_n^4}{2}\right) \qquad \text{(B-4)}$$

Appendix C

Evaluation of the Mean and Variance of m_{ss}

The mean of m_{ss} is

$$E\{m_{ss}\} = E\left\{\frac{1}{N}\sum_{k=1}^{N}|Y_k|^2\right\}$$

$$= m^2 C_2^2 + \frac{1}{N}\sum_{k=1}^{N}E|n_k|^2 + \frac{1}{N}\sum_{k=1}^{N}E\{\Re\{n_k s_k^*\}\}$$

$$= m^2 C_2^2 + 2\sigma_n^2 \tag{C-1}$$

To find the variance of m_{ss}, let

$$V_k = |Y_k|^2$$

$$= Y_{I_k}^2 + Y_{Q_k}^2$$

$$= (s_{I_k} + n_{I_k})^2 + (s_{Q_k} + n_{Q_k})^2 \tag{C-2}$$

where Y_{I_k}, Y_{Q_k}, s_{I_k}, s_{Q_k}, n_{I_k}, and n_{Q_k} are defined in Appendix A. Because the $\{V_k\}$ are independent, the variance of m_{ss} can be expressed in terms of the variance of V_k [4, p. 352],

$$\sigma_{m_{ss}}^2 = \frac{1}{N^2}\sum_{k=1}^{N}\sigma_{V_k}^2 \tag{C-3}$$

$$\sigma_{V_k}^2 = E\{V_k^2\} - (E\{V_k\})^2 \tag{C-4}$$

where

$$E^2\{V\} = E^2\{m_{ss}\}$$

$$= (m^2 C_2^2 + 2\sigma_n^2)^2 \tag{C-5}$$

Expanding V_k^2, take the expectation of V_k^2; apply $E\{n_{I_k}^2\} = E\{n_{Q_k}^2\} = \sigma_n^2$, $E\{n_{I_k}^3\} = E\{n_{Q_k}^3\} = 0$, and $E\{n_{I_k}^4\} = E\{n_{Q_k}^4\} = 3\sigma_n^4$ [3, p. 147]; and use the independence of n_{I_k} and n_{Q_k}, and one will obtain

$$E\{V_k^2\} = 8\sigma_n^4 + 8\sigma_n^2(s_{I_k}^2 + s_{Q_k}^2) + (s_{I_k}^2 + s_{Q_k}^2)^2$$

$$\times\; 8\sigma_n^4 + 8m^2 C_2^2\sigma_n^2 + m^4 C_2^4 \tag{C-6}$$

The variance of V_k becomes

$$\sigma_{V_k}^2 = 4m^2 C_2^2\sigma_n^2 + 4\sigma_n^4 \tag{C-7}$$

Finally, the variance of m_{ss} can be written as

$$\sigma_{m_{ss}}^2 = \frac{1}{N}\left[4m^2 C_2^2\sigma_n^2 + 4\sigma_n^4\right] \tag{C-8}$$

Appendix D

Evaluation of the Covariance of m_p and m_{ss}

The covariance of m_p and m_{ss} is defined as

$$\text{cov}\,(m_p, m_{ss}) = E\{(m_p - \overline{m}_p)(m_{ss} - \overline{m}_{ss})\}$$

$$= E\{m_p m_{ss}\} - \overline{m}_p \overline{m}_{ss} \qquad (\text{D-1})$$

As in Appendices B and C, let

$$U_k = \Re\{Y_{\alpha_k} Y_{\beta_k}^*\}$$

$$= Y_{\alpha I_k} Y_{\beta I_k} + Y_{\alpha Q_k} Y_{\beta Q_k}$$

$$= (s_{\alpha_{I_k}} + n_{\alpha_{I_k}})(s_{\beta_{I_k}} + n_{\beta_{I_k}})$$

$$+ (s_{\alpha_{Q_k}} + n_{\alpha_{Q_k}})(s_{\beta_{Q_k}} + n_{\beta_{Q_k}}) \qquad (\text{D-2})$$

and

$$V_k = |Y_k|^2$$

$$= s_{I_k}^2 + 2s_{I_k} n_{I_k} + n_{I_k}^2 + s_{Q_k}^2 + 2s_{Q_k} n_{Q_k} + n_{Q_k}^2$$

$$(\text{D-3})$$

The expectation of the product of $U_k V_k$ can be expressed as

$$E\{U_k V_k\} =$$

$$\sigma_n^4 + 2\sigma_n^2 E\{s_{\alpha_{I_k}} s_{\beta_{I_k}} + s_{\alpha_{Q_k}} s_{\beta_{Q_k}}\} + E\{s_{I_k}^2 + s_{Q_k}^2\}\sigma_n^2$$

$$+ E\{(s_{I_k}^2 + s_{Q_k}^2)(s_{\alpha_{I_k}} s_{\beta_{I_k}} + s_{\alpha_{Q_k}} s_{\beta_{Q_k}})\}$$

$$= \sigma_n^4 + 2\sigma_n^2 \left(\frac{1}{4} m^2 C_1^2 \cos \frac{z}{2}\right)$$

$$+ \sigma_n^2 (m^2 C_2^2) + m^2 C_2^2 \left(\frac{1}{4} m^2 C_1^2 \cos \frac{z}{2}\right) \qquad (\text{D-4})$$

where

$$C_1 = \frac{\text{sinc}\,(\omega_o T_{sy}/4)}{\text{sinc}\,(\omega_o T_s/2)}$$

$$C_2 = \frac{\text{sinc}\,(\omega_o T_{sy}/2)}{\text{sinc}\,(\omega_o T_s/2)}$$

and

$$z = \omega_o T_{sy}$$

Note that

$$C_2^2 = \frac{1}{2} C_1^2 \left[1 + \cos\left(\frac{\omega_o T_{sy}}{2}\right)\right]$$

Then

$$E\{m_p m_{ss}\} =$$

$$E\left\{\frac{1}{N}\sum_{k=1}^{N} U_k \frac{1}{N}\sum_{l=1}^{N} V_l\right\}$$

$$= \frac{1}{N^2}\left[\sum_{k=1}^{N} E\{U_k V_k\} + \sum_{k=1}^{N} E\{U_k\} \sum_{l=1, l\neq k}^{N} E\{V_l\}\right]$$

$$= \frac{1}{N^2}\left[N E\{U_k V_k\} + N\overline{m}_p(N-1)\overline{m}_{ss}\right] \qquad (\text{D-5})$$

Finally, the covariance of m_p and m_{ss} is

$$\text{cov}\,(m_p, m_{ss}) = \frac{1}{N}[E\{U_k V_k\} - \overline{m}_p \overline{m}_{ss}]$$

$$= \frac{1}{N}\left[m^2 C_2^2 \sigma_n^2 + \sigma_n^4\right] \qquad (\text{D-6})$$

Appendix E

Evaluation of the Partial Derivatives

Denote the following parameters in a convenient form,

$$K_1 = \frac{4\text{sinc}^2\ (\omega_o T_s/2)}{\text{sinc}^2\ (\omega_o T_{sy}/4)\ \cos\ (\omega_o T_{sy}/2)}$$

$$K_2 = \frac{2}{\cos\ (\omega_o T_{sy}/2)} + 2$$

and

$$C_1 = \frac{\text{sinc}\ (\omega_o T_{sy}/4)}{\text{sinc}\ (\omega_o T_s/2)}$$

$$z = \omega_o T_{sy}$$

In terms of K_1 and K_2, the estimate SNR* defined in Eq. (12) is given by

$$\text{SNR}^* = g(m_p, m_{ss})$$

$$= \frac{K_1 m_p}{m_{ss} - K_2 m_p} \qquad \text{(E-1)}$$

Then $g(\overline{m}_p, \overline{m}_{ss}) = \text{SNR}$, and the partial derivatives of this estimator function are

$$\left.\frac{\partial g}{\partial m_p}\right|_{\overline{m}_p, \overline{m}_{ss}} = \frac{K_1 K_2 \overline{m}_p}{(\overline{m}_{ss} - K_2 \overline{m}_p)^2} + \frac{K_1}{\overline{m}_{ss} - K_2 \overline{m}_p} =$$

$$\frac{1}{\sigma_n^2 \cos\ (z/2)}\left[\text{SNR}\left(1 + \cos\frac{z}{2}\right) + \frac{2}{C_1^2}\right]$$

$$\text{(E-2)}$$

$$\left.\frac{\partial^2 g}{\partial m_p^2}\right|_{\overline{m}_p, \overline{m}_{ss}} = \frac{2K_1 K_2^2 \overline{m}_p}{(\overline{m}_{ss} - K_2 \overline{m}_p)^3} + \frac{2K_1 K_2}{(\overline{m}_{ss} - K_2 \overline{m}_p)^2} =$$

$$\frac{2}{\sigma_n^4}\frac{\cos^2(z/4)}{\cos^2(z/2)}\left[2\text{SNR}\left(1 + \cos\frac{z}{2}\right) + \frac{4}{C_1^2}\right] \qquad \text{(E-3)}$$

$$\left.\frac{\partial g}{\partial m_{ss}}\right|_{\overline{m}_p, \overline{m}_{ss}} = -\frac{K_1 \overline{m}_p}{(\overline{m}_{ss} - K_2 \overline{m}_p)^2}$$

$$= -\frac{1}{2\sigma_n^2}\text{SNR} \qquad \text{(E-4)}$$

$$\left.\frac{\partial^2 g}{\partial m_{ss}^2}\right|_{\overline{m}_p, \overline{m}_{ss}} = \frac{2K_1 \overline{m}_p}{(\overline{m}_{ss} - K_2 \overline{m}_p)^3}$$

$$= \frac{1}{2\sigma_n^4}\text{SNR} \qquad \text{(E-5)}$$

$$\left.\frac{\partial^2 g}{\partial m_p \partial m_{ss}}\right|_{\overline{m}_p, \overline{m}_{ss}} =$$

$$-\frac{2K_1 K_2 \overline{m}_p}{(\overline{m}_{ss} - K_2 \overline{m}_p)^3} - \frac{K_1}{(\overline{m}_{ss} - K_2 \overline{m}_p)^2} =$$

$$-\frac{1}{\cos(z/2)\sigma_n^4}\left[\text{SNR}\left(1 + \cos\frac{z}{2}\right) + \frac{1}{C_1^2}\right] \qquad \text{(E-6)}$$

where the second lines of Eqs. (E-2) through (E-6) are obtained by substituting the expressions for $\overline{m}_p = E\{m_p\}$ and $\overline{m}_{ss} = E\{m_{ss}\}$ from Appendices B and C.

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