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IDENTIFICATION AND CONTROL OF STRUCTURES IN SPACE

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Submitted to

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Abstract

During the last phase of the project, emphasis has changed to flexible space robotics, by mutual agreement between Dr. R. C. Montgomery, NASA Technical Officer, and the Principal Investigator.

Significant advances have been achieved over the period covered by this report. Research has been concerned with two main subjects: 1) the maneuvering and control of freely floating flexible space robots and 2) the development of a theory for the motion of flexible multibody systems. Work on the first subject has resulted in two papers, both of them concerned with planar maneuvers. The first is concerned with the maneuvering and delivery of a payload to a certain point and in a certain orientation in space. The second is concerned with the docking maneuver with a target whose motion is not known a priori. Both papers will appear in the *Journal of Guidance, Control, and Dynamics.* The second subject is concerned with the development of hybrid (ordinary and partial) differential equations for the three-dimensional motion of flexible multibody systems, a subject of vital interest in flexible space robotics. The paper will appear in the *Journal of Guidance, Control and Dynamics* in an issue dedicated to the memory of Lawrence W. Taylor, Jr.

Abstracts and copies of the papers are hereby included.

1. Meirovitch, L. and Lim, S., "Maneuvering and Control of Flexible Space Robots," *NASA Workshop on Distributed Parameter Modeling and Control of Flexible Aerospace Systems*, Williamsburg, VA, June 8-10, 1992. Also *Journal of Guidance, Control, and Dynamics* (in press).

This paper is concerned with a flexible space robot capable of maneuvering payloads. The robot is assumed to consist of two hinge-connected flexible arms and a rigid end-effector holding a payload; the robot is mounted on a rigid platform floating in space. The equations of motion are nonlinear and of high order. Based on the assumption that the maneuvering motions are one order of magnitude larger than the elastic vibrations, a perturbation approach permits design of controls for the two types of motion separately. The rigid-body maneuvering is carried out open loop, but the elastic motions are controlled closed loop, by means of discrete-time linear quadratic regulator theory with prescribed degree of stability. A numerical example demonstrates the approach. In the example, the controls derived by the perturbation approach are applied to the original nonlinear system and errors are found to be relatively small.

2. Chen, Y. and Meirovitch, L., "Control of a Flexible Space Robot Executing a Docking Maneuver," *AAS/AIAA Astrodynamics Conference*, Victoria, B.C., Canada, August 16-19, 1993. Also *Journal of Guidance, Control, and Dynamics* (to appear).

This paper is concerned with a flexible space robot executing a docking maneuver with a target whose motion is not known a priori. The dynamical equations of the space robot are first derived by means of Lagrange's equations and then separated into two sets of equations suitable for rigid-body maneuver and vibration suppression control. For the rigid-body maneuver, on-line feedback tracking control is carried out by means of an algorithm based on Liapunov-like methodology and using on-line measurements of the target motion. For the vibration suppression, LQR feedback control in conjunction with disturbance compensation is carried out by means of piezoelectric sensor/actuator pairs dispersed along the flexible arms. Problems related to the digital implementation of the control algorithms, such as the bursting phenomenon and system instability, are discussed and a modified discrete-time control scheme is developed. A numerical example demonstrates the control algorithms.

3. Meirovitch, L. and Stemple, T. "Hybrid Equations of Motion for Flexible Multibody Systems Using Quasi-Coordinates," *AIAA Guidance, Navigation, and Control Conference*, Monterey, CA, August 9-11, 1993. Also *Journal of Guidance, Control, and dynamics* - Issue dedicated to L. W. Taylor, Jr. (to appear).

A variety of engineering systems, such as automobiles, aircraft, rotorcraft, robots, spacecraft, etc., can be modeled as flexible multibody systems. The individual flexible bodies are in general characterized by distributed parameters. In most earlier investigations they were approximated by some spatial discretization procedure, such as the classical Rayleigh-Ritz method or the finite element method. This paper presents a mathematical formulation for distributed-parameter multibody systems consisting of a set of hybrid (ordinary and partial) differential equations of motion in terms of quasi-coordinates. Moreover, the equations for the elastic motions include rotatory inertia and shear deformation effects. The hybrid set is cast in state form, thus making it suitable for control design.

MANEUVERING AND CONTROL OF FLEXIBLE SPACE ROBOTS[†]

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ABSTRACT

This paper is concerned with a flexible space robot capable of maneuvering payloads. The robot is assumed to consist of two hinge-connected flexible arms and a rigid end-effector holding a payload; the robot is mounted on a rigid platform floating in space. The equations of motion are nonlinear and of high order. Based on the assumption that the maneuvering motions are one order of magnitude larger than the elastic vibrations, a perturbation approach permits design of controls for the two types of motion separately. The rigid-body maneuvering is carried out open loop, but the elastic motions are controlled closed loop, by means of discrete-time linear quadratic regulator theory with prescribed degree of stability. A numerical example demonstrates the approach. In the example, the controls derived by the perturbation approach are applied to the original nonlinear system and errors are found to be relatively small.

1. INTRODUCTION

A variety of space missions can be carried out effectively by space robots. These missions include the collection of space debris, recovery of spacecraft stranded in a useless orbit, repair of malfunctioning spacecraft, construction of a space station in orbit and servicing the space station while in operation. To maximize the usefulness of the space robot, the manipulator arms should be reasonably long. On the other hand, because of weight limitations, they must be relatively light. To satisfy both requirements, the manipulator arms must be

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highly flexible. Hence, space robots share some of the dynamics and control technology with articulated space structures.

Robotics has been an active research area for the past few decades, but applications have been concerned primarily with industrial robots, which are ground based and tend to be very stiff and bulky. In contrast, space robots are based on a floating platform and tend to be highly flexible. Hence, whereas industrial and space robots have a number of things in common, the differences are significant. More recent investigations have been concerned with flexible industrial robots.[1–4] On the other hand, some investigations are concerned with space robots consisting of rigid links.[5–7] Research on flexible space robots has come to light only recently.[8,9]

This paper is concerned with a flexible space robot capable of maneuvering payloads. The robot is assumed to consist of two hinge-connected flexible arms and a rigid end-effector holding a payload; the robot is mounted on a rigid platform floating in space (Fig. 1). The platform is capable of translations and rotations, the flexible arms are capable of rotations and elastic deformations and the end-effector/payload can undergo rotations relative to the connecting flexible arm. Based on a consistent kinematical synthesis, the motions of one body in the chain takes into consideration the motions of the preceding body in the chain. This permits the derivation of the equations of motion without the imposition of constraints. The equations of motion are derived by the Lagrangian approach. The equations are nonlinear and of relatively high order.

Ideally, the maneuvering of payloads should be carried out without exciting elastic vibration, which is not possible in general. However, the elastic motions tend to be small compared to the rigid-body maneuvering motions. Under such circumstances, a perturbation approach permits separation of the problem into a zero-order problem (in a perturbation theory sense) for the rigid-body maneuvering of the space robot and a first-order problem for the control of the elastic motions and the perturbations from the rigid-body motions. The maneuvering can be carried out open loop, but the elastic and rigid-body perturbations are controlled closed loop.

The robot mission consists of carrying a payload over a prescribed trajectory and placing it in a certain orientation relative to the inertial space. For planar motion, the end-effector/payload configuration is defined by three variables, two translations and one rotation. At the end of the mission, the vibration should be damped out, so that the robot can be regarded as rigid at that time. Still, the rigid robot possesses six degrees of freedom, two translations of the platform and one rotation of each of the four bodies, including the platform. This implies that a kinematic redundancy exists. This redundancy can be used to optimize the robot trajectory[10] in the context of trajectory planning. A simpler approach is to remove the redundancy by imposing certain constraints on the robot trajectory, such as prescribing the motion of the platform.[11] Then, for a given end-effector/payload trajectory, the rigid-body maneuvering configuration of the robot can be obtained by means of inverse kinematics. Finally, the forces and torques required for the robot trajectory realization are obtained from the zero-order equations by means of inverse dynamics.

The first-order equations for the elastic motions and the perturbations in the rigid-body maneuvering motions are linear, but of high order, time-varying and they are subjected to persistent disturbances. The persistent disturbances arise from the zero-order solution, and hence are known; they are treated by means of feedforward control. All other disturbances are controlled closed loop, with the feedback control being designed by means of discrete-time linear quadratic regulator (LQR) theory with prescribed degree of stability. A numerical example demonstrates the approach. In the example, the controls derived by the perturbation approach are applied to the original nonlinear system and the errors in the end effector/payload configuration were found to be relatively small during the maneuver and to vanish soon after the termination of the maneuver.

2. A CONSISTENT KINEMATICAL SYNTHESIS

To describe the motion of the space robot, it is convenient to adopt a consistent kinematical synthesis whereby the system is regarded as a chain of articulated flexible bodies and the motion of one body is defined with due consideration to the motion of the preceeding body in the chain. Figure 1 shows the mathematical model of a planar space robot. The robot consists of a rigid platform (Body 1), two hinge-connected flexible arms (Bodies 2 and 3) and a rigid end-effector holding the payload (Body 4). The various motions are referred to a set of inertial axes and sets of body axes to be defined shortly.

The object is to derive the system equations of motion, which can be done by means of Lagrange's equations in terms of quasi-coordinates.[12] Because in the case at hand the motion is planar, it is more expedient to use the standard Lagrange's equations. This requires the kinetic energy, potential energy and virtual work. The kinetic energy, in turn, requires the velocity of a typical point in each of the bodies.

The position of a nominal point on the platform is given by

$$\mathbf{R}_1 = \mathbf{R}_0 + \mathbf{r}_1 \tag{1}$$

where $\mathbf{R}_0 = [X\ Y]^T$ is the position vector of the origin O_1 of the body axes x_1, y_1 (Fig. 1) relative to the inertial axes X, Y and in terms of X, Y components and $\mathbf{r}_1 = [x_1\ y_1]^T$ is the position vector of the nominal point on the platform relative to the body axes x_1, y_1 and in terms of x_1, y_1 components. The velocity vector of a point on the platform can be expressed in terms of x_1, y_1 components as follows:

$$\mathbf{V}_1 = C_1 \dot{\mathbf{R}}_0 + \tilde{\omega}_1 \mathbf{r}_1 \tag{2}$$

where

$$C_1 = \begin{bmatrix} c\theta_1 & s\theta_1 \\ -s\theta_1 & c\theta_1 \end{bmatrix} \tag{3}$$

is a matrix of direction cosines between axes x_1, y_1, and X, Y, in which $s\theta_1 = \sin\theta_1$, $c\theta_1 = \cos\theta_1$,

$$\dot{\mathbf{R}}_0 = \left[\dot{X}\ \ \dot{Y}\right]^T \tag{4}$$

is the velocity vector of O_1 in terms of X, Y components and

$$\tilde{\omega}_1 = \begin{bmatrix} 0 & -\dot{\theta}_1 \\ \dot{\theta}_1 & 0 \end{bmatrix} \tag{5}$$

The second body is flexible, so that we must resolve the question of body axes. We define the body axes x_2, y_2 as a set of axes with the origin at the hinge O_2 and embedded in the undeformed body such that x_2 is tangent to the body at O_2 (Fig. 2). Then, we define the motion of axes x_2, y_2 as the rigid-body motion of Body 2 and measure the elastic motion relative to x_2, y_2. Hence, the velocity of a point on Body 2 (first flexible arm) in terms of x_2, y_2 components is

$$\begin{aligned}\mathbf{V}_2 =& C_{2-1}\mathbf{V}_1(O_2) + \tilde{\omega}_2(\mathbf{r}_2 + \mathbf{u}_2) + \dot{\mathbf{u}}_{2\mathrm{rel}} \\ =& C_2\dot{\mathbf{R}}_0 + C_{2-1}\tilde{\omega}_1\mathbf{r}_1(O_2) + \tilde{\omega}_2(\mathbf{r}_2 + \mathbf{u}_2) + \dot{\mathbf{u}}_{2\mathrm{rel}}\end{aligned} \tag{6}$$

where C_{2-1} and C_2 are matrices similar to C_1, Eq. (3), except that θ_1 is replaced by $\theta_2 - \theta_1$ and θ_2, respectively, $\tilde{\omega}_2$ has the same structure as $\tilde{\omega}_1$ but with $\dot{\theta}_2$ replacing $\dot{\theta}_1$, $\mathbf{r}_1(O_2) = [d_1 \; h_1]^T$, $\mathbf{r}_2 = [x_2 \; 0]^T$, $\mathbf{u}_2 = [0 \; u_2]^T$ and $\dot{\mathbf{u}}_{2\mathrm{rel}} = [0 \; \dot{u}_2]$, in which $u_2 = u_2(x_2, t)$ and $\dot{u}_2 = \dot{u}_2(x_2, t)$ are the elastic displacement and velocity, respectively.

Using the analogy with Body 2, the velocity of a point on Body 3 (second flexible arm) in terms of x_3, y_3 components can be shown to be

$$\begin{aligned}\mathbf{V}_3 =& C_{3-2}\mathbf{V}_2(L_2) + \tilde{\omega}_3(\mathbf{r}_3 + \mathbf{u}_3) + \dot{\mathbf{u}}_{3\mathrm{rel}} \\ =& C_3\dot{\mathbf{R}}_0 + C_{3-1}\tilde{\omega}_1\mathbf{r}_1(O_2) + C_{3-2}\left\{\tilde{\omega}_2\left[\mathbf{r}_2(L_2) + \mathbf{u}_2(L_2, t)\right] + \dot{\mathbf{u}}_{2\mathrm{rel}}(L_2, t)\right\} \\ &+ \tilde{\omega}_3(\mathbf{r}_3 + \mathbf{u}_3) + \dot{\mathbf{u}}_{3\mathrm{rel}}\end{aligned} \tag{7}$$

The fourth body consists of the end-effector and payload combined, and is treated as rigid. Following the established pattern, the velocity of a point on Body 4 in terms of x_4, y_4 components is

$$\begin{aligned}\mathbf{V}_4 =& C_{4-3}\mathbf{V}_3(L_3) + \tilde{\omega}_4\mathbf{r}_4 \\ =& C_4\dot{\mathbf{R}}_0 + C_{4-1}\tilde{\omega}_1\mathbf{r}_1(O_2) + C_{4-2}\left\{\tilde{\omega}_2\left[\mathbf{r}_2(L_2) + \mathbf{u}_2(L_2, t)\right] + \dot{\mathbf{u}}_{2\mathrm{rel}}(L_2, t)\right\} \\ &+ C_{4-3}\left\{\tilde{\omega}_3\left[\mathbf{r}_3(L_3) + \mathbf{u}_3(L_3, t)\right] + \dot{\mathbf{u}}_{3\mathrm{rel}}(L_3, t)\right\} + \tilde{\omega}_4\mathbf{r}_4\end{aligned} \tag{8}$$

The consistent kinematical synthesis just described permits the formulation of the equations of motion for the whole system without invoking constraint equations. Such constraint equations must be used to eliminate redundant coordinates in a formulation in which equations of motion are derived separately for each body.

3. SPATIAL DISCRETIZATION OF THE FLEXIBLE ARMS

The velocity expressions derived in Sec. 2 involved rigid-body motions depending on time alone and elastic motions depending on the spatial position and time. Equations of motion based on such formulations are hybrid, in the sense that the equations for the rigid-body motions are ordinary differential equations and the ones for the elastic motions are partial differential equations. Designing maneuvers and controls on the basis of hybrid differential equations is likely to cause serious difficulties, so that the only viable alternative is to transform the hybrid system into one consisting of ordinary differential equations alone. This amounts to discretization in space of the elastic displacements, which can be done by means of series expansions. Assuming that the flexible arms act as beams in bending, the elastic displacements can be expanded in the series

$$u_i(x_i,t) = \sum_{j=1}^{n_i} \phi_{ij}(x_i)\,\eta_{ij}(t) = \boldsymbol{\phi}_i^T(x_i)\,\boldsymbol{\eta}_i(t)\,,\ i = 2,3 \tag{9}$$

where $\phi_{ij}(x_i)$ are *admissible functions*, often referred to as *shape functions*, and $\eta_{ij}(t)$ are generalized coordinates ($i = 2,3;\ j = 1,2,\ldots,n_i$); $\boldsymbol{\phi}_i$ and $\boldsymbol{\eta}_i$ are corresponding n_i-dimensional vectors.

The question arises as to the nature of the admissible functions. Clearly, the object is to approximate the displacements with as few terms in the series as possible. This is not a new problem in structural dynamics, and the very same subject has been investigated recently in Ref. 13, in which a new class of functions, referred to as *quasi-comparison functions*, has been introduced. Quasi-comparison functions are defined as linear combinations of admissible functions capable of satisfying the boundary conditions of the elastic member. As shown in Fig. 2, the beam is tangent to axis x_i at $O_i\,(i = 2,3)$. Hence, the admissible functions must be zero and their slope must be zero at $x_i = 0$. At $x_i = L_i$, the displacement, slope, bending moment and shearing force are generally nonzero. Quasi-comparison functions are linear combinations of functions possessing these characteristics. Admissible functions from a single family of functions do not possess the characteristics, but admissible functions from several suitable families can be combined to obtain them. In the case at hand, quasi-comparison functions can be obtained in the form of suitable linear combinations of clamped-free and clamped-clamped shape functions.

4. LAGRANGE'S EQUATIONS

Before we can derive Lagrange's equations, we must produce expressions for the kinetic energy, potential energy and virtual work. To this end, and following the spatial discretization indicated by Eqs. (9), we introduce the configuration vector $\mathbf{q}(t) = [X(t)\ Y(t)\ \theta_1(t)\ \theta_2(t)\ \ \theta_3(t)\ \theta_4(t)\ \boldsymbol{\eta}_2^T(t)\ \boldsymbol{\eta}_3^T(t)]^T$ so that the velocity vectors, Eqs. (2), (6)-(8), can be written in the compact form

$$\mathbf{V}_i = D_i\dot{\mathbf{q}},\ i = 1,2,3,4 \tag{10}$$

where

$$\begin{aligned} D_1 &= \begin{bmatrix} c\theta_1 & s\theta_1 & -y_1 & 0 & \dots & \mathbf{0}^T \\ -s\theta_1 & c\theta_1 & x_1 & 0 & \dots & \mathbf{0}^T \end{bmatrix} \\ D_2 &= \begin{bmatrix} c\theta_2 & s\theta_2 & d_1 s(\theta_2-\theta_1) - h_1 c(\theta_2-\theta_1) & -\boldsymbol{\phi}_2^T\boldsymbol{\eta}_2 & 0 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ -s\theta_2 & c\theta_2 & d_1 c(\theta_2-\theta_1) + h_1 s(\theta_2-\theta_1) & x_2 & 0 & 0 & \boldsymbol{\phi}_2^T & \mathbf{0}^T \end{bmatrix} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{11}$$

Then, the kinetic energy is simply

$$T = \frac{1}{2}\sum_{i=1}^{4}\int_{m_i}\mathbf{V}_i^T\mathbf{V}_i dm_i = \frac{1}{2}\dot{\mathbf{q}}^T M\dot{\mathbf{q}} \tag{12}$$

where

$$M = \sum_{i=1}^{4}\int_{m_i} D_i^T D_i dm_i \tag{13}$$

is the mass matrix. Typical entries in the mass matrix are

$$\begin{aligned} m_{11} &= m,\ m_{12} = 0,\ m_{13} = -(m_2+m_3+m_4)(h_1 c\theta_1 + d_1 s\theta_1) \\ m_{14} &= -\left[\overline{\boldsymbol{\phi}}_2^T + (m_3+m_4)\,\boldsymbol{\phi}_2^T(L_2)\right]\boldsymbol{\eta}_2 c\theta_2 - [S_2 + (m_3+m_4)L_2]\,s\theta_2 \end{aligned}$$

. . . .

$$\mathbf{m}_{18} = -\left[\overline{\phi}_3^T + m_4\phi_3^T(L_3)\right]s\theta_3$$

$$m_{22} = m,\ m_{23} = -(m_2 + m_3 + m_4)(h_1 s\theta_1 - d_1 c\theta_1) \tag{14}$$

. . . .

$$\mathbf{m}_{28} = \left[\overline{\phi}_3^T + m_4\phi_3^T(L_3)\right]c\theta_3$$

. . . .

$$m_{88} = \int_{\text{Body } 3}\phi_3\phi_3^T dm_3 + m_4\phi_3(L_3)\phi_3^T(L_3)$$

in which

$$m = \sum_{i=1}^{4} m_i,\ \overline{\phi}_i = \int_{m_i}\phi_i dm_i,\ i = 2,3,\ S_i = \int_{m_i} x_i dm_i,\ i = 1,2,3,4 \tag{15}$$

The potential energy, assumed to be entirely due to bending, has the form

$$V = \sum_{i=1}^{4}\int_0^{L_2} EI_2\left[u''(x_2,t)\right]^2 dx_2 + \frac{1}{2}\int_0^{L_3} EI_3\left[u_3''(x_3,t)\right]^2 dx_3 = \frac{1}{2}\mathbf{q}^T K\mathbf{q} \tag{16}$$

in which $EI_i\,(i = 2,3)$ are bending stiffnesses and primes denote spatial derivatives. Moreover,

$$K = \text{block-diag}\left[0\ K_2\ K_3\right] \tag{17}$$

is the stiffness matrix, where

$$K_i = \int_0^{L_i} EI_i\phi_i''\phi_i''^T dx_i,\ i = 2,3 \tag{18}$$

are stiffness matrices for the flexible arms.

Next, we propose to derive the virtual work expression. To this end, we must specify first the actuators to be used. There are three actuators acting on the platform, two thrusters F_{x1} and F_{y1} acting at O_1 in directions aligned with the body axes and a torquer M_1. Three other torquers M_2, M_3 and M_4 are located at the hinges O_2, O_3 and O_4, respectively, the first acting on the platform and first arm, the second acting on the first and second arm and the third acting on the second arm and end-effector. Moreover, there are torquers $M_5,\ M_6,\ M_7$

and M_8 acting at $x_2 = L_2/3$, $x_2 = 2L_2/3$, $x_3 = L_3/3$ and $x_4 = 2L_3/3$, respectively. In view of this, the virtual work can be written as follows:

$$\begin{aligned}\delta W =&F_{x1}\left(\cos\theta_1\delta X + \sin\theta_1\delta Y\right) + F_{y1}\left(-\sin\theta_1\delta X + \cos\theta_1\delta Y\right) + M_1\delta\theta_1\\&+ M_2\delta\left(\theta_2 - \theta_1\right) + M_3\delta\psi_3 + M_4\delta\psi_4 + M_5\delta\left[\theta_2 + \boldsymbol{\phi}_2'^T\left(L_2/3\right)\boldsymbol{\eta}_2\right]\\&+ M_6\delta\left[\theta_2 + \boldsymbol{\phi}_2'^T\left(2L_2/3\right)\boldsymbol{\eta}_2\right] + M_7\delta\left[\theta_3 + \boldsymbol{\phi}_3'^T\left(L_3/3\right)\boldsymbol{\eta}_3\right]\\&+ M_8\left[\theta_3 + \boldsymbol{\phi}_3'^T\left(2L_3/3\right)\boldsymbol{\eta}_3\right]\end{aligned} \tag{19}$$

where $\delta X, \delta Y, \ldots$ are virtual displacements. Moreover, denoting the angles between the two arms and between the second arm and the end-effector by

$$\begin{aligned}\psi_3 =&\theta_3 - \theta_2 - \frac{\partial u_2}{\partial x_2}\Big|_{x_2=L_2} = \theta_3 - \theta_2 - \boldsymbol{\phi}_2'^T\left(L_2\right)\boldsymbol{\eta}_2\\\psi_4 =&\theta_4 - \theta_3 - \frac{\partial u_3}{\partial x_3}\Big|_{x_3=L_3} = \theta_4 - \theta_3 - \boldsymbol{\phi}_3'^T\left(L_3\right)\boldsymbol{\eta}_3\end{aligned} \tag{20}$$

we can write

$$\delta\psi_3 = \delta\theta_3 - \delta\theta_2 - \boldsymbol{\phi}_2'^T\left(L_2\right)\delta\boldsymbol{\eta}_2,\ \delta\psi_4 - \delta\theta_3 - \boldsymbol{\phi}_3'^T\left(L_3\right)\delta\boldsymbol{\eta}_3 \tag{21}$$

Inserting Eqs. (21) into Eq. (19), we can express the virtual work in terms of generalized forces and generalized virtual displacements in the form

$$\delta W = \mathbf{Q}^T\delta\mathbf{q} \tag{22}$$

where $\mathbf{Q} = \left[F_X\ F_Y\ \Theta_1\ \Theta_2\ \Theta_3\ \Theta_4\ \mathbf{N}_2^T\ \mathbf{N}_3^T\right]^T$ is the generalized force vector, in which

$$\begin{aligned}&F_X = F_{x1}\cos\theta_1 - F_{y1}\sin\theta_1,\ F_Y = F_{x1}\sin\theta_1 + F_{y1}\cos\theta_1\\&\Theta_1 = M_1 - M_2,\ \Theta_2 = M_2 - M_3 + M_3 + M_5 + M_6\\&\Theta_3 = M_3 - M_4 + M_7 + M_8,\ \Theta_4 = M_4\\&\mathbf{N}_2 = -M_3\boldsymbol{\phi}_2'\left(L_2\right) + M_5\boldsymbol{\phi}_2'\left(L_2/3\right) + M_6\boldsymbol{\phi}_2'\left(2L_2/3\right)\\&\mathbf{N}_3 = -M_4\boldsymbol{\phi}_3'\left(L_3\right) + M_7\boldsymbol{\phi}_3'\left(L_3/3\right) + M_8\boldsymbol{\phi}_3'\left(2L_3/3\right)\end{aligned} \tag{23}$$

and $\delta\mathbf{q} = \left[\delta X \; \delta Y \; \delta\theta_1 \; \delta\theta_2 \; \delta\theta_3 \; \delta\theta_4 \; \delta\eta_2^T \; \delta\eta_3^T\right]^T$ is the generalized virtual displacement vector. Equations (23) express the generalized forces and torques in terms of the actual actuator forces and torques and can be written in the compact form

$$\mathbf{Q} = E\mathbf{F} \tag{24}$$

where $\mathbf{F} = [F_{x1} \; F_{y1} \; M_1 \; M_2 \; \ldots \; M_8]^T$ is the actual control vector and

$$E = E(\theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\phi_2'(L_2) & \mathbf{0} & \phi_2'\left(\frac{L_2}{3}\right) & \phi_2'\left(\frac{2L_2}{3}\right) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\phi_3'(L_3) & \mathbf{0} & \mathbf{0} & \phi_3'\left(\frac{L_3}{3}\right) & \phi_3'\left(\frac{2L_3}{3}\right) \end{bmatrix} \tag{25}$$

where $s_1 = \sin\theta_1$, $c_1 = \cos\theta_1$. Note that E is a time-varying coefficient matrix, because θ_1 varies with time.

Lagrange's equations can be expressed in the general symbolic vector form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial T}{\partial q} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q} \tag{26}$$

Observing that $M = M(\mathbf{q})$, we can write

$$\begin{aligned} &\frac{\partial T}{\partial \dot{q}} = M\dot{\mathbf{q}} \; , \; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}} \\ &\frac{\partial T}{\partial \mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial M}{\partial \mathbf{q}}\dot{\mathbf{q}} \; , \; \frac{\partial V}{\partial \mathbf{q}} = K\mathbf{q} \end{aligned} \tag{27}$$

Inserting Eqs. (27) into Eq. (26), we obtain Lagrange's equations in the more explicit form

$$M\ddot{\mathbf{q}} + \left(\dot{M} - \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial M}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{Q} \tag{28}$$

in which

$$\dot{M} = \sum_{j=1}^{6+2n} \frac{\partial M}{\partial q_j}\dot{q}_j, \ \dot{\mathbf{q}}^T \frac{\partial M}{\partial \mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}^T \partial M/\partial q_1 \\ \dot{\mathbf{q}}^T \partial M/\partial q_2 \\ \vdots \\ \dot{\mathbf{q}}^T \partial M/\partial q_{6+2n} \end{bmatrix} \tag{29}$$

5. A PERTURBATION APPROACH TO THE CONTROL DESIGN

Equation (29) represents a high-order system of nonlinear differential equations, and is not very suitable for control design. Hence, an approach capable of coping with the problems of high-dimensionality and nonlinearity is highly desirable. Such an approach must be based on the physics of the problem. The ideal maneuver is that in which the robot acts as if its arms were rigid. In reality, the arms are flexible, so that some elastic vibration is likely to take place. It is reasonable to assume, however, that the elastic motions are one order of magnitude smaller that the maneuvering motions. This permits treatment of the elastic motions as perturbations on the maneuvering motions. In turn, the elastic perturbations give rise to perturbations in the "rigid-body" maneuvering motions. This suggests a perturbation approach, whereby the problem is separated into a zero-order problem for the "rigid-body" maneuvering of the payload and a first-order problem for the control of the elastic motions and the perturbations in the rigid-body maneuvering motions. The zero-order problem is nonlinear, albeit of relatively low dimension. It can be solved independently and the control can be open loop. On the other hand, the first-order problem is linear, but of relatively high dimension. It is affected by the solution to the zero-order problem, where the effect is in the form of time-varying coefficients and persistent disturbances. The control for the first-order problem is to be closed loop.

We consider a first-order perturbation solution characterized by

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}_1, \ \mathbf{Q} = \mathbf{Q}_0 + \mathbf{Q}_1 \tag{30}$$

where the subscripts 0 and 1 denote zero-order and first-order quantities, with the zero-order quantities being one order of magnitude larger than the first-order ones. Inserting Eqs. (30)

into Eq. (28), separating quantities of different orders of magnitude and ignoring terms of order two and higher, we obtain the equation for the zero-order problem

$$M_0\ddot{\mathbf{q}}_0 + \left(M_v - \frac{1}{2}M_v^T\right)\dot{\mathbf{q}}_0 = \mathbf{Q}_0 = E_0\mathbf{F}_0 \tag{31}$$

where $\mathbf{q}_0 = \left[X_0\ Y_0\ \theta_{10}\ \theta_{20}\ \theta_{30}\ \theta_{40}\ \mathbf{0}^T\ \mathbf{0}^T\right]^T$, $\mathbf{Q}_0 = \left[F_{X0}\ F_{Y0}\ \Theta_{10}\ \Theta_{20}\ \Theta_{30}\ \Theta_{40}\ \mathbf{0}^T\ \mathbf{0}^T\right]^T$ are zero-order displacement and generalized control vectors, $E_0 = E(\theta_{10})$ is the matrix E, Eq. (25), evaluated at $\theta_1 = \theta_{10}$, $\mathbf{F}_0 = [F_{x0}\ F_{y0}\ M_{10}\ M_{20}\ \dots\ M_{80}]^T$ and

$$M_0 = M(\mathbf{q}_0)\,,\ M_v = \left[\frac{\partial M}{\partial q_1}\dot{\mathbf{q}}_0\ \ \frac{\partial M}{\partial q_2}\dot{\mathbf{q}}_0\ \ \cdots\ \ \frac{\partial M}{\partial q_{6+2n}}\dot{\mathbf{q}}_0\right]\Big|_{\mathbf{q}=\mathbf{q}_0} \tag{32a, b}$$

Moreover, we obtain the equation for the first-order problem

$$M_0\ddot{\mathbf{q}}_1 + \left(M_v + M' - M_v^T\right)\dot{\mathbf{q}}_1 + \left(M_a + M_{vv} - \frac{1}{2}M'_{vv} + K\right)\mathbf{q}_1 = \mathbf{Q}_1 + \mathbf{Q}_d \tag{33}$$

where $\mathbf{q}_1 = \left[X_1\ Y_1\ \theta_{11}\ \theta_{21}\ \theta_{31}\ \theta_{41}\ \boldsymbol{\eta}_2^T\ \boldsymbol{\eta}_3^T\right]^T$, $\mathbf{Q}_1 = \left[F_{X1}\ F_{Y1}\ \Theta_{11}\ \Theta_{21}\ \Theta_{31}\ \Theta_{41}\ \mathbf{N}_2^T\ \mathbf{N}_3^T\right]^T$ are first-order displacement and generalized control vectors, $\mathbf{Q}_d = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ \mathbf{F}_{d2}^T\ \mathbf{F}_{d3}^T\right]^T$ is a persistent disturbance vector and

$$M_a = \left[\frac{\partial M}{\partial q_1}\ddot{\mathbf{q}}_0\ \ \frac{\partial M}{\partial q_2}\ddot{\mathbf{q}}_0\ \ \cdots\ \ \frac{\partial M}{\partial q_{6+2n}}\ddot{\mathbf{q}}_0\right]\Big|_{\mathbf{q}=\mathbf{q}_0} \tag{34a}$$

$$M' = \sum_{j=1}^{6+2n}\frac{\partial M}{\partial q_j}\Big|_{\mathbf{q}=\mathbf{q}_0}\dot{q}_{0j} \tag{34b}$$

$$M_{vv}\mathbf{q}_1 = \sum_{j=1}^{6+2n}\sum_{k=1}^{6+2n}\frac{\partial^2 M}{\partial q_j \partial q_k}\Big|_{\mathbf{q}=\mathbf{q}_0} q_{1k}\dot{q}_{0j}\dot{\mathbf{q}}_0 \tag{34c}$$

$$M'_{vv}\mathbf{q}_1 = \dot{\mathbf{q}}_0^T\sum_{k=1}^{6+2n}\frac{\partial^2 M}{\partial \mathbf{q}\partial q_k}\Big|_{\mathbf{q}=\mathbf{q}_0} q_{1k}\dot{\mathbf{q}}_0 \tag{34d}$$

From Eqs. (24) and (25), however, we can write

$$\mathbf{Q}_1 = E_0\mathbf{F}_1 + E_1\mathbf{F}_0 = E_0\mathbf{F}_1 + \mathbf{F}_0^*\mathbf{q}_1 \tag{35}$$

where

$$E_1 = \frac{\partial E}{\partial \theta_1}\Big|_{\theta_1=\theta_{10}}\theta_{11} \tag{36}$$

Moreover, the matrix F_0^* has the entries

$$F_{011}^* = -\left(F_{x10}\sin\theta_{10} + F_{y10}\cos\theta_{10}\right)$$

$$F_{021}^* = F_{x10}\cos\theta_{10} - F_{y10}\sin\theta_{10} \tag{37}$$

$$F_{0ij}^* = 0,\ i = 3,4,\ldots,\ y + n_2 + n_3;\ j = 2,3,\ldots,6 + n_2 + n_3$$

In view of this, the equation for the first-order problem, can be rewritten as

$$M_0\ddot{\mathbf{q}}_1 + \left(M_v + M' - M_v^T\right)\dot{\mathbf{q}}_1 + \left(M_1 + M_{vv} - \frac{1}{2}M'_{vv} + K - F_0^*\right)\mathbf{q}_1 = E_0\mathbf{F}_1 + \mathbf{Q}_d \tag{38}$$

6. TRAJECTORY PLANNING

The mission consists of delivering the payload to a certain point in space and placing it in a certain orientation. For planar motion, the final payload configuration is defined by three variables, two translations and one rotation. The trajectory planning, designed to realize this final configuration, will be carried out as if the robot system were rigid, with the expectation that all elastic motions and perturbations in the rigid-body maneuvering motions will be annihilated by the end of the maneuver. The rigid-body motion of the robot is described by the zero-order problem and it consists of six components, two translations of the platform and one rotation of each of the four bodies. This implies that a kinematical redundancy exists, as there is an infinity of ways a six-dimensional configuration can generate a three-dimensional trajectory. This redundancy can be removed by controlling surplus variables, perhaps in an optimal fashion. In this study, we prescribe three of the configuration variables, such as the translations and rotation of the platform. Under these circumstances, the rigid space robot can be treated as a nonredundant manipulator.

Next, we denote the end-effector configuration by $\mathbf{X}_E$, so that from kinematics we can write

$$\mathbf{X}_E = \mathbf{f}\left(\mathbf{q}_0\right) \tag{39}$$

where $\mathbf{f}$ is a three-dimensional vector function. Differentiating Eq. (39) with respect to time, we obtain

$$\dot{\mathbf{X}}_E = J\left(\mathbf{q}_0\right)\dot{\mathbf{q}}_0 \tag{40}$$

where

$$J(\mathbf{q}_0) = [\partial \mathbf{f} / \partial \mathbf{q}_0] \tag{41}$$

is the 3×6 Jacobian matrix. Introducing the notation

$$\mathbf{q}_0 = \left[\mathbf{q}_S^T \;\vdots\; \mathbf{q}_M^T \right]^T \tag{42}$$

where $\mathbf{q}_S = [X_0 \; Y_0 \; \theta_{10}]^T$ and $\mathbf{q}_M = [\theta_{20} \; \theta_{30} \; \theta_{40}]^T$ are the controlled platform configuration vector and the open-loop controlled manipulator configuration vector, respectively, and partitioning the Jacobian matrix accordingly, or

$$J = \left[J_X \;\vdots\; J_M \right] \tag{43}$$

Eq. (40) can be rewritten as

$$\dot{\mathbf{X}}_E = J_s \dot{\mathbf{q}}_S + J_M \dot{\mathbf{q}}_M \tag{44}$$

Then, on the assumption that $\dot{\mathbf{q}}_S$ is prescribed and for a given end-effector trajectory $\mathbf{X}_E$, we can determine the manipulator velocity vector from

$$\dot{\mathbf{q}}_M = J_M^{-1} \left(\dot{\mathbf{X}}_E - J_S \dot{\mathbf{q}}_S \right) \tag{45}$$

The end-effector trajectory was taken in the form of a sinusoidal function so as to prevent excessive vibration. Finally, with $\mathbf{q}_0$ given, we can obtain the required open-loop control $\mathbf{F}_0$ by inverse dynamics, which amounts to using Eq. (31).

7. FEEDBACK CONTROL OF THE ELASTIC MOTIONS AND RIGID-BODY PERTURBATIONS

The elastic motions and the perturbations in the rigid-body maneuvering motions are governed by the equation defining the first-order problem, Eq. (38). The persistent disturbances are controlled open loop and all other disturbances are controlled closed loop. To this end, we express the control vector in the form

$$\mathbf{F}_1 = \mathbf{F}_{1o} + \mathbf{F}_{1c} \tag{46}$$

where the subscripts o and c indicate open loop and closed loop, respectively. Recognizing that E_0 is a rectangular matrix, the open-loop control can be written as

$$\mathbf{F}_{1o} = E_0^{\dagger}\mathbf{Q}_d \tag{47}$$

in which

$$E_0^{\dagger} = \left(E_0^T E_0\right)^{-1} E_0^T \tag{48}$$

is the psuedo-inverse of E_0.

For the closed-loop control, we consider LQR control, which requires recasting the equations of motion in state form. Adjoining the identity $\dot{\mathbf{q}}_1 = \dot{\mathbf{q}}_1$, the state equations can be expressed as

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)E_0\mathbf{u}_c(t) + B(t)D\mathbf{d}(t) \tag{49}$$

where $\mathbf{x} = \left[\mathbf{q}_1^T \ \ \dot{\mathbf{q}}_1^T\right]^T$ is the state vector, $\mathbf{u}_c = \mathbf{F}_{1c}$ is the control vector, $\mathbf{d} = \mathbf{Q}_d$ is the disturbance vector and

$$A = \begin{bmatrix} 0 & I \\ -M_0^{-1}\left(M_a + M_{vv} - \frac{1}{2}M'_{vv} + K - F_0^*\right) & -M_0^{-1}\left(M_v + M' - M_v^T\right) \end{bmatrix} \tag{50a}$$

$$B = \begin{bmatrix} 0 \\ M_0^{-1} \end{bmatrix}, \ D = \left(I - E_0 E_0^{\dagger}\right) \tag{50b,c}$$

are coefficient matrices. It should be noted here that, if the matrix E_0 is not square, the matrix D is not zero, so that the open-loop control does not annihilate the persistent disturbances completely. As the number of actuators approaches the number of degrees of freedom of the system, the matrix E_0 tends to become square. When the number of actuators coincides with the number of degrees of freedom the matrix E_0 is square, in which case the pseudo-inverse becomes an exact inverse and the matrix D reduces to zero.

The state equations, Eq. (49), possess time-varying coefficients and are subject to residual persistent disturbances. Due to difficulties in treating such systems in continuous time, we propose to discretize the state equations in time. Following the usual steps,[14] the state equations in discrete time can be shown to be

$$\mathbf{x}_{k+1} = \Phi_k\mathbf{x}_k + \Gamma_k E_{0k}\mathbf{u}_{ck} + \Gamma_k D_k\mathbf{d}_k, \ k = 0, 1, \ldots \tag{51}$$

where

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{x}\,(kT)\,,\ \mathbf{u}_{ck} = \mathbf{u}_c\,(kT)\,,\ \mathbf{d}_k = \mathbf{d}\,(kT)\,,\ k = 0.1, \ldots \\
\Phi_k &= \exp\ A_k T,\ \Gamma_k = (\exp\ A_k T - I)\, A_k^{-1} B_k,\ k = 0, 1, \ldots \\
E_{0k} &= E_0\,(kT)\,,\ D_k = D\,(kT)\,,\ k = 0, 1, \ldots
\end{aligned} \tag{52}
$$

in which T is the sampling period and

$$A_k = A(kT),\ B_k = B(kT) \tag{53}$$

In view of the above discussion, we assume that the effect of the persistent disturbances has been reduced drastically by the feedforward control, and design the feedback control in its absence. This design is according to a discrete-time LQR with prescribed degree of stability. To this end, we consider the performance measure

$$J = \mathbf{x}_N^T P_N \mathbf{x}_N + \sum_{k=0}^{N-1} e^{2\alpha k} \left(\mathbf{x}_k^T Q_k \mathbf{x}_k + \mathbf{u}_{ck}^T R_k \mathbf{u}_{ck} \right) \tag{54}$$

where P_N and Q_k are symmetric positive semidefinite matrices, R_k is a symmetric positive definite matrix, α is a nonnegative constant defining the degree of stability and NT is the final sampling time.

The optimization process using the performance measure given by Eq. (54) can be reduced to a standard discrete-time LQR form by means of the transformation

$$\hat{\mathbf{x}}_k = e^{\alpha k} \mathbf{x}_k,\ \hat{\mathbf{u}}_{ck} = e^{\alpha k} \mathbf{u}_{ck},\ \hat{P}_N = e^{-2\alpha N} P_N \tag{55a, b, c}$$

Multiplying Eqs. (51) through by $e^{\alpha(k+1)}$ using Eqs. (55a,b) and ignoring the small perturbing term, we obtain the new state equations

$$\hat{\mathbf{x}}_{k+1} = e^{\alpha} \left(\Phi_k \hat{\mathbf{x}}_k + \gamma_k E_{0k} \hat{\mathbf{u}}_{ck} \right),\ k = 0, 1, \ldots, N-1 \tag{56}$$

Similarly, inserting Eqs. (55) into Eq. (54), we obtain the new performance measure

$$J = \hat{\mathbf{x}}_N^T \hat{P}_N \hat{\mathbf{x}}_N + \sum_{k=0}^{N-1} \left(\hat{\mathbf{x}}_k^T Q_k \hat{\mathbf{x}}_k + \hat{\mathbf{u}}_{ck}^T R_k \hat{\mathbf{u}}_{ck} \right) \tag{57}$$

It can be shown that the optimal control law has the form[14]

$$\hat{\mathbf{u}}_{ck} = G_k \hat{\mathbf{x}}_k, \ k = 0, 1, \ldots, N-1 \tag{58}$$

where G_k are gain matrices obtained from the discrete-time Riccati equations

$$G_{N-i} = -\left(e^{2\alpha} E_{0,N-i}^T \Gamma_{N-i}^T \hat{P}_{N+1-i} \Gamma_{N-i} E_{0,N-i} + R_{N-i}\right)^{-1} e^{2\alpha} E_{0,N-i}^T \Gamma_{N-i}^T \hat{P}_{N+1-i} \Phi_{N-i},$$
$$i = 1, 2, \ldots, N; \ \hat{P} = e^{-2\alpha N} P_N \tag{59a}$$

$$\begin{aligned}\hat{P}_{N-i} =& e^{2\alpha}\left(\Phi_{N-i} + \Gamma_{N-i} E_{0,N-i} G_{N-i}\right)^T \hat{P}_{N+1-i}\left(\Phi_{N-i} + \Gamma_{N-1} E_{0,N-i} G_{N-i}\right) \\ &+ G_{N-i}^T R_{N-i} G_{N-i} + Q_{N-i}, \ i = 1, 2, \ldots, N; \ \hat{P}_N = e^{-2\alpha N} P_N\end{aligned} \tag{59b}$$

Equations (59a) and (59b) are evaluated alternately for $G_{N-1}, \ \hat{P}_{N-1}, \ G_{N-2}, \ \hat{P}_{N-2}, \ldots, G_0$, given the final value of $\hat{P}_N$.

Inserting the control law, Eqs. (58), into Eqs. (56), we obtain the closed-loop transformed state equations

$$\hat{\mathbf{x}}_{k+1} = e^{\alpha}\left(\Phi_k + \Gamma_k E_{0k} G_k\right) \hat{\mathbf{x}}_k, \ k = 0, 1, \ldots \tag{60}$$

Then, recalling Eq. (55a) and restoring the persistent disturbance term, the closed-loop state equations for the original system can be written in the form

$$\mathbf{x}_{k+1} = \left(\Phi_k + \Gamma_k E_{0k} G_k\right) \mathbf{x}_k + \Gamma_k D_k \mathbf{d}_k, \ k = 0, 1, \ldots \tag{61}$$

8. NUMERICAL EXAMPLE

The example involves the flexible space robot shown in Fig. 1. Numerical values for the system parameters are as follows:

$$L_1 = 1 \text{ m}, d_1 = 0.5 \text{ m}, L_2 = L_3 = 5m, \ L_4 = 1.66\text{m}$$

$$m_1 = 10 \text{ kg}, \ m_2 = m_3 = 1\text{kg}, \ m_4 = 0.1 \text{ kg}$$

$$J_1 = 20 \text{ kgm}^2, \ J_2 = 3 \text{ kgm}^2, \ EI_2 = EI_3 = 122.28 \text{ Nm}^2$$

The quasi-comparison functions for the flexible arm were chosen as a linear combination of clamped-free and clamped-clamped shape functions. Both families of shape functions have the functional form

$$\phi_i = \frac{1}{\sqrt{L}} \left[\cosh \lambda_i x/L - \cos \lambda_i x/L - \sigma_i \left(\sinh \lambda_i x/L - \sin \lambda_i x/L\right)\right], \; i = 1, 2, \ldots, n$$

The values of λ_i and σ_i for each family are given in Table 1. They correspond to two clamped-free and three clamped-clamped shape functions, for a total of $n = 5$ for each flexible arm.

The initial and final end-effector positions are defined by

$$X_i = 9.757 \text{ m}, \; Y_i = 1.914 \text{ m}, \; \theta_{4i} = 0 \text{ rad}$$

$$X_f = 5.000 \text{ m}, \; Y_f = 1.914 \text{ m}, \; \theta_{4f} = -\pi/2 \text{ rad}$$

and we note that the path from the initial to the final position represents a straight-line translation, while the orientation undergoes a 90° change. In terms of time, the translational and rotational accelerations represent one-cycle sinusoidal curves.

The maneuver time is $t_f = 2.5$ s. The zero-order actuator forces and torques to carry out the maneuver are shown in Fig. 3.

The control of the elastic motions and the perturbations in the rigid-body motions was extended to $t = 4$ s. Note that for $2.5 \text{ s} < t < 4$ s the system is time-invariant, during which time the control gains can be regarded as constant. The weighting matrices in the performance measure are

$$Q_k = 10I, \; R_k = I, \; P_N = 10I$$

The degree of stability constant is $\alpha = 0.1$. Moreover, the samping period is $T = 0.01$ s and the number of time increments is $N = 350$.

Time-lapse pictures of the uncontrolled and controlled robot configuration are shown in Figs. 4a and 4b, respectively, at the instants 0, 1, 1.5 and 2.5 s. Figures 5 and 6 show time histories of the errors in the end-effector position. The discrete-time open-loop and closed-loop poles for $\alpha = 0.01$ are given in Tables 2 and 3. For comparison, Fig. 7 shows the time history of the errors and Table 4 gives the closed-loop poles for $\alpha = 1$.

It should be pointed out that the actuator dynamics was also included in the computer simulation, but the effect turned out to be small.[11]

9. CONCLUSIONS

An orderly kinematic synthesis in conjunction with the Lagrangian approach permits the derivation of the equations of motion for an articulated multibody system, such as those describing the dynamical behavior of a flexible space robot, without the imposition of constraints. The equations are nonlinear and of relatively high order. A perturbation approach permits the separation of the problem into a zero-order problem (in a perturbation sense) for the rigid-body maneuvering of the space robot and a first-order problem for the control of the elastic motions and the perturbations from the rigid-body motions. The robot mission consists of carrying a payload over a prescribed trajectory and placing it in a certain orientation relative to the inertial space. This represents the zero-order problem and the control can be carried out open loop. The first-order equations defining the first-order problem (in a perturbation sense) are linear, time-varying, of high-order and subject to persistent disturbances. The persistent disturbances are treated by means of feedforward control. All other disturbances are controlled closed loop, with the feedback control being designed by means of discrete-time LQR theory with prescribed degree of stability. In a numerical example, the controls derived by the perturbation approach are found to work satisfactorily when applied to the original nonlinear system.

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Table 1. Shape Function Coefficients

i	λ_i	σ_i
1	1.8751	0.7341
2	4.6941	1.0185
3	7.8548	0.9992
4	10.9955	1.0000
5	14.1372	1.0000

Table 2. Discrete-Time Open-Loop Poles

No.	Pole Location	Mag.	No.	Pole Location	Mag.
1,2	$-0.840 \pm 0.543i$	1.000	17,18	$0.991 \pm 0.135i$	1.000
3,4	$-0.778 \pm 0.629i$	1.000	19,20	$0.994 \pm 0.107i$	1.000
5,6	$-0.700 \pm 0.714i$	1.000	21,22	1.000	1.000
7,8	$-0.690 \pm 0.724i$	1.000	23,24	1.000	1.000
9,10	$0.586 \pm 0.810i$	1.000	25,26	1.000	1.000
11,12	$0.629 \pm 0.778i$	1.000	27,28	1.000	1.000
13,14	$0.902 \pm 0.431i$	1.000	29,30	1.000	1.000
15,16	$0.921 \pm 0.390i$	1.000	31,32	1.000	1.000

Table 3. Discrete-Time Closed-Loop Poles for $\alpha = 0.1$

No.	Pole Location	Mag.	No.	Pole Location	Mag.
1,2	$-0.169 \pm 0.546i$	0.572	18,19	$0.803 \pm 0.976 \times 10^{-1}i$	0.809
3	0.493×10^{-2}	0.005	20	0.805	0.805
4	0.120×10^{-1}	0.012	21	0.807	0.807
5	0.125	0.125	22,23	$0.814 \pm 0.362 \times 10^{-2}i$	0.814
6	0.204	0.204	24,25	0.817	0.817
7,8	$0.302 \pm 0.148i$	0.336	26	0.817	0.817
9,10	$0.454 \pm 0.493i$	0.670	27	0.819	0.819
11,12	$0.468 \pm 0.323i$	0.569	28,29	$0.821 \pm 0.366 \times 10^{-2}i$	0.821
12,13	$0.536 \pm 0.500i$	0.733	30	0.822	0.822
15,16	$0.749 \pm 0.860 \times 10^{-1}i$	0.754	31	0.822	0.822
17	0.792	0.792	32	0.827	0.827

Table 4. Discrete-Time Closed-Loop Poles for $\alpha = 1$

No.	Pole Location	Mag.	No.	Pole Location	Mag.
1	-0.566	0.566	17,18	$0.139 \pm 0.844 \times 10^{-2}i$	0.139
2,3	$-0.160 \pm 0.186i$	0.246	19,20	$0.150 \pm 0.022i$	0.152
4,5	$-0.109 \pm 0.275i$	0.296	21,22	$0.187 \pm 0.145i$	0.236
6,7	$0.062 \pm 0.088i$	0.108	23,24	$0.198 \pm 0.288 \times 10^{-1}i$	0.200
8	-0.177×10^{-1}	0.018	25	0.251	0.251
9,10	$0.779 \times 10^{-2} \pm 0.209i$	0.209	26,27	$0.252 \pm 0.180i$	0.310
11,12	$0.072 \pm 0.088i$	0.114	28,29	$0.279 \pm 0.490i$	0.564
13,14	$0.118 \pm 0.016i$	0.119	30,31	$0.328 \pm 0.148i$	0.360
15,16	$0.132 \pm 0.920 \times 10^{-2}i$	0.132	32	0.430	0.430

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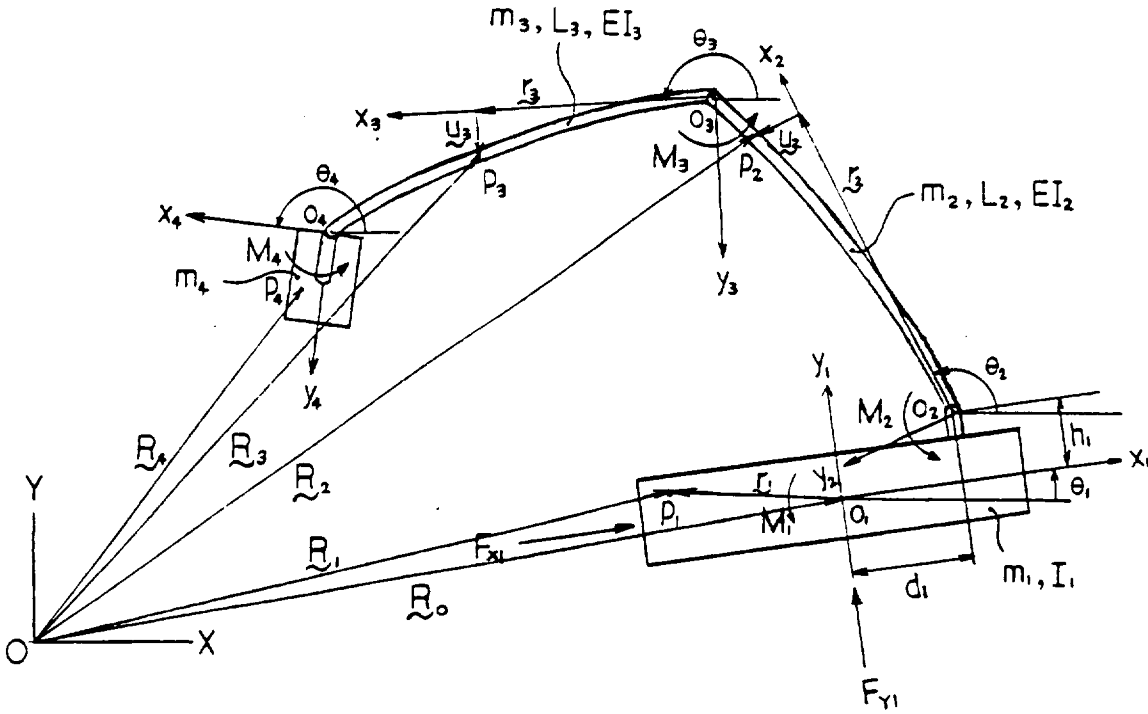

Figure 1. Flexible Space Robot

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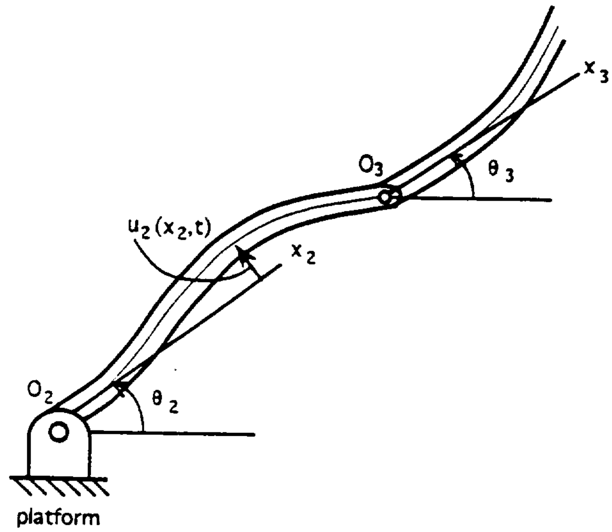

Figure 2. Displacements for Body 2

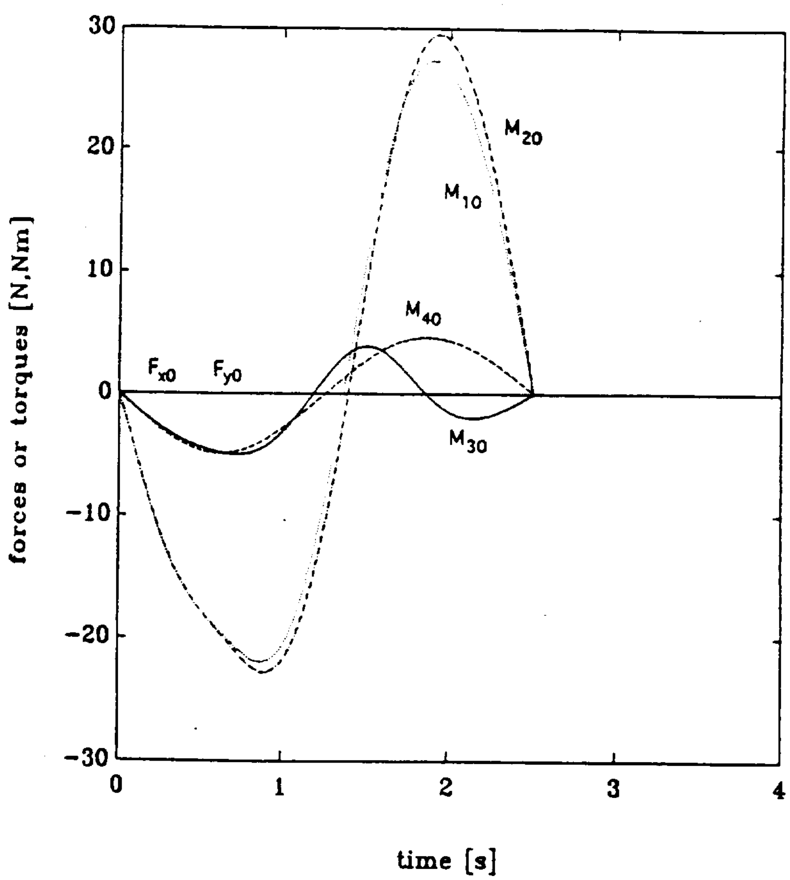

Figure 3. Zero-Order Forces and Torques

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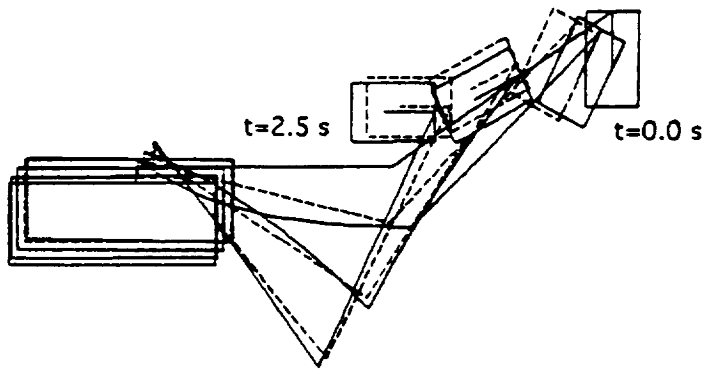

Figure 4a. Time-Lapse Picture of the Uncontrolled Robot Maneuver

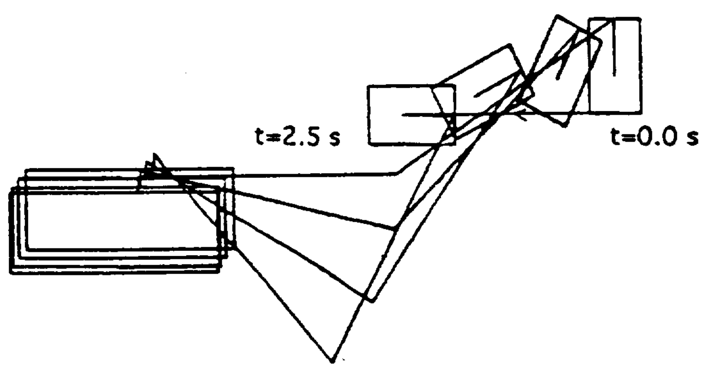

Figure 4b. Time-Lapse Picture of the LQR-Controlled Robot Maneuver

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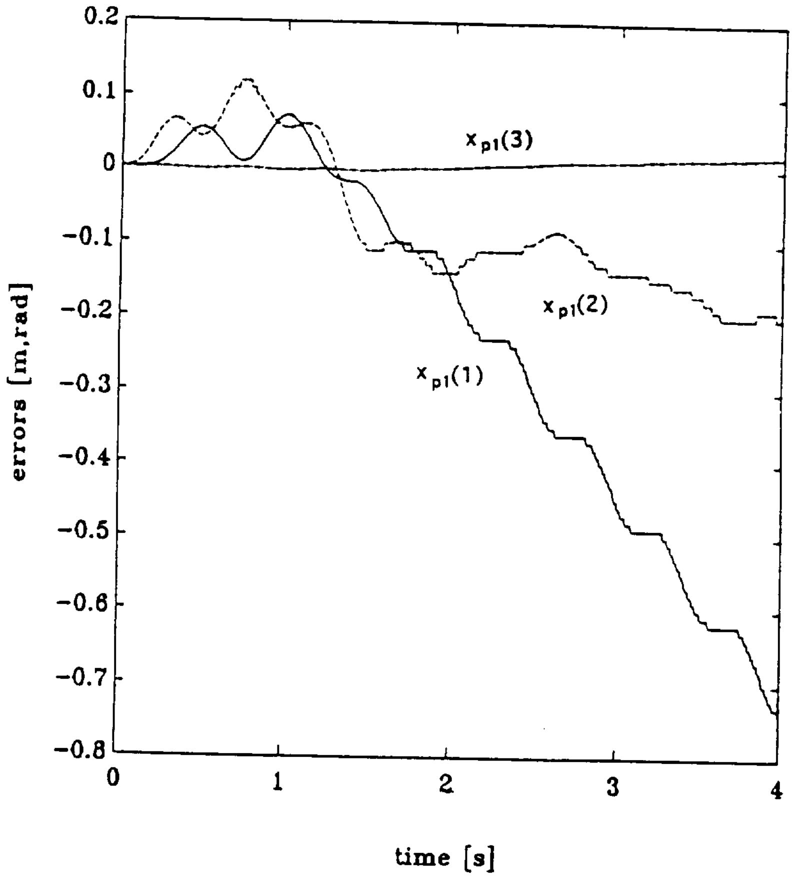

Figure 5. Uncontrolled End-Effector Position Errors

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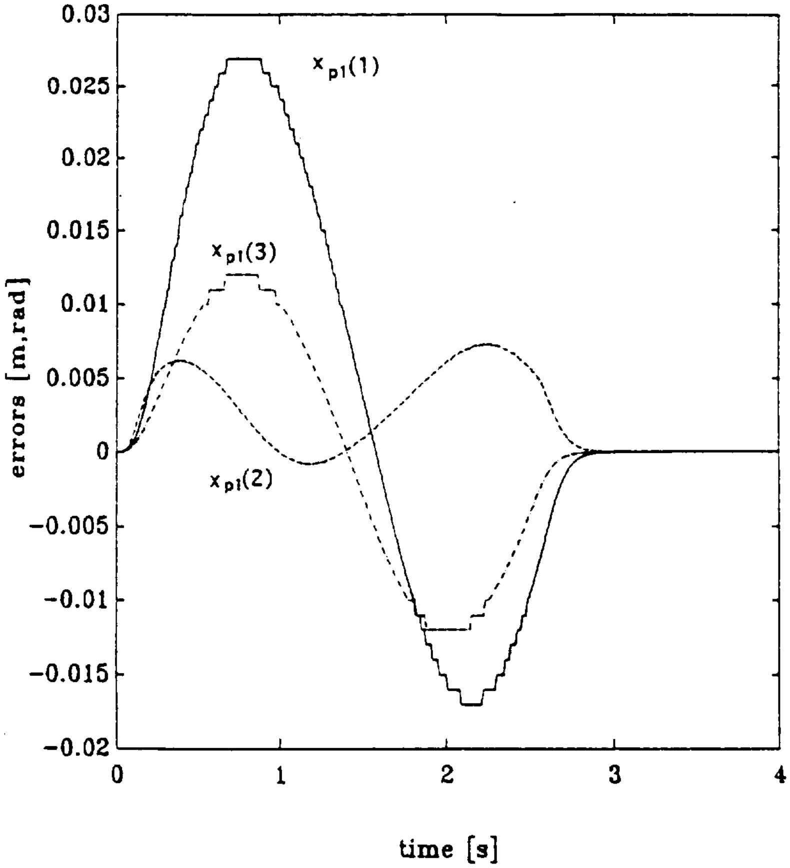

Figure 6. LQR-Controlled End-Effector Position Errors for $\alpha = 0.1$

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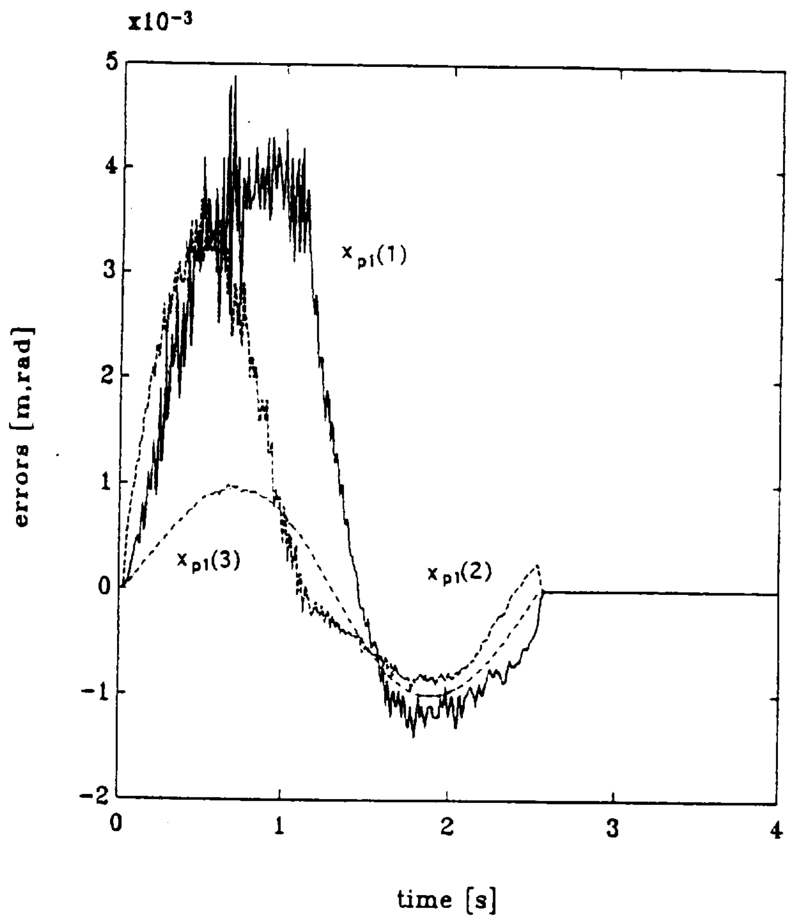

Figure 7. LQR-Controlled End-Effector Position Errors for $\alpha = 1$

G 3882

CONTROL OF A FLEXIBLE SPACE ROBOT EXECUTING A DOCKING MANEUVER[†]

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ABSTRACT

This paper is concerned with a flexible space robot executing a docking maneuver with a target whose motion is not known a priori. The dynamical equations of the space robot are first derived by means of Lagrange's equations and then separated into two sets of equations suitable for rigid-body maneuver and vibration suppression control. For the rigid-body maneuver, on-line feedback tracking control is carried out by means of an algorithm based on Liapunov-like methodology and using on-line measurements of the target motion. For the vibration suppression, LQR feedback control in conjunction with disturbance compensation is carried out by means of piezoelectric sensor/actuator pairs dispersed along the flexible arms. Problems related to the digital implementation of the control algorithms, such as the bursting phenomenon and system instability, are discussed and a modified discrete-time control scheme is developed. A numerical example demonstrates the control algorithms.

1. INTRODUCTION

One of the functions of a space robot is to deliver payloads accurately and smoothly to a moving target. An example of such a space robot is shown in Fig. 1. The robot consists of a rigid base, two flexible arms attached to the base in series and an end-effector/payload. To carry out the mission described, the space robot must have its own control system enabling the platform to translate and rotate and its arms to rotate. In this paper, the target motion is assumed not to be known a priori, so that the control permitting the space robot to execute

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the docking maneuver must be based on on-line measurements.

The equations governing the behavior of space robots are nonlinear and can be expressed in the general form of the state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1a}$$

and the output equation

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) \tag{1b}$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the control force vector and $\mathbf{y}$ is the output vector, usually defined as the position and orientation variables of the end-effector. The target output vector $\mathbf{y}_t$ is defined as the position and orientation variables of the target. We can then define the error vector as

$$\mathbf{e} = \mathbf{y}_t - \mathbf{y} \tag{2}$$

The problem reduces to that of designing a control law $\mathbf{u}(t)$ so that $\mathbf{e}$ and its time derivative $\dot{\mathbf{e}}$ are driven to zero.

There are two significant differences between industrial robots in current use and space robots considered here. In the first place, industrial robots are mounted on a fixed base, whereas space robots are mounted on space platforms capable of translations and rotations. The second significant difference is that space robots must be very light, and hence very flexible, unlike industrial robots characterized by very bulky and stiff arms. The flexibility of the robot arms causes elastic vibration, which tends to affect adversely the performance of the end-effector. Both a floating platform and flexibility are being considered in this paper.

In the case of space-based robots, research has been carried out on the assumption that the platform floats freely,[1–6] i.e., that there are no external forces and torques acting on the system, which implies that the system linear and angular momentum are conserved. For a space robot tracking a moving target, it is unrealistic to make such an assumption, so that algorithms concerned with free-floating space robots are not applicable to the problem considered here.

The most commonly used approach to robotics can be described as follows: first, inverse kinematics is performed to obtain the desired robot configuration trajectory $\mathbf{q}_d(t)$ from the desired end-effector trajectory $\mathbf{y}_d(t)$. Then, using the system equations of motion, inverse dynamics is performed to obtain the control force realizing $\mathbf{q}_d(t)$. If the target motion is known a priori, the end-effector's trajectory, as well as the robot trajectory, can be determined by an off-line planning algorithm. For a kinematically- redundant robot, such as the one considered here, the robot redundancy can be used to achieve optimality.[7]

If the target motion is not known a priori, planning is impossible. Even when the target motion is known, it is very likely that some unexpected disturbance can cause errors. In view of this, on-line feedback control for the tracking problem, whereby the control decision is based on measurements of the current output error, appears more attractive. The technical literature on this subject is not very abundant. For tracking control, the Liapunov stability concept appears quite useful. Wang[8] used it to design a guidance law for a spacecraft docking with another spacecraft. The two docking objects are assumed to be three-dimensional rigid bodies and to have their own control system on board. Another assumption used in Ref. 8 is that the motion of the target decays to zero with time. Recently, Novakovic[9] presented a technique using Liapunov-like methodology for robot tracking control problem. In this paper, the algorithm presented in Ref. 9 is adopted and modified for the tracking control of flexible space robots.

In the case of flexible space structures, maneuvering motions excite vibration of the flexible members. There are two major control schemes for flexible manipulators. The first is based on linearized models derived from the nonlinear equations of motion of the flexible manipulator on the assumption that maneuver motions are much larger than elastic motions. Such a perturbation approach was developed by Meirovitch and Quinn[10,11] and applied by Meirovitch and Kwak[12,13] to the maneuvering and control of articulated flexible spacecraft and by Modi and Chang[14] and Meirovitch and Lim[15] to the maneuvering and control of flexible robots. The second is the adaptive control,[16] which does not need dynamical models. Instead, an auto-regressive-moving average (ARMA) model of system identification is used.

A control law for flexible manipulators based on the Liapunov method was proposed by Bang and Junkins.[17] It represents proportional and derivative control and includes a boundary force as a feedback force. This control scheme is valid only for problems in which the system approaches an equilibrium point in the state space.

References 15 and 18 are concerned with flexible space robots of the type considered here, but the mission is more modest in scope. Indeed, in Ref. 15 the mission is to place a payload in a certain position and orientation in space and in Ref. 18 the objective is to dock with a target whose motion is known a priori.

In this paper, a control scheme permitting a flexible space robot to track and dock with a moving target whose motion is not known a priori is presented. For the robot maneuver, on-line feedback tracking control is carried out by means of an algorithm based on Liapunov-like methodology and using on-line measurements of the target motion. For the vibration suppression, linear quadratic regulator (LQR) control in conjunction with disturbance compensation is carried out by means of sensor/actuator pairs dispersed along the flexible arms. A modified discrete-time control scheme is developed, and problems related to the digital implementation of the control algorithms are discussed. The control algorithms are demonstrated by means of a numerical example.

2. EQUATIONS OF MOTION

The flexible space robot and the coordinate systems are shown in Fig. 2. Body 0 represents the robot base, assumed to be rigid. Bodies 1 and 2 are the robot manipulator arms attached in series to Body 0 and they are flexible. Body 3 is the end-effector/payload, also assumed to rigid. For planar motion, the robot base is capable of two translations, x_0 and y_0, and one rotation, θ_0; the two flexible arms are capable of the rotations θ_1 and θ_2 and the elastic vibrations u_1 and u_2 and the end-effector is capable of the rotation θ_3. Referring to Fig. 2, the displacement vector $\mathbf{U}_0$ and velocity vector $\mathbf{V}_0$ for a typical point in Body 0 are as follows:

$$\mathbf{U}_0 = \mathbf{R} + C_0^T \mathbf{R}_0 \tag{3a}$$

$$\mathbf{V}_0 = \dot{\mathbf{R}} + C_0^T \tilde{\omega}_0 \mathbf{R}_0 \tag{3b}$$

Similarly, for Body 1

$$\mathbf{U}_1 = \mathbf{R} + C_0^T \mathbf{L}_0 + C_1^T (\mathbf{r}_1 + \mathbf{u}_1) \tag{4a}$$

$$\mathbf{V}_1 = \dot{\mathbf{R}} + C_0^T \tilde{\omega}_0 \mathbf{L}_0 + C_1^T \tilde{\omega}_1 (\mathbf{r}_1 + \mathbf{u}_1) + C_1^T \dot{\mathbf{u}}_1 \tag{4b}$$

for Body 2

$$\mathbf{U}_2 = \mathbf{R} + C_0^T \mathbf{L}_0 + C_1^T (\mathbf{L}_1 + \mathbf{u}_{12}) + C_2^T (\mathbf{r}_2 + \mathbf{u}_2) \tag{5a}$$

$$\begin{aligned}\mathbf{V}_2 = \dot{\mathbf{R}} &+ C_0^T \tilde{\omega}_0 \mathbf{L}_0 + C_1^T \tilde{\omega}_1 (\mathbf{L}_1 + \mathbf{u}_{12}) + C_1^T \dot{\mathbf{u}}_{12} \\ &+ C_2^T \tilde{\omega}_2 (\mathbf{r}_2 + \mathbf{u}_2) + C_2^T \dot{\mathbf{u}}_2\end{aligned} \tag{5b}$$

and for Body 3

$$\mathbf{U}_3 = \mathbf{R} + C_0^T \mathbf{L}_0 + C_1^T (\mathbf{L}_1 + \mathbf{u}_{12}) + C_2^T (\mathbf{L}_2 + \mathbf{u}_{23}) + C_3^T \mathbf{r}_3 \tag{6a}$$

$$\begin{aligned}\mathbf{V}_3 = \dot{\mathbf{R}} &+ C_0^T \tilde{\omega}_0 \mathbf{L}_0 + C_1^T \tilde{\omega}_1 (\mathbf{L}_1 + \mathbf{u}_{12}) + C_1^T \dot{\mathbf{u}}_{12} \\ &+ C_2^T \tilde{\omega}_2 (\mathbf{L}_2 + \mathbf{u}_{23}) + C_2^T \dot{\mathbf{u}}_{23} + C_3^T \tilde{\omega}_e \mathbf{r}_3\end{aligned} \tag{6b}$$

where

$$C_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \qquad i = 0, 1, 2, 3 \tag{7}$$

are matrices of direction cosines,

$$\tilde{\omega}_i = \begin{bmatrix} 0 & -\dot{\theta}_i \\ \dot{\theta}_i & 0 \end{bmatrix} \qquad i = 0, 1, 2, 3 \tag{8}$$

are skew symmetric angular velocity matrices,

$$\mathbf{R} = [x_0 \;\; y_0]^T, \quad \mathbf{r}_1 = [x_1 \;\; 0]^T, \quad \mathbf{r}_2 = [x_2 \;\; 0]^T \tag{9}$$

are position vectors and

$$\mathbf{u}_1 = [0 \;\; u_1]^T, \quad \mathbf{u}_2 = [0 \;\; u_2]^T \tag{10}$$

are elastic displacement vectors. Moreover,

$$u_{12} = u_1|_{x_1 = L_1}, \quad u_{23} = u_2|_{x_2 = L_2} \tag{11}$$

The elastic displacements are discretized as follows:

$$u_i\left(x_i,t\right) = \mathbf{\Phi}_i^T\left(x_i\right)\boldsymbol{\xi}_i\left(t\right), \quad i = 1,2 \tag{12}$$

where $\mathbf{\Phi}_i(x)$ are vectors of quasi-comparison functions[19] and $\boldsymbol{\xi}_i(t)$ are vectors of generalized displacements. Regarding the robot arms as beams in bending, the quasi-comparison functions can be chosen as linear combination of the admissible functions

$$\phi_k = \cosh\frac{\lambda_k x}{L} - \cos\frac{\lambda_k x}{L} - \sigma_k\left(\sinh\frac{\lambda_k x}{L} - \sin\frac{\lambda_k x}{L}\right), \quad k = 1,2,\ldots \tag{13}$$

which represent the eigenfunctions of a clamped-free beam for k odd and clamped-clamped beam for k even, where λ_k and σ_k are nondimensional parameters.

Using Eqs. (3)-(13), the kinetic energy of the system can be written as

$$T = \sum_{i=0}^{3} T_i = \frac{1}{2}\sum_{i=0}^{3}\int_{\text{Body } i}\rho_i \mathbf{V}_i^T\mathbf{V}_i dD_i = \frac{1}{2}\dot{\mathbf{q}}^T M\dot{\mathbf{q}} \tag{14}$$

where $\mathbf{q} = \left[\mathbf{R}^T\ \theta_0\ \theta_1\ \theta_2\ \theta_3\ \ \boldsymbol{\xi}_1^T\ \boldsymbol{\xi}_2^T\right]^T$ is the configuration vector and M is the mass matrix with entries given in Appendix A.

The potential energy for the system is due entirely to the elasticity of the robot arms and can be written in the form

$$V = \sum_{i=1}^{2}\frac{1}{2}\boldsymbol{\xi}_i^T K_i\boldsymbol{\xi}_i = \frac{1}{2}\mathbf{q}^T K\mathbf{q} \tag{15}$$

where

$$K = \text{block-diag}\left[0\ \ \bar{K}_1\ \ \bar{K}_2\right] \tag{16}$$

in which

$$\bar{K}_i = \int_0^{L_i} EI_i\mathbf{\Phi}_i''\left(\mathbf{\Phi}_i''\right)^T dx_i, \quad i = 1,2 \tag{17}$$

are the stiffness matrices for Bodies i, in which EI_i denotes bending stiffnesses. Note that the gravitational potential is ignored here on the assumption that it represents a second-order effect.

The control forces acting on the robot system include the horizontal and vertical thrusts F_x and F_y acting at the base center, the external torque M_0 acting on the base, the internal joint torques M_1, M_2 and M_3 acting at the joints and the distributed internal moments $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ generated by piezoelectric actuators on links 1 and 2. We define the control force vector as $\mathbf{F} = \begin{bmatrix} F_x & F_y & M_0 & M_1 & M_2 & M_3 & \boldsymbol{\tau}_1^T & \boldsymbol{\tau}_2^T \end{bmatrix}^T$. Then, the virtual work of the system can be written in the form

$$\begin{aligned} \delta W =& F_x \delta x_0 + F_y \delta y_0 + M_0 \delta\theta_0 + M_1 \left(\delta\theta_1 - \delta\theta_0\right) \\ &+ M_2 \left(\delta\theta_2 - \delta\theta_1 - \Phi_1'^T (L_1) \delta\boldsymbol{\xi}_1\right) + M_3 \left(\delta\theta_3 - \delta\theta_2 - \Phi_2'^T (L_2) \delta\boldsymbol{\xi}_2\right) \\ &+ \sum_{i=1}^{m_1} \tau_{1i} \Phi_1''^T (x_{1i}) \delta\boldsymbol{\xi}_1 + \sum_{i=1}^{m_2} \tau_{2i} \Phi_2''^T (x_{2i}) \delta\boldsymbol{\xi}_2 = \mathbf{Q}^T \delta\mathbf{q} \end{aligned} \tag{18}$$

where $\mathbf{Q}$ is a generalized force vector defined as

$$\mathbf{Q} = G\mathbf{F} \tag{19}$$

The entries of the matrix G are given in Appendix A.

Lagrange's equations for the system can be expressed in the symbolic vector form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q} \tag{20}$$

Inserting Eqs. (15), (16) and (19) into Eq. (20), we obtain the system equations in the matrix form

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{Q} \tag{21}$$

The entries of the matrix C are also given in Appendix A.

Equation (21) represents the equation governing the motion of the flexible space robot. It is used for computer simulation of the dynamical system. For the purpose of control design, Eq. (21) is conveniently separated into two sets of equations, rigid-body motion equations and elastic vibration equations. To this end, we write $\mathbf{q} = \begin{bmatrix} \mathbf{q}_r^T & \mathbf{q}_e^T \end{bmatrix}^T$ and $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_r^T & \mathbf{Q}_e^T \end{bmatrix}^T$, where $\mathbf{q}_r = \begin{bmatrix} x_0 & y_0 & \theta_0 & \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T$ is a rigid-body displacement vector, $\mathbf{q}_e = \begin{bmatrix} \boldsymbol{\xi}_1^T & \boldsymbol{\xi}_2^T \end{bmatrix}^T$ is an elastic displacement vector and $\mathbf{Q}_r$ and $\mathbf{Q}_e$ are corresponding generalized force vectors. Then Eq. (21) can be written in the partitioned matrix form

$$\begin{bmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} C_{rr} & C_{re} \\ C_{er} & C_{ee} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_r \\ \dot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_r \\ \mathbf{Q}_e \end{bmatrix} \tag{22}$$

After some algebraic manipulations, and ignoring higher-order terms in the elastic displacements, Eq. (22) can be separated into

$$M_r(\mathbf{q}_r)\ddot{\mathbf{q}}_r + C_r(\mathbf{q}_r, \dot{\mathbf{q}}_r)\dot{\mathbf{q}}_r + \mathbf{d}_e(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{Q}_r \tag{23}$$

and

$$M_e(\mathbf{q}_r)\ddot{\mathbf{q}}_e + C_e(\mathbf{q}_r, \dot{\mathbf{q}}_r)\dot{\mathbf{q}}_e + K_e(\mathbf{q}_r, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\mathbf{q}_e + \mathbf{d}_r(\mathbf{q}_r, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r) = \mathbf{Q}_e \tag{24}$$

where M_r is the rigid-body part of the mass matrix M_{rr} and C_r is the rigid-body part of C_{rr}. Moreover, $M_e = M_{ee}$, $C_e = C_{ee}$, K_e consists of the stiffness matrix K and the part due to elasticity in the matrices M_{re} and C_{re} and $\mathbf{d}_e$ and $\mathbf{d}_r$ are disturbance vectors. The entries of the various matrices are given in Appendix B. The term $\mathbf{d}_e$ in Eq. (23) is a linear combination of $\mathbf{q}_e$, $\dot{\mathbf{q}}_e$ and $\ddot{\mathbf{q}}_e$. It can be regarded as a disturbance due to the flexibility of the robot arms. The term $\mathbf{d}_r$ in Eq. (24) is a function of $\mathbf{q}_r$, $\dot{\mathbf{q}}_r$ and $\ddot{\mathbf{q}}_r$. It can be regarded as a disturbance due to the rigid-body maneuvering of the robot. Equations (23) and (24) are coupled. The coupling between rigid-body motions and flexible vibration is provided in Eq. (24) by the persistent disturbance $\mathbf{d}_r$ from the rigid-body motion, which causes the elastic motion $\mathbf{q}_e$, $\dot{\mathbf{q}}_e$ and $\ddot{\mathbf{q}}_e$. In turn, the elastic motion disturbs the rigid body motion through $\mathbf{d}_e$ in Eq. (23). Equation (23) is used for the design of the maneuver control for tracking a moving target and Eq. (24) is used for design of control for vibration suppression.

3. TRACKING CONTROL ALGORITHM USING LIAPUNOV-LIKE METHODOLOGY

In this section, the general idea of Liapunov-like methodology for tracking control developed for rigid robots[9] is introduced.

The dynamical equation of a *rigid* robot is given by

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{Q} \tag{25}$$

and the kinematic relation between the robot configuration vector $\mathbf{q}$ and robot output vector $\mathbf{y}_e$ is given by

$$\mathbf{y}_e = \mathbf{f}(\mathbf{q}) \tag{26}$$

so that

$$\dot{\mathbf{y}}_e = J(\mathbf{q})\,\dot{\mathbf{q}} \tag{27}$$

and

$$\ddot{\mathbf{y}}_e = J(\mathbf{q})\,\ddot{\mathbf{q}} + \dot{J}(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} \tag{28}$$

where $J(\mathbf{q}) = [\partial \mathbf{f}/\partial \mathbf{q}]$ is the Jacobian matrix.

Because tracking is carried out by the end-effector, the tracking problem consists of driving the error $\mathbf{e} = \mathbf{y}_t - \mathbf{y}_e$ and its time derivative $\dot{\mathbf{e}}$ to zero. To this end, a Liapunov function is defined by

$$V = \frac{1}{2}\mathbf{z}^T\mathbf{z} \tag{29a}$$

where

$$\mathbf{z} = (\dot{\mathbf{e}} + \beta\mathbf{e}) \tag{29b}$$

in which β is a positive scalar. If the control is designed in such a way that

$$\dot{V} = -\sigma V, \quad \sigma = \ln\left(\frac{V_0}{\epsilon}\right)/t_s \tag{30a, b}$$

where ϵ is an arbitrarily small positive scalar and V_0 is the initial value of V, it is guaranteed that the function V remains in the ϵ-neighborhood of zero for $t > t_s$, no matter how the target motion changes. This ensures that the error $\mathbf{e}$ and its derivative $\dot{\mathbf{e}}$ are also very close to zero.

We consider the nonlinear control law

$$\mathbf{Q} = M(\mathbf{q})\,\mathbf{u}_r + C(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} \tag{31}$$

where $\mathbf{u}_r$ is chosen in the form

$$\mathbf{u}_r = \mathbf{w}\frac{h_1 + h_2}{\mathbf{z}^T J\mathbf{w}} \tag{32}$$

in which $\mathbf{w}$ is an arbitrarily chosen vector and

$$h_1 = \mathbf{z}^T\left(\ddot{\mathbf{y}}_t - \dot{J}\dot{\mathbf{q}} + \beta\dot{\mathbf{e}}\right), \quad h_2 = 0.5\sigma\mathbf{z}^T\mathbf{z} \equiv \sigma V \tag{33a, b}$$

It can be shown that the control algorithm described above yields the desired result, i.e., Eqs. (30a,b).

The control algorithm possesses the following advantages:

1) The control decision is made using on-line information of the current robot state $(\mathbf{q}, \dot{\mathbf{q}})$ and target state ($\mathbf{e}, \dot{\mathbf{e}}$ and $\ddot{\mathbf{y}}_t$). The feedback control can automatically counteract adverse disturbances in space and achieve the final docking in an accurate and smooth way.

2) The on-line calculation is relatively simple, as it involves neither inverse kinematics nor matrix inversions.

3) Stability is always guaranteed by Liapunov stability theorem, as can be seen from Eqs. (30), no matter how the target motion changes.

However, after applying the above algorithm directly to our space robot system and simulating the system in both continuous time and discrete time, the results from discrete-time system exhibited some undesirable phenomenon, although the performance of the continuous system was good. As shown in Fig. 3, in which the solid line denotes continuous-time results and the dashed line denotes discrete-time results, the control force in discrete time exhibits periods of oscillatory behavior. Further numerical simulations show that the magnitude of the control force during chattering is bounded, although very large, and its mean value is close to the results of the corresponding continuous time system. Moreover, the occurence of the oscillating period is random, and the length of the oscillating periods and the length of the "good performance" periods are both unpredictable. This phenomenon is similar to the so-called "bursting", which appears frequently in discrete-time adaptive systems and has been reported for almost a decade.[20] It is important to keep the control force from bursting. Otherwise the possibility exists that the control cannot be realized. To this end, a modified version of the above algorithm is presented, which also takes into account the flexibility of the robot arms.

4. MODIFIED TRACKING CONTROL ALGORITHM FOR FLEXIBLE SPACE ROBOTS

To apply Liapunov-like methodology to flexible space robots, we first extend the kinematical relation given by Eq. (26) to flexible space robots as follows:

$$\begin{aligned} x_e &= x_0 - L_0 \sin\theta_0 + L_1 \cos\theta_1 + L_2 \cos\theta_2 + L_3 \cos\theta_3 - u_{12} \sin\theta_1 - u_{23} \sin\theta_2 \\ y_e &= y_0 + L_0 \cos\theta_0 + L_1 \sin\theta_1 + L_2 \sin\theta_2 + L_3 \sin\theta_3 + u_{12} \cos\theta_1 + u_{23} \cos\theta_2 \\ \theta_e &= \theta_3 \end{aligned} \tag{34}$$

For kinematical analysis, we define $\bar{\mathbf{q}} = \begin{bmatrix} \mathbf{q}_r^T & \mathbf{q}_u^T \end{bmatrix}^T$, where $\mathbf{q}_r$ was defined earlier and $\mathbf{q}_u = [u_{12} \;\; u_{23}]^T$. The Jacobian matrix $\bar{J}$, obtained by differentiating Eq. (34) with respect to $\bar{\mathbf{q}}$, has the form

$$\bar{J} = [J_r \;\; J_u] \tag{35}$$

where

$$J_r = \begin{bmatrix} 1 & 0 & -L_0\cos\theta_0 & -L_1\sin\theta_1 - u_{12}\cos\theta_1 & -L_2\sin\theta_2 - u_{23}\cos\theta_2 & -L_3\sin\theta_3 \\ 0 & 1 & -L_0\sin\theta_0 & L_1\cos\theta_1 - u_{12}\sin\theta_1 & L_2\cos\theta_2 - u_{23}\sin\theta_2 & L_3\cos\theta_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{36a}$$

$$J_u = \begin{bmatrix} -\sin\theta_1 & -\sin\theta_2 \\ \cos\theta_1 & \cos\theta_2 \\ 0 & 0 \end{bmatrix} \tag{36b}$$

Hence, we can write the relations

$$\dot{\mathbf{y}}_e = \bar{J}\dot{\bar{\mathbf{q}}} \tag{37}$$

$$\ddot{\mathbf{y}}_e = \bar{J}\ddot{\bar{\mathbf{q}}} + \dot{\bar{J}}\dot{\bar{\mathbf{q}}} \tag{38}$$

The dynamical equation for the rigid-body motion of the space robot is given by Eq. (23). We first define a nonlinear control law for $\mathbf{Q}_r$ as follows:

$$\mathbf{Q}_r = M_r(\mathbf{q}_r)\,\mathbf{u}_r + C_r(\mathbf{q}_r, \dot{\mathbf{q}}_r)\,\dot{\mathbf{q}}_r \tag{39}$$

Substituting Eq. (39) into Eq. (23), we obtain

$$\ddot{\mathbf{q}}_r = \mathbf{u}_r - M_r^{-1}\mathbf{d}_e \tag{40}$$

To prevent the bursting phenomenon, we propose a decoupled Liapunov function defined by

$$V_i = \frac{1}{2}z_i^2, \quad z_i = \dot{e}_i + \beta e_i, \quad i = 1,2,3 \tag{41a, b}$$

Taking the derivative of Eq. (41a) and using Eqs. (37), (38) and (40), we obtain

$$\dot{V}_i = z_i h_i - z_i \left([J_r \mathbf{u}_r]_i - \left[J_r M_r^{-1} \mathbf{d}_e \right]_i \right), \quad i = 1, 2, 3 \tag{42}$$

where $[\ ]_i$ denotes the $i-th$ element of a vector and h_i are the components of the vector

$$\mathbf{h} = \ddot{\mathbf{y}}_t - \dot{J}\dot{\mathbf{q}} + \beta \dot{\mathbf{e}} - J_u \ddot{\mathbf{q}}_u \tag{43}$$

Because M_r is a positive definite matrix, M_r^{-1} is bounded, and we note that J_r is also bounded. Moreover, from Eq. (B.3) in Appendix B, we see that $\mathbf{d}_e$ is a linear combination of $\mathbf{q}_e$, $\dot{\mathbf{q}}_e$ and $\ddot{\mathbf{q}}_e$. We then assume that $\mathbf{d}_e$ is bounded in accordance with our ultimate goal of vibration suppression. Hence, we can assume that the term $\left[J_r M_r^{-1} \mathbf{d}_e \right]_i$ is bounded and satisfies the relation

$$\left[J_r M_r^{-1} \mathbf{d}_e \right]_i < \delta_i, \quad i = 1, 2, 3 \tag{44}$$

From Eq. (44), we have

$$z_i \left[J_r M_r^{-1} \mathbf{d}_e \right]_i < |z_i| \delta_i, \quad i = 1, 2, 3 \tag{45}$$

If we can determine a vector $\mathbf{u}_r$ that satisfies the following conditions:

$$z_i [J_r \mathbf{u}]_r = z_i h_i + \frac{1}{2} \alpha_i z_i^2 + |z_i| \delta_i, \quad i = 1, 2, 3 \tag{46}$$

then

$$\dot{V}_i = \frac{1}{2} \alpha_i z_i^2 + \left[J_r M_r^{-1} \mathbf{d}_e \right]_i - |z_i| \delta_i < -\frac{1}{2} \alpha_i z_i^2 = -\alpha_i V_i, \quad i = 1, 2, 3 \tag{47}$$

According to the Liapunov stability theorem, Eq. (46) is the sufficient condition for our tracking problem. We further simplify Eq. (46) by assuming $z_i \neq 0$, so that

$$[J_r \mathbf{u}_r]_i = h_i + \frac{1}{2} \alpha_i z_i + \operatorname{sgn}(z_i) \delta_i, \quad i = 1, 2, 3 \tag{48}$$

or

$$[J_r \mathbf{u}_r]_i = s_i, \quad i = 1, 2, 3 \tag{49}$$

with

$$s_i = \ddot{y}_{ti} - \left[\dot{J} \dot{\mathbf{q}} \right]_i + \beta \dot{e}_i - [J_u \ddot{\mathbf{q}}_u]_i + \frac{1}{2} \alpha_i z_i + \operatorname{sgn}(z_i) \delta_i \tag{50}$$

Equation (49) can be expressed in the matrix form

$$J_r \mathbf{u}_r = \mathbf{s} \tag{51}$$

where $\mathbf{s} = [s_1 \;\; s_2 \;\; s_3]^T$ and J_r is a 3×6 matrix. The solution of Eq. (51) does not yield a unique $\mathbf{u}_r$. This agrees with Eq. (32) in the original control scheme in which $\mathbf{w}$ is an arbitrarily chosen vector. Here we can simply prescribe the redundant degrees of freedom and then solve Eq. (51) accordingly.

As a simple example, we constrain three components of $\mathbf{u}_r$ by taking

$$u_{r3} = u_{r4} = u_{r5} = 0 \tag{52}$$

for the entire tracking period and use Eqs. (51) to solve for the other three components of $\mathbf{u}_r$ on-line, with the result

$$\begin{aligned} u_{r1} &= s_1 + L_3 \sin\theta_3 u_{r6} \\ u_{r2} &= s_2 - L_3 \cos\theta_3 u_{r6} \\ u_{r6} &= s_3 \end{aligned} \tag{53}$$

The above algorithm for $\mathbf{u}_r$, together with Eq. (39), represents the maneuver control for a flexible space robot tracking a moving target whose motion is not known a priori. The control algorithm requires that the following conditions be satisfied:

1) The output error vector $\mathbf{e}$ and its time derivative $\dot{\mathbf{e}}$ can be measured on-line.
2) The target output acceleration $\ddot{\mathbf{y}}_t$ can be measured or estimated on-line.
3) The robot rigid-body displacement vector $\mathbf{q}_r$ and its time derivative $\dot{\mathbf{q}}_r$ can be measured on-line.
4) The elastic tip displacement vector $\mathbf{q}_u$ and its time derivatives $\dot{\mathbf{q}}_u$ and $\ddot{\mathbf{q}}_u$ can be measured on-line.
5) The elastic vibration of the robot arms should be controlled so that a reasonable value for the upper bound δ_i can be set.

In addition to the advantages of the original algorithm mentioned in Sec. 3, the modified control algorithm presented here provides two extensions from the original one.[9] The first extension is that the flexible effect of the robot arms is incorporated into the control algorithm.

It is reflected in the kinematic relations expressed by Eqs. (34) and in the term $\mathrm{sgn}(z_i)\,\delta_i$ in Eq. (50) which is associated with the vibration disturbance vector $\mathbf{d}_e$ in Eq. (23). The second extension consists of the use of decoupled Liapunov functions, Eqs. (41), to eliminate the bursting phenomenon (Sec. 3) when the control algorithm is implemented in discrete-time.

5. VIBRATION CONTROL

Because of coupling between the rigid-body motions and the elastic vibration, the performance of the tracking control depends on how well the vibration suppression is carried out. Without vibration control, the tracking cannot be truly realized for a flexible space robot. Our objective is to drive the elastic motion state $\mathbf{q}_e$, $\dot{\mathbf{q}}_e$ close to zero during the tracking and docking operation. We recall that the motion of the elastic vibration of the space robot is described by Eq. (24), which represents a linear time-varying system with a persistent disturbance term $\mathbf{d}_r$ due to the rigid-body motions.

We propose to control the vibration in discrete time. To this end, we separate the generalized control force $\mathbf{Q}_e$ into

$$\mathbf{Q}_e(k) = \mathbf{Q}_{er}(k) + \mathbf{Q}_{ee}(k) \tag{54}$$

The discrete-time control algorithm for disturbance compensation is expressed by

$$\mathbf{Q}_{er}(k) = \mathbf{d}_r(\mathbf{q}_r(k), \dot{\mathbf{q}}_r(k), \ddot{\mathbf{q}}_r(k)) \tag{55}$$

If the disturbance is cancelled perfectly, Eq. (24) becomes

$$M_e(\mathbf{q}_r)\,\ddot{\mathbf{q}}_e + C_e(\mathbf{q}_r, \dot{\mathbf{q}}_r)\,\dot{\mathbf{q}}_e + K_e(\mathbf{q}_r, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\,\mathbf{q}_e = \mathbf{Q}_{ee} \tag{56}$$

Letting $\mathbf{x}(k) = \left[\mathbf{q}_e(k)^T \quad \dot{\mathbf{q}}_e(k)^T\right]^T$ be the state vector and $\mathbf{u}(k) = \mathbf{Q}_{ee}(k)$ the control vector, the discrete-time state space counterpart of Eq. (56) can be written as

$$\mathbf{x}(k+1) = \hat{A}(k)\,\mathbf{x}(k) + \hat{B}(k)\,\mathbf{u}(k) \tag{57}$$

where the coefficient matrices are given by

$$\hat{A}(k) = e^{A(kT)},\ \hat{B}(k) = \left(e^{A(kT)} - I\right) A^T(kT)\,B(kT) \tag{58a, b}$$

in which

$$A(t) = \begin{bmatrix} 0 & I \\ -M_e^{-1}K_e & -M_e^{-1}C_e \end{bmatrix} \quad B(t) = \begin{bmatrix} 0 \\ M_e^{-1} \end{bmatrix} \tag{59a, b}$$

The performance index for the discrete-time LQR is given by[21]

$$\hat{J} = \frac{1}{2}\sum_{k=0}^{N}\left[\mathbf{x}^T(k)\,Q\,(k) + \mathbf{u}^T(k)\,R\mathbf{u}\,(k)\right] \tag{60}$$

yielding the control law

$$\mathbf{u}(k) = -\left(R + \hat{B}(k)\,\hat{K}(k)\,\hat{B}(k)\right)^{-1}\hat{B}^T(k)\,\hat{K}(k)\,\hat{A}(k)\,\mathbf{x}(k) \tag{61}$$

where $\hat{K}(k)$ satisfies the discrete-time algebraic Riccati equation

$$\hat{K}(k) = \hat{A}^T(k)\left[\hat{K}(k) - \hat{K}(k)\,\hat{B}(k)\left(R + \hat{B}^T(k)\,\hat{K}(k)\,\hat{B}(k)\right)^{-1}\hat{B}^T(k)\,\hat{K}(k)\right]\hat{A}(k) + Q \tag{62}$$

Direct application of the discrete-time control algorithm described by Eqs. (55) and (61) to our problem causes severe instability. The reason is that the discrete-time control force $\mathbf{Q}_{er}$ in Eq. (55) is not able to cancel the continuous disturbance term $\mathbf{d}_r$ in Eq. (24) perfectly. Hence, the LQR control design based on Eq. (56), in which the disturbance is absent, is no longer appropriate. The error accumulates with time and it finally results in instability. To resolve this problem, a modified vibration control algorithm is proposed in the next section.

6. MODIFIED DISCRETE-TIME VIBRATION CONTROL ALGORITHM

An examination of the disturbance term $\mathbf{d}_r$ in Eq. (B.14) of Appendix B, i.e., an examination of

$$\mathbf{d}_r = M_{re}^T\ddot{\mathbf{q}}_r + C_{er}\dot{\mathbf{q}}_r \tag{63}$$

reveals that $\ddot{\mathbf{q}}_r$ in the first term is the major cause of the system instability. Usually $\ddot{\mathbf{q}}_r(k)$ is not available and $\ddot{\mathbf{q}}_r(k-1)$ is used as an estimate of $\ddot{\mathbf{q}}_r(k)$. Stable performance of the system can be achieved only if $\ddot{\mathbf{q}}_r(k)$ can be measured or estimated perfectly. Even a very small error in $\ddot{\mathbf{q}}_r$ appearing in Eq. (63) can result in failure of the LQR design. To avoid use of $\ddot{\mathbf{q}}_r$ in Eq. (63), we replace $\ddot{\mathbf{q}}_r$ by $\mathbf{u}_r$, so that the disturbance compensation scheme becomes

$$\begin{aligned}\mathbf{Q}_{er}(k) =& \mathbf{d}_r(\mathbf{q}_r(k), \dot{\mathbf{q}}_r(k), \mathbf{u}_r(k)) \\ =& M_{re}^T(\mathbf{q}_r(k))\,\mathbf{u}_r(k) + C_{er}(\mathbf{q}_r(k), \dot{\mathbf{q}}_r(k))\,\dot{\mathbf{q}}_r(k)\end{aligned} \tag{64}$$

where $\mathbf{u}_r(k)$ is calculated by the tracking control algorithm given by Eq. (51). We then substitute Eqs. (63), (64) and (40) into Eq. (24) and obtain the system equation as follows:

$$M_e(\mathbf{q}_r)\,\ddot{\mathbf{q}}_e + C_e(\mathbf{q}_r, \dot{\mathbf{q}}_r)\,\dot{\mathbf{q}}_e + K_e(\mathbf{q}_r, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\,\mathbf{q}_e - M_{re}^T M_r^{-1}\mathbf{d}_e = \mathbf{Q}_{ee} \tag{65}$$

As shown in Appendix B, $\mathbf{d}_e$ can be expressed as

$$\mathbf{d}_e = M_{re}\ddot{\mathbf{q}}_e + C_{re}\dot{\mathbf{q}}_e + (K_M^e + K_C^e)\,\mathbf{q}_e \tag{66}$$

where K_M^e and K_C^e are given by Eqs. (B.6) and (B.8), respectively. Substituting Eq. (66) into Eq. (65), we obtain the modified linear time-varying system

$$M_e^*(\mathbf{q}_r)\,\ddot{\mathbf{q}}_e + C_e^*(\mathbf{q}_r, \dot{\mathbf{q}}_r)\,\dot{\mathbf{q}}_e + K_e^*(\mathbf{q}_r, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\,\mathbf{q}_e = \mathbf{Q}_{ee} \tag{67}$$

where, comparing Eqs. (56) and (67), we observe that matrices M_e^*, C_e^* and K_e^* represent modified coefficient matrices given by

$$M_e^* = M_e - M_{re}^T M_r^{-1} M_{re} \tag{68a}$$

$$C_e^* = C_e - M_{re}^T M_r^{-1} C_{re} \tag{68b}$$

$$K_e^* = K_e - M_{re}^T M_r^{-1}(K_M^e + K_C^e) \tag{68c}$$

Based on Eqs. (67) and (68), we can follow the same procedure as in Sec. 5 and obtain the control law for $\mathbf{Q}_{ee}$. The simulation results using the modified control scheme showed stable performance. Further numerical simulations showed that even in the case of a system with only the mass matrix M_e modified, i.e., a system described by

$$M_e^*(\mathbf{q}_r)\,\ddot{\mathbf{q}}_e + C_e(\mathbf{q}_r, \dot{\mathbf{q}}_r)\,\dot{\mathbf{q}}_e + K_e(\mathbf{q}_R, \dot{\mathbf{q}}_r, \ddot{\mathbf{q}}_r)\,\mathbf{q}_e = \mathbf{Q}_{ee} \tag{69}$$

the LQR control law is still able to produce good system performance. This is because the first term on the right side of Eq. (66) is dominant, so that using C_e and K_e instead of C_e^*

and K_e^*, respectively, is equivalent to dropping the second and third terms in Eq. (66), which does not affect the system performance very much.

7. NUMERICAL EXAMPLE

We assume that the parameters for the flexible space robot shown in Fig. 1 have the values

$$\begin{aligned} m_0 = 40.0 \text{ kg}, \quad m_1 = m_2 = 10.0 \text{ kg}, \quad m_3 = 2.0 \text{ kg} \\ L_0 = 2.5 \text{ m}, \quad L_1 = L_2 = 10.0 \text{ m}, \quad L_3 = 2.0 \text{ m} \\ S_x = S_y = 0, \quad I_x = 83.333 \text{ kg m}^2, \quad I_y = 333.333 \text{ kg m}^2 \\ EI_1 = EI_2 = 10^4 \text{ kg m}^2 \end{aligned} \tag{70}$$

The target motion is not known a priori and must be measured on-line. However, for simulation purposes, we choose an example target trajectory as follows:

$$\begin{aligned} x_t(t) &= 10.0 \sin\left(\frac{\pi}{10}t\right) \\ y_t(t) &= 10.0 + 10.0 \sin\left(\frac{\pi}{10}t\right), \quad t \in [0, 5.0 \text{ s}] \\ \theta_t(t) &= \frac{3\pi}{20}t \end{aligned} \tag{71}$$

The initial conditions of the space robot are given by:

$$\begin{aligned} \mathbf{q}_r(0) &= [0.0 \quad -15.0 \quad 0.0 \quad 0.5\pi \quad 0.4775\pi \quad 0.25\pi]^T, \ \dot{\mathbf{q}}_r(0) = \mathbf{0} \\ \mathbf{q}_e(0) &= [0.01 \ \ldots \ 0.01]^T, \ \dot{\mathbf{q}}_e(0) = \mathbf{0} \end{aligned} \tag{72}$$

The parameters of the control synthesis design are

$$\beta = 20.0, \quad \epsilon = 10^{-3}, \quad t_s = 2.5 \text{ s}, \quad \delta_i = 20, \quad i = 1, 2, 3 \tag{73}$$

We designate the three redundant degrees of freedom in $\mathbf{u}_r$ as u_{r3}, u_{r4} and u_{r5}. They are defined for two different cases as follows:

Case 1:

$$u_{r3} = u_{r4} = u_{r5} = 0, \qquad 0 \le t \le 5 \text{ s} \tag{74}$$

Case 2:

$$u_{r3} = \begin{cases} 0, & t \leq 0 \\ 4\Delta\theta_0/t_f^2, & 0 < t \leq t_f/2 \\ -4\Delta\theta_0/t_f^2, & t_f/2 < t \leq t_f \\ 0, & t > t_f \end{cases} \tag{75a}$$

$$u_{r4} = \begin{cases} 0, & t \leq 0 \\ 4\Delta\theta_1/t_f^2, & 0 < t \leq t_f/2 \\ -4\Delta\theta_1/t_f^2, & t_f/2, t \leq t_f \\ 0, & t > t_f \end{cases} \tag{75b}$$

$$u_{r5} = \begin{cases} 0, & t \leq 0 \\ 4\Delta\theta_2/t_f^2, & 0 < t \leq t_f/2 \\ -4\Delta\theta_2/t_f^2, & t_f/2 < t \leq t_f \\ 0, & t > t_f \end{cases} \tag{75c}$$

where $t_f = 4.0$ s, $\Delta\theta_0 = \frac{\pi}{6}$ rad, $\Delta\theta_1 = \frac{\pi}{4}$ rad, and $\Delta\theta_2 = -\frac{\pi}{6}$ rad.

For a rigid space robot, Eqs. (74) and (75) represent constraints on the acceleration of the robot configuration. In Case 1, the mission amounts to keeping the base attitude θ_0 and the two joint angles θ_1 and θ_2 constant while tracking a moving target. In Case 2, the mission implies bang-bang maneuvers involving a base attitude change of $\Delta\theta_0$ and arms angle changes of $\Delta\theta_1$ and $\Delta\theta_2$ while tracking a moving target.

The constraints cannot be realized perfectly for a flexible space robot due to disturbance-causing vibration. However, the performance can be improved by vibration control. Because the major objective here is to track the moving target, we use the constraint equations, Eqs. (74) and (75), to eliminate the robot redundancy.

For vibration control, the LQR design parameters are chosen as

$$R = \text{diag}\left[I_{n\times n} \quad I_{n\times n}\right]$$
$$Q = \text{diag}\left[2.0\times 10^4 I_{n\times n} \quad 10^4 I_{n\times n} \quad 2.0\times 10^4 I_{n\times n} \quad 10^4 I_{n\times n}\right] \tag{76}$$

The elastic displacement for each of the two arms was modeled by means of five quasi-comparison functions.[19]

The system performance under the tracking and docking maneuver is simulated over 5 s. To this end, the tracking control algorithm presented in Sec. 4 and the vibration control algorithm presented in Sec. 6 are used. The simulation is performed in discrete-time with a sampling period $T = 0.001$ s.

Figures 4a and 4b show time-lapse pictures of the robot configuration for Cases 1 and 2, respectively. For Case 2, time histories of the tracking error $\mathbf{e}$ and its time derivative $\dot{\mathbf{e}}$ are shown in Figs. 5a-5c, time histories of the tip elastic displacements of the two flexible links are shown in Fig. 6 and time histories of the control forces and torques for the rigid-body maneuver are displayed in Figs. 7a-7c. Time histories of the control torques acting on the flexible bodies for disturbance rejection and LQR control are shown in Fig. 8 and Fig. 9, respectively. The results are very satisfactory, with control achieved in less than one second.

8. SUMMARY AND CONCLUSIONS

This paper is concerned with the control of a flexible space robot executing a docking maneuver with a target whose motion is not known a priori. The control is based on on-line measurements of the target motion. The dynamical equations of the space robot are first derived by means of Lagrange's equations and then separated into two coupled sets of equations suitable for rigid-body maneuvers and vibration suppression. Controls for the rigid-body maneuver and vibration suppression are developed and implemented in discrete time. Problems arising from digital implementation of the control algorithms are discussed. Then, modifications of the control algorithms so as to prevent the problems are made.

The control scheme presented can be applied to two-dimensional, as well as three-dimensional problems. Furthermore, it has the flexibility of solving different problems by defining appropriate output vectors other than the end-effector output vector. For example, if the mission involves tracking and docking with an orbiting target while its base attitude is to be kept constant, we can define the output vector as $\mathbf{y}_e = [x_e \;\; y_e \;\; \theta_e \;\; \theta_0]^T$ and the target output vector as $\mathbf{y}_t = [x_t \;\; y_t \;\; \theta_t \;\; 0]^T$, and then the proposed tracking control algorithm can be used to drive the error vector $\mathbf{e} = \mathbf{y}_t - \mathbf{y}_e$ and its time derivative $\dot{\mathbf{e}}$ to zero.

A numerical example is used to demonstrate the control scheme. The simulation results have shown very good system performance in both the tracking maneuver and the vibration suppression.

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APPENDIX A - Matrices in the Equations of Motion

The mass matrix M appearing in Eq. (14), as well as in Eq. (21), is defined as

$$M = \begin{bmatrix} & & \mathbf{m}_{17}^T & \mathbf{m}_{18}^T \\ & & \mathbf{m}_{27}^T & \mathbf{m}_{28}^T \\ & \bar{M}_0 & \mathbf{m}_{37}^T & \mathbf{m}_{38}^T \\ & & \mathbf{m}_{47}^T & \mathbf{m}_{48}^T + \mathbf{b}_1^T \\ & & \mathbf{m}_{57}^T + \mathbf{b}_2^T & \mathbf{m}_{58}^T \\ & & \mathbf{m}_{67}^T & \mathbf{m}_{68}^T \\ \mathbf{m}_{17} & \ldots \mathbf{m}_{67} & m_{77} & m_{78}^T \\ \mathbf{m}_{18} & \ldots \mathbf{m}_{68} & m_{78} & m_{88} \end{bmatrix} \tag{A.1}$$

with

$$\bar{M}_0 = \begin{bmatrix} m_t & 0 & -S_{tx} & a_1 & a_2 & -S_3 s_3 \\ 0 & m_t & -S_{ty} & a_3 & a_4 & S_3 c_3 \\ -S_{tx} & -S_{ty} & I_{t0} & a_5 & a_6 & S_3 L_0 s_{30} \\ a_1 & a_3 & a_5 & \bar{I}_1 & a_7 & a_8 \\ a_2 & a_4 & a_6 & a_7 & \bar{I}_2 & a_9 \\ -S_3 s_3 & S_3 c_3 & S_3 L_0 s_{30} & a_8 & a_9 & I_3 \end{bmatrix} \tag{A.2}$$

in which

$$\begin{aligned}
&a_1 = -S_{t1} s_1 - \bar{\boldsymbol{\Phi}}_{t1}^T \boldsymbol{\xi}_1 c_1, \quad a_2 = -S_{t2} s_2 - \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 c_2 \\
&a_3 = S_{t1} c_1 - \bar{\boldsymbol{\Phi}}_{t1}^T \boldsymbol{\xi}_1 s_1, \quad a_4 = S_{t2} c_2 - \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 s_2 \\
&a_5 = S_{t1} L_0 s_{10} + \bar{\boldsymbol{\Phi}}_{t1}^T \boldsymbol{\xi}_1 L_0 c_{10}, \quad a_6 = S_{t2} L_0 s_{20} + \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_0 c_{20} \\
&a_7 = S_{t2} L_1 c_{21} + S_{t2} \boldsymbol{\Phi}_{12}^T \boldsymbol{\xi}_1 s_{21} - \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_1 s_{21} + \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 \boldsymbol{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{21} \\
&a_8 = S_3 L_1 c_{31} + S_3 \boldsymbol{\Phi}_{12}^T \boldsymbol{\xi}_1 s_{31}, \quad a_9 = S_3 L_2 c_{32} + S_3 \boldsymbol{\Phi}_{23}^T \boldsymbol{\xi}_2 s_{32} \\
&\mathbf{b}_1 = \bar{\boldsymbol{\Phi}}_{t2} \boldsymbol{\Phi}_{12}^T \boldsymbol{\xi}_1 s_{21}, \quad \mathbf{b}_2 = -\boldsymbol{\Phi}_{12} \bar{\boldsymbol{\Phi}}_{t2}^T \boldsymbol{\xi}_2 s_{21} \\
&\bar{I}_1 = I_{t1} + \boldsymbol{\xi}_1^T m_{77} \boldsymbol{\xi}_1, \quad \bar{I}_2 = I_{t2} + \boldsymbol{\xi}_2^T m_{88} \boldsymbol{\xi}_2
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
&\mathrm{m}_{17} = -\bar{\mathbf{\Phi}}_{t1}s_1, \quad \mathrm{m}_{27} = \bar{\mathbf{\Phi}}_{t1}c_1, \quad \mathrm{m}_{37} = \bar{\mathbf{\Phi}}_{t1}L_0 s_{10} \\
&\mathrm{m}_{47} = \tilde{\bar{\mathbf{\Phi}}}_1 + (m_2+m_3)\,L_1\mathbf{\Phi}_{12}, \quad \mathrm{m}_{57} = S_{t2}\mathbf{\Phi}_{12}c_{21}, \quad \mathrm{m}_{67} = S_3\mathbf{\Phi}_{12}c_{31} \\
&\mathrm{m}_{18} = -\bar{\mathbf{\Phi}}_{t2}s_2, \quad \mathrm{m}_{28} = \bar{\mathbf{\Phi}}_{t2}c_2, \quad \mathrm{m}_{38} = \bar{\mathbf{\Phi}}_{12}c_{21}, \quad \mathrm{m}_{67} = S_3\mathbf{\Phi}_{12}c_{31} \\
&\mathrm{m}_{48} = \bar{\mathbf{\Phi}}_{t2}L_1c_{21}, \quad \mathrm{m}_{58} = \tilde{\bar{\mathbf{\Phi}}}_2 + m_3L_2\mathbf{\Phi}_{23}, \quad \mathrm{m}_{68} = S_3\mathbf{\Phi}_{23}c_{32} \\
&m_{77} = \Lambda_1 + (m_2+m_3)\,\mathbf{\Phi}_{12}\mathbf{\Phi}_{12}^T, \quad \mathrm{m}_{78} = \mathbf{\Phi}_{12}\bar{\mathbf{\Phi}}_{t2}^Tc_{21} \\
&m_{88} = \Lambda_2 + m_3\mathbf{\Phi}_{23}\mathbf{\Phi}_{23}^T
\end{aligned} \tag{A.4}$$

and we note that $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij} = \sin(\theta_i - \theta_j)$ and $c_{ij} = \cos(\theta_i - \theta_j)$. Moreover, we have used the following definitions:

$$\begin{aligned}
m_t &= m_0 + m_1 + m_2 + m_3 \\
S_{tx} &= S_{0x}\sin\theta_0 + S_{0y}\cos\theta_0 + (m_1+m_2+m_3)\,L_0\cos\theta_0 \\
S_{ty} &= -S_{0x}\cos\theta_0 + S_{0y}\sin\theta_0 + (m_1+m_2+m_3)\,L_0\sin\theta_0 \\
S_{t1} &= S_1 + (m_2+m_3)\,L_1, \quad S_{t2} = S_2 + m_3L_2 \\
I_{t0} &= I_{0x} + I_{0y} + (m_1+m_2+m_3)\,L_0^2 \\
I_{t1} &= I_1 + (m_2+m_3)\,L_1^2, \quad I_{t2} = I_2 + m_3L_2^2 \\
\bar{\mathbf{\Phi}}_{t1} &= \bar{\mathbf{\Phi}}_1 + (m_2+m_3)\,\mathbf{\Phi}_{12}, \quad \bar{\mathbf{\Phi}}_{t2} = \bar{\mathbf{\Phi}}_2 + m_3\mathbf{\Phi}_{23}
\end{aligned} \tag{A.5}$$

in which

$$\begin{aligned}
&m_i = \int_{\mathrm{Body}\ i}\rho_i dD_i \quad i = 0,1,2,3 \\
&S_i = \int_{\mathrm{Body}\ i}\rho_i x_i dD_i, \quad I_i = \int_{\mathrm{Body}\ i}\rho_i x_i^2 dD_i, \quad i = 1,2,3 \\
&S_{0x} = \int_{\mathrm{Body}\ 0}\rho_0 x dD_0, \quad S_{0y} = \int_{\mathrm{Body}\ 0}\rho_0 y dD_0 \\
&I_{0x} = \int_{\mathrm{Body}\ 0}\rho_0 x^2 dD_0, \quad I_{0y} = \int_{\mathrm{Body}\ 0}\rho_0 y^2 dD_0 \\
&\bar{\mathbf{\Phi}}_i = \int_{\mathrm{Body}\ i}\rho_i\mathbf{\Phi}_i dD_i, \quad \tilde{\bar{\mathbf{\Phi}}}_i = \int_{\mathrm{Body}\ i}\rho_i x_i\mathbf{\Phi}_i dD_i \\
&\Lambda_i = \int_{\mathrm{Body}\ i}\rho_i\mathbf{\Phi}_i\mathbf{\Phi}_i^T dD_i, \quad i = 1,2 \\
&\mathbf{\Phi}_{12} = \mathbf{\Phi}_1(x_1)\Big|_{x_1=L_1}, \quad \mathbf{\Phi}_{23} = \mathbf{\Phi}_2(x_2)\Big|_{x_2=L_2}
\end{aligned} \tag{A.6}$$

The matrix G in Eq. (19) is defined as

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & 1 & 0 & 0 & 0 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & 0 & 1 & -1 & 0 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & 0 & 0 & 1 & -1 & 0 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & 0 & 0 & 0 & 1 & -1 & \mathbf{0}^T & \mathbf{0}^T \\ 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_1'(L_1) & \mathbf{0} & G_1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_2'(L_2) & 0 & G_2 \end{bmatrix} \tag{A.7}$$

where primes denote spatial derivatives and

$$G_i = \left[\mathbf{\Phi}_i''(x_{i1}) \ \ldots \ \mathbf{\Phi}_i''(x_{im}) \right] \quad i = 1, 2 \tag{A.8}$$

in which m is the number of actuators on each link. Here m is equal to the number of modes and G_i are square matrices.

The coefficient matrix C in Eq. (21) is defined as

$$C = \begin{bmatrix} 0 & 0 & C_{13} & C_{14} & C_{15} & C_{16} & \mathbf{C}_{17} & \mathbf{C}_{18} \\ 0 & 0 & C_{23} & C_{24} & C_{25} & C_{26} & \mathbf{C}_{27} & \mathbf{C}_{28} \\ 0 & 0 & 0 & C_{34} & C_{35} & C_{36} & \mathbf{C}_{37} & \mathbf{C}_{38} \\ 0 & 0 & C_{43} & 0 & C_{45} & C_{46} & \mathbf{C}_{47} & \mathbf{C}_{48} \\ 0 & 0 & C_{53} & C_{54} & 0 & C_{56} & \mathbf{C}_{57} & \mathbf{C}_{58} \\ 0 & 0 & C_{63} & C_{64} & C_{65} & 0 & \mathbf{C}_{67} & \mathbf{C}_{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{73} & \mathbf{C}_{74} & \mathbf{C}_{75} & \mathbf{C}_{76} & 0 & C_{78} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{83} & \mathbf{C}_{84} & \mathbf{C}_{85} & \mathbf{C}_{86} & C_{87} & 0 \end{bmatrix} \tag{A.9}$$

where

$$C_{13} = S_{ty}\dot{\theta}_0, \ C_{14} = \left(-S_{t1}c_1 + \bar{\mathbf{\Phi}}_{t1}^T \boldsymbol{\xi}_1 s_1\right)\dot{\theta}_1, \ C_{15} = \left(-S_{t2}c_2 + \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 s_2\right)\dot{\theta}_2$$

$$C_{16} = -S_3 c_3 \dot{\theta}_3, \ \mathbf{C}_{17} = -2\bar{\mathbf{\Phi}}_{t1}^T c_1 \dot{\theta}_1, \ \mathbf{C}_{18} = -2\bar{\mathbf{\Phi}}_{t2}^T c_2 \dot{\theta}_2$$

$$C_{23} = -S_{tx}\dot{\theta}_0, \ C_{24} = \left(-S_{t1}s_1 - \bar{\mathbf{\Phi}}_{t1}^T \boldsymbol{\xi}_1 c_1\right)\dot{\theta}_1, C_{25} = \left(-S_{t2}s_2 - \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 c_2\right)\dot{\theta}_2$$

$$C_{26} = -S_3 s_3 \dot{\theta}_3, \ \mathbf{C}_{27} = -2\bar{\mathbf{\Phi}}_{t1}^T s_1 \dot{\theta}_1, \ \mathbf{C}_{28} = -2\bar{\mathbf{\Phi}}_{t2}^T s_2 \dot{\theta}_2$$

$$C_{34} = \left(S_{t1}L_0 c_{10} - \bar{\mathbf{\Phi}}_{t1}^T \boldsymbol{\xi}_1 L_0 s_{10}\right)\dot{\theta}_1, C_{35} = \left(S_{t2}L_0 c_{20} - \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_0 s_{20}\right)\dot{\theta}_2$$

$$C_{36} = S_3 L_0 c_{30}\dot{\theta}_3, \ \mathbf{C}_{37} = 2\bar{\mathbf{\Phi}}_{t1}^T L_0 c_{10}\dot{\theta}_1, \ \mathbf{C}_{38} = 2.0\bar{\mathbf{\Phi}}_{t2}^T L_0 c_{20}\dot{\theta}_2$$

$$C_{43} = \left(-S_{t1}L_0 c_{10} + \bar{\mathbf{\Phi}}_{t1}^T \boldsymbol{\xi}_1 L_0 s_{10}\right)\dot{\theta}_0$$

$$C_{45} = \left(-S_{t2}L_1 s_{21} - \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_1 c_{21} + S_{t2}\mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{21} - \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 s_{21}\right)\dot{\theta}_2$$

$$
\begin{aligned}
&C_{46} = \left(-S_3 L_1 s_{31} + S_3 \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{31}\right) \dot{\theta}_3 \\
&\mathbf{C}_{47} = 2\boldsymbol{\xi}_1^T \left(\Lambda_1 + (m_2 + m_3) \mathbf{\Phi}_{12} \mathbf{\Phi}_{12}^T\right) \dot{\theta}_1 \\
&\mathbf{C}_{48} = 2\left(-L_1 s_{21} \bar{\mathbf{\Phi}}_{t2}^T + \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{21} \bar{\mathbf{\Phi}}_{t2}^T\right) \dot{\theta}_2 \\
&C_{53} = \left(-S_{t2} L_0 c_{20} + \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_0 s_{20}\right) \dot{\theta}_0 \\
&C_{54} = \left(S_{t2} L_1 s_{21} + \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 L_1 c_{21} - S_{t2} \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{21} + \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 s_{21}\right) \dot{\theta}_1 \\
&C_{56} = \left(-S_3 L_2 s_{32} + S_3 \mathbf{\Phi}_{23}^T \boldsymbol{\xi}_2 c_{32}\right) \dot{\theta}_3, \; \mathbf{C}_{57} = 2\left(S_{t2} s_{21} \mathbf{\Phi}_{12}^T + \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 c_{21} \mathbf{\Phi}_{12}^T\right) \dot{\theta}_1 \\
&\mathbf{C}_{58} = 2\boldsymbol{\xi}_2^T \left(\Lambda_2 + m_3 \mathbf{\Phi}_{23} \mathbf{\Phi}_{23}^T\right) \dot{\theta}_1, \; C_{63} = -S_3 L_0 c_{30} \dot{\theta}_0 \\
&C_{64} = \left(S_3 L_1 s_{31} - S_3 \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{31}\right) \dot{\theta}_1, \; C_{65} = \left(S_3 L_2 s_{32} - S_3 \mathbf{\Phi}_{23}^T \boldsymbol{\xi}_2 c_{32}\right) \dot{\theta}_2 \\
&\mathbf{C}_{67} = 2 S_3 s_{31} \mathbf{\Phi}_{23}^T \dot{\theta}_2, \; \mathbf{C}_{68} = 2 S_3 s_{32} \mathbf{\Phi}_{23}^T \dot{\theta}_2, \; \mathbf{C}_{73} = -\bar{\mathbf{\Phi}}_{t1} L_0 c_{10} \dot{\theta}_0 \\
&\mathbf{C}_{74} = -\left(\Lambda_1 + (m_2 + m_3) \mathbf{\Phi}_{12} \mathbf{\Phi}_{12}^T\right) \boldsymbol{\xi}_1 \dot{\theta}_1, \; \mathbf{C}_{75} = \left(-S_{t2} s_{21} \mathbf{\Phi}_{12} - \bar{\mathbf{\Phi}}_{t2}^T \boldsymbol{\xi}_2 c_{21} \mathbf{\Phi}_{12}\right) \dot{\theta}_2 \\
&\mathbf{C}_{76} = -S_3 s_{31} \mathbf{\Phi}_{12} \dot{\theta}_3, \; C_{78} = -2\mathbf{\Phi}_{12} \bar{\mathbf{\Phi}}_{t2}^T s_{21} \dot{\theta}_2, \; \mathbf{C}_{83} = -\bar{\mathbf{\Phi}}_{t2} L_0 c_{20} \dot{\theta}_0 \\
&\mathbf{C}_{84} = \left(L_1 s_{21} \bar{\mathbf{\Phi}}_{t2} - \mathbf{\Phi}_{12}^T \boldsymbol{\xi}_1 c_{21} \bar{\mathbf{\Phi}}_{t2}\right) \dot{\theta}_1, \; \mathbf{C}_{85} = -\left(\Lambda_2 + m_3 \mathbf{\Phi}_{23} \mathbf{\Phi}_{23}^T\right) \boldsymbol{\xi}_2 \dot{\theta}_2 \\
&\mathbf{C}_{86} = -S_3 s_{32} \mathbf{\Phi}_{23} \dot{\theta}_3, \; C_{87} = 2\bar{\mathbf{\Phi}}_{t2} \mathbf{\Phi}_{12}^T s_{21} \dot{\theta}_1
\end{aligned}
\tag{A.10}
$$

APPENDIX B - Matrices in the Partitioned Equations of Motion

The mass matrix M_r and the coefficient matrix C_r in Eq. (23) are defined as

$$
M_r = \begin{bmatrix}
m_t & 0 & -S_{tx} & -S_{t1} s_1 & -S_{t2} s_2 & -S_3 s_3 \\
0 & m_t & -S_{ty} & S_{t1} c_1 & S_{t2} c_2 & S_3 c_3 \\
-S_{tx} & -S_{ty} & I_{t0} & S_{t1} L_0 s_{10} & S_{t2} L_0 s_{20} & S_3 L_0 s_{30} \\
-S_{t1} s_1 & S_{t1} c_1 & S_{t1} L_0 s_{10} & I_{t1} & S_{t2} L_1 c_{21} & S_3 L_1 c_{31} \\
-S_{t2} s_2 & S_{t2} c_2 & S_{t2} L_0 s_{20} & S_{t2} L_1 c_{21} & I_{t2} & S_3 L_2 c_{32} \\
-S_3 s_3 & S_3 c_3 & S_3 L_0 s_{30} & S_3 L_1 c_{31} & S_3 L_2 c_{32} & I_3
\end{bmatrix}
\tag{B.1}
$$

$$
C_r = \begin{bmatrix}
0 & 0 & S_{ty} \dot{\theta}_0 & -S_{t1} c_1 \dot{\theta}_1 & -S_{t2} c_2 \dot{\theta}_2 & -S_3 c_3 \dot{\theta}_3 \\
0 & 0 & -S_{tx} \dot{\theta}_0 & -S_{t1} s_1 \dot{\theta}_1 & -S_{t2} s_2 \dot{\theta}_2 & -S_3 s_3 \dot{\theta}_3 \\
0 & 0 & 0 & S_{t1} L_0 c_{10} \dot{\theta}_1 & S_{t2} L_0 c_{20} \dot{\theta}_2 & S_3 L_0 c_{30} \dot{\theta}_3 \\
0 & 0 & -S_{t1} L_0 c_{10} \dot{\theta}_0 & 0 & -S_{t2} L_1 s_{21} \dot{\theta}_2 & -S_3 L_1 s_{31} \dot{\theta}_3 \\
0 & 0 & -S_{t2} L_0 c_{20} \dot{\theta}_0 & S_{t2} L_1 s_{21} \dot{\theta}_1 & 0 & -S_3 L_2 s_{32} \dot{\theta}_3 \\
0 & 0 & -S_3 L_0 c_{30} \dot{\theta}_0 & S_3 L_1 s_{31} \dot{\theta}_1 & S_3 L_2 s_{32} \dot{\theta}_2 & 0
\end{bmatrix}
\tag{B.2}
$$

The disturbance vector $\mathbf{d}_e$ in Eq. (23) is defined as

$$
\mathbf{d}_e = M_{re} \ddot{\mathbf{q}}_e + C_{re} \dot{\mathbf{q}}_e + \left(K_M^e + K_C^e\right) \mathbf{q}_e
\tag{B.3}
$$

where

$$M_{re} = \begin{bmatrix} \mathbf{m}_{17} & \mathbf{m}_{18} \\ \mathbf{m}_{27} & \mathbf{m}_{28} \\ \vdots & \vdots \\ \mathbf{m}_{67} & \mathbf{m}_{68} \end{bmatrix} \tag{B.4}$$

$$C_{re} = \begin{bmatrix} -2\bar{\Phi}_{t1}^T c_1 \dot{\theta}_1 & -2\bar{\Phi}_{t2}^T c_2 \dot{\theta}_2 \\ -2\bar{\Phi}_{t1}^T s_1 \dot{\theta}_1 & -2\bar{\Phi}_{t2}^T s_2 \dot{\theta}_2 \\ 2\bar{\Phi}_{t1}^T L_0 c_{10} \dot{\theta}_1 & 2\bar{\Phi}_{t2}^T L_0 c_{20} \dot{\theta}_2 \\ 0 & -2\bar{\Phi}_{t2}^T L_1 s_{21} \dot{\theta}_2 \\ 2\Phi_{12}^T S_{t2} s_{21} \dot{\theta}_1 & 0 \\ 2\Phi_{12}^T S_3 s_{31} \dot{\theta}_1 & 2\Phi_{23}^T S_3 s_{32} \dot{\theta}_2 \end{bmatrix} \tag{B.5}$$

Moreover,

$$K_M^e = \begin{bmatrix} -\bar{\Phi}_{t1}^T c_1 \ddot{\theta}_1 & -\bar{\Phi}_{t2}^T c_2 \ddot{\theta}_2 \\ -\bar{\Phi}_{t1}^T s_1 \ddot{\theta}_1 & -\bar{\Phi}_{t2}^T s_2 \ddot{\theta}_2 \\ \bar{\Phi}_{t1}^T L_0 c_{10} \ddot{\theta}_1 & \bar{\Phi}_{t2}^T L_0 c_{20} \ddot{\theta}_2 \\ \mathbf{k}_{M1} & -\bar{\Phi}_{t2}^T L_1 s_{21} \ddot{\theta}_2 \\ \Phi_{12}^T S_{t2} s_{21} \ddot{\theta}_1 & \mathbf{k}_{M2} \\ \Phi_{12}^T S_3 s_{31} \ddot{\theta}_1 & \Phi_{23}^T S_3 s_{32} \ddot{\theta}_2 \end{bmatrix} \tag{B.6}$$

in which

$$\begin{aligned} \mathbf{k}_{M1} &= -\bar{\Phi}_{t1}^T \left(c_1 \ddot{x}_0 + s_1 \ddot{y}_0 - L_0 c_{10} \ddot{\theta}_0 \right) + \Phi_{12}^T \left(S_{t2} s_{21} \ddot{\theta}_2 + S_3 s_{31} \ddot{\theta}_3 \right) \\ \mathbf{k}_{M2} &= -\bar{\Phi}_{t2}^T \left(c_2 \ddot{x}_0 + s_2 \ddot{y}_0 - L_0 c_{20} \ddot{\theta}_0 \right) - \bar{\Phi}_{t2}^T L_1 s_{21} \ddot{\theta}_1 + \Phi_{23} S_3 s_{32} \ddot{\theta}_3 \end{aligned} \tag{B.7}$$

and

$$K_C^e = \begin{bmatrix} \bar{\Phi}_{t1}^T s_1 \dot{\theta}_1^2 & \bar{\Phi}_{t2}^T s_2 \dot{\theta}_2^2 \\ -\bar{\Phi}_{t1}^T c_1 \dot{\theta}_1^2 & -\bar{\Phi}_{t2}^T c_2 \dot{\theta}_2^2 \\ -\bar{\Phi}_{t1}^T L_0 s_{10} \dot{\theta}_1^2 & -\bar{\Phi}_{t2}^T L_0 s_{20} \dot{\theta}_2^2 \\ \mathbf{k}_{C1} & -\bar{\Phi}_{t2}^T L_1 c_{21} \dot{\theta}_2^2 \\ -\Phi_{12}^T S_{t2} c_{21} \dot{\theta}_1^2 & \mathbf{k}_{C2} \\ -\Phi_{12}^T S_3 c_{31} \dot{\theta}_1^2 & -\Phi_{23}^T S_3 c_{32} \dot{\theta}_2^2 \end{bmatrix} \tag{B.8}$$

in which

$$\begin{aligned} \mathbf{k}_{C1} &= \bar{\Phi}_{t1}^T L_0 s_{10} \dot{\theta}_0^2 + \Phi_{12}^T S_{t2} c_{21} \dot{\theta}_2^2 + \Phi_{12}^T S_3 c_{31} \dot{\theta}_3^2 \\ \mathbf{k}_{C2} &= \bar{\Phi}_{t2}^T L_0 s_{20} \dot{\theta}_0^2 + \bar{\Phi}_{t2}^T L_1 c_{21} \dot{\theta}_1^2 + \Phi_{23}^T S_3 c_{32} \dot{\theta}_3^2 \end{aligned} \tag{B.9}$$

The mass matrix M_e and the coefficient matrix C_e are defined as

$$M_e = \begin{bmatrix} \Lambda_1 + (m_2 + m_3) \Phi_{12} \Phi_{12}^T & \Phi_{12} \bar{\Phi}_{t2}^T c_{21} \\ \bar{\Phi}_{t2} \Phi_{12}^T c_{21} & \Lambda_2 + m_3 \Phi_{23} \Phi_{23}^T \end{bmatrix} \tag{B.10}$$

$$C_e = \begin{bmatrix} 0 & -2\Phi_{12} \bar{\Phi}_{t2}^T s_{21} \dot{\theta}_2 \\ 2\bar{\Phi}_{t2} \Phi_{12}^T s_{21} \dot{\theta}_1 & 0 \end{bmatrix} \tag{B.11}$$

and the coefficient matrix K_e is defined as

$$K_e = K + K_M + K_C \tag{B.12}$$

where

$$K = \begin{bmatrix} \bar{K}_1 & 0 \\ 0 & \bar{K}_2 \end{bmatrix} \tag{B.13}$$

$$K_M = \begin{bmatrix} 0 & -\Phi_{12}\bar{\Phi}_{t2}^T s_{21}\ddot{\theta}_2 \\ \bar{\Phi}_{t2}\Phi_{12}^T s_{21}\ddot{\theta}_1 & 0 \end{bmatrix} \tag{B.14}$$

and

$$K_C = \begin{bmatrix} -\left(\Lambda_1 + (m_2 + m_3)\,\Phi_{12}\Phi_{12}^T\right)\dot{\theta}_1^2 & -\Phi_{12}\bar{\Phi}_{t2}^T c_{21}\dot{\theta}_2^2 \\ -\bar{\Phi}_{t2}\Phi_{12}^T c_{21}\dot{\theta}_1^2 & -\left(\Lambda_2 + m_3\Phi_{23}\Phi_{23}^T\right)\dot{\theta}_2^2 \end{bmatrix} \tag{B.15}$$

The disturbance vector $\mathbf{d}_r$ is defined as

$$\mathbf{d}_r = M_{re}^T\ddot{\mathbf{q}}_r + C_{er}\dot{\mathbf{q}}_r \tag{B.16}$$

where M_{re} is given by Eq. (B.4) and

$$C_{er} = \begin{bmatrix} 0 & 0 & -\bar{\Phi}_{t1}L_0c_{10}\dot{\theta}_0 & 0 & -\Phi_{12}S_{t2}s_{21}\dot{\theta}_2 & -\Phi_{12}S_3s_{31}\dot{\theta}_3 \\ 0 & 0 & -\bar{\Phi}_{t2}L_0c_{20}\dot{\theta}_0 & \bar{\Phi}_{t2}L_1s_{21}\dot{\theta}_1 & 0 & -\Phi_{23}S_3s_{32}\dot{\theta}_3 \end{bmatrix} \tag{B.17}$$

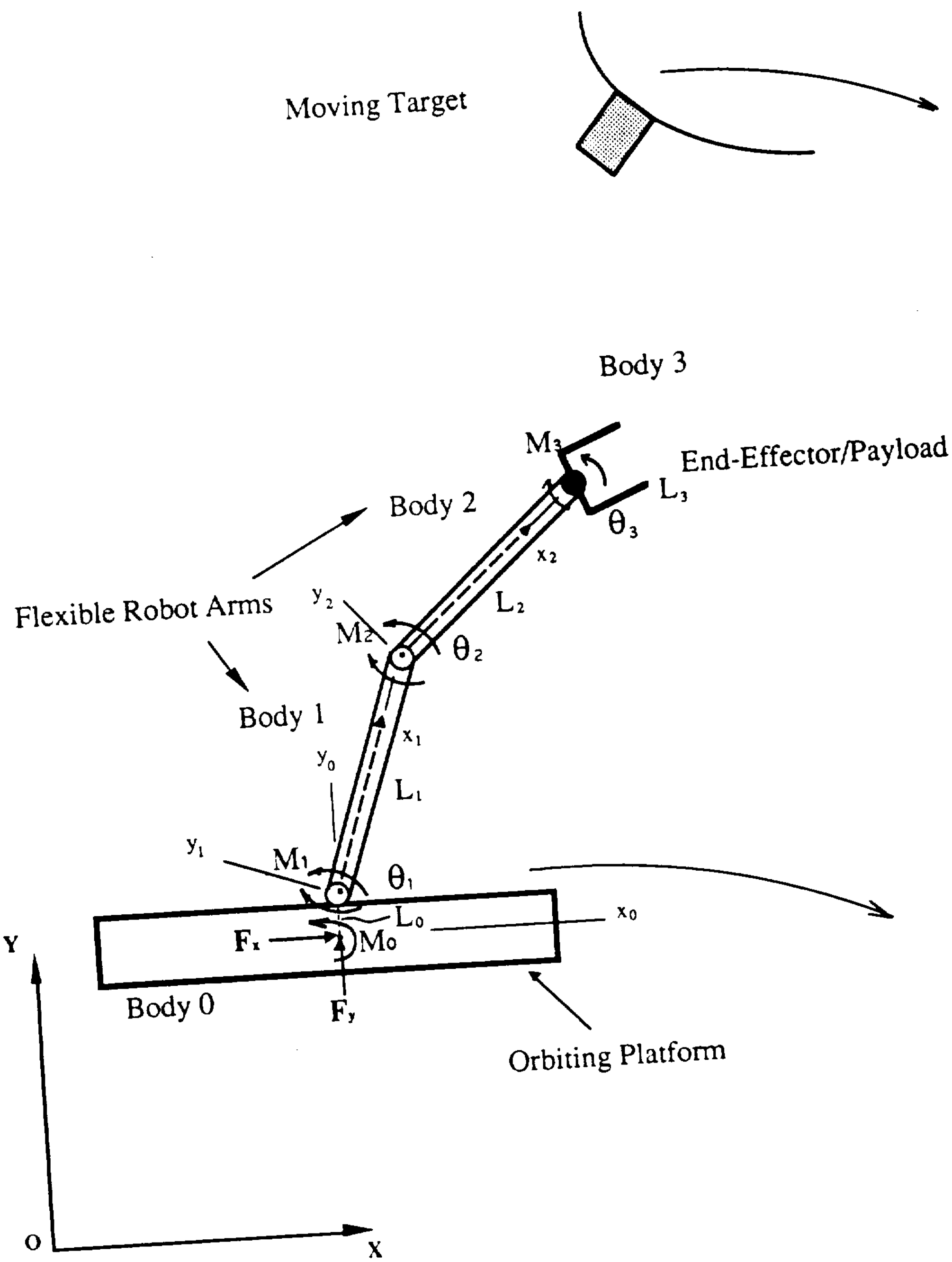

Fig. 1 - Flexible Space Robot

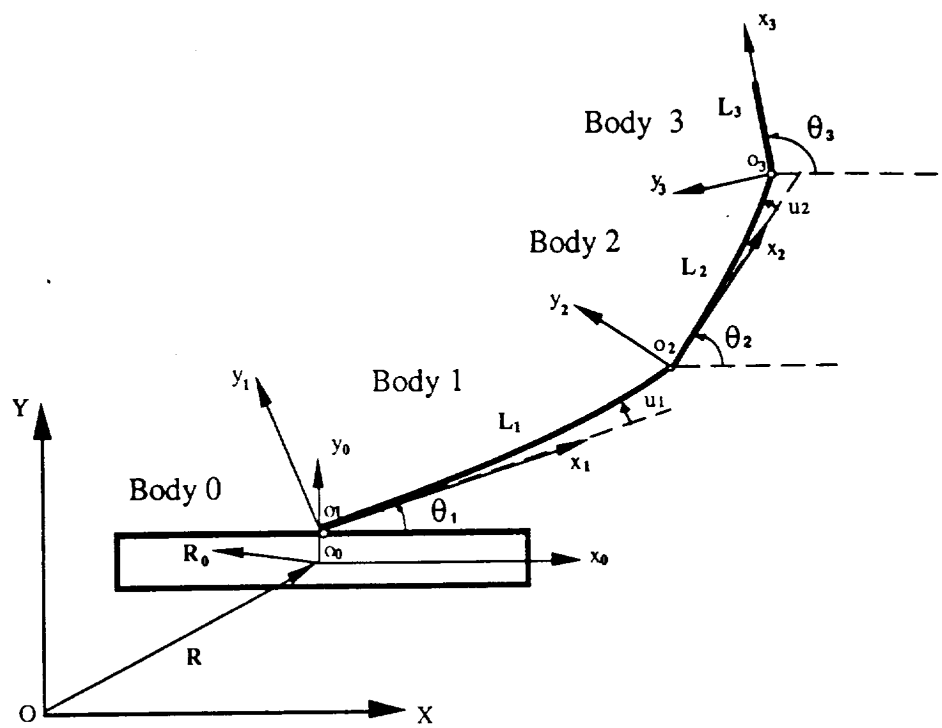

Fig. 2 - Coordinate Systems for the Space Robot

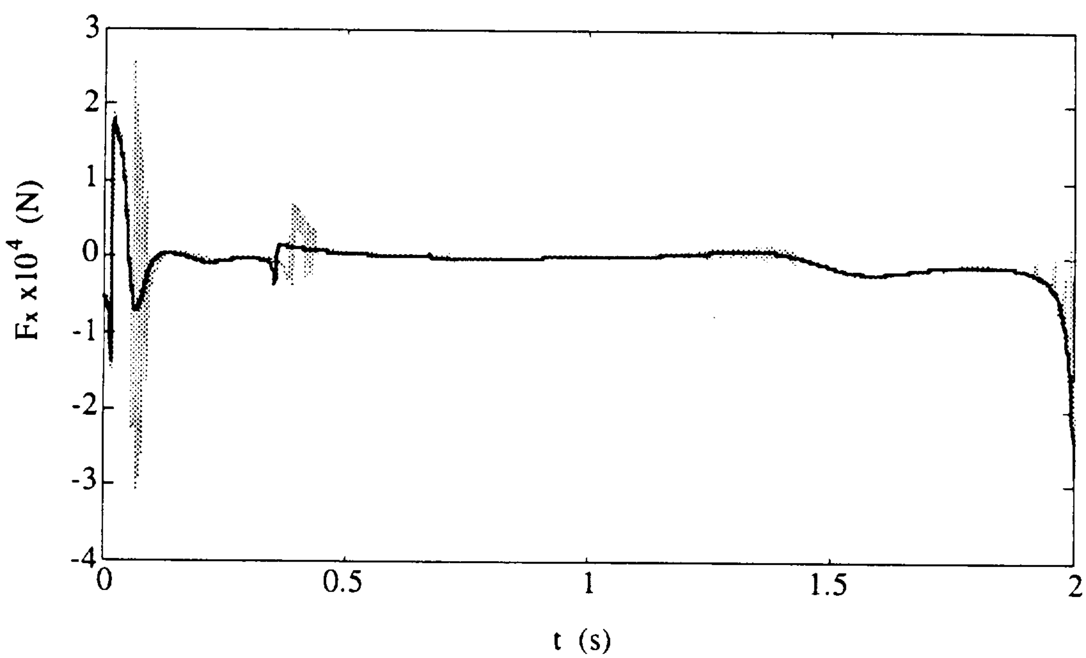

Fig. 3 - The Bursting Phenomenon

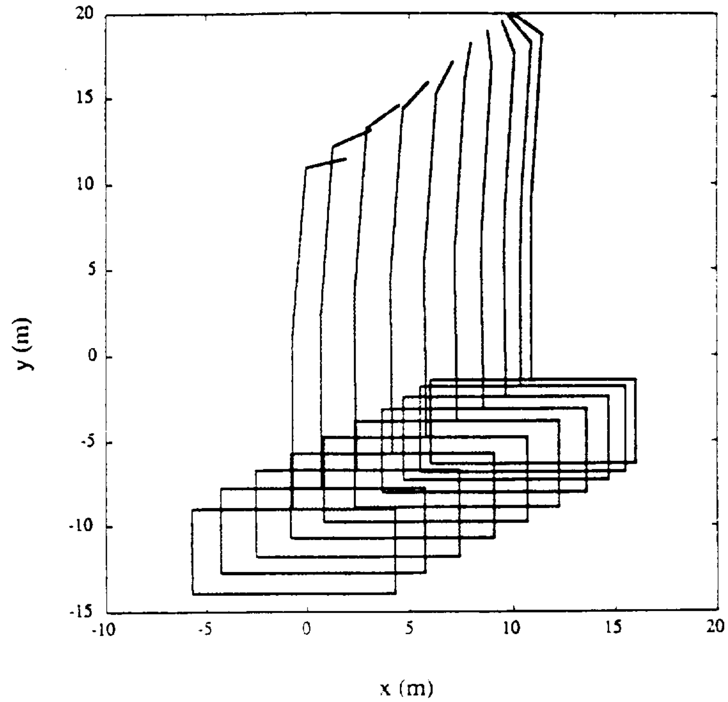

Fig. 4a - Time-Lapse Picture of the Robot Configuration - Case 1

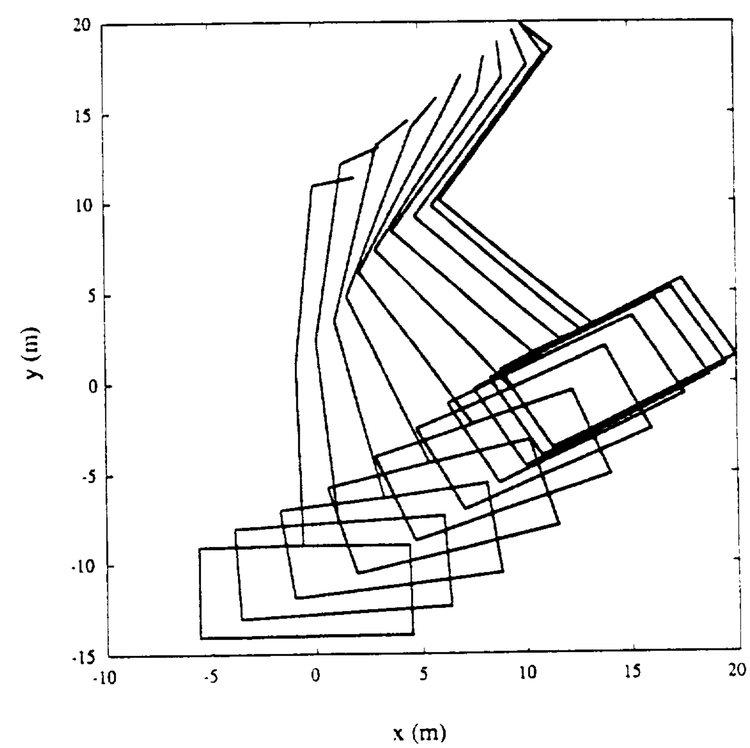

Fig. 4b - Time-Lapse Picture of the Robot Configuration - Case 2

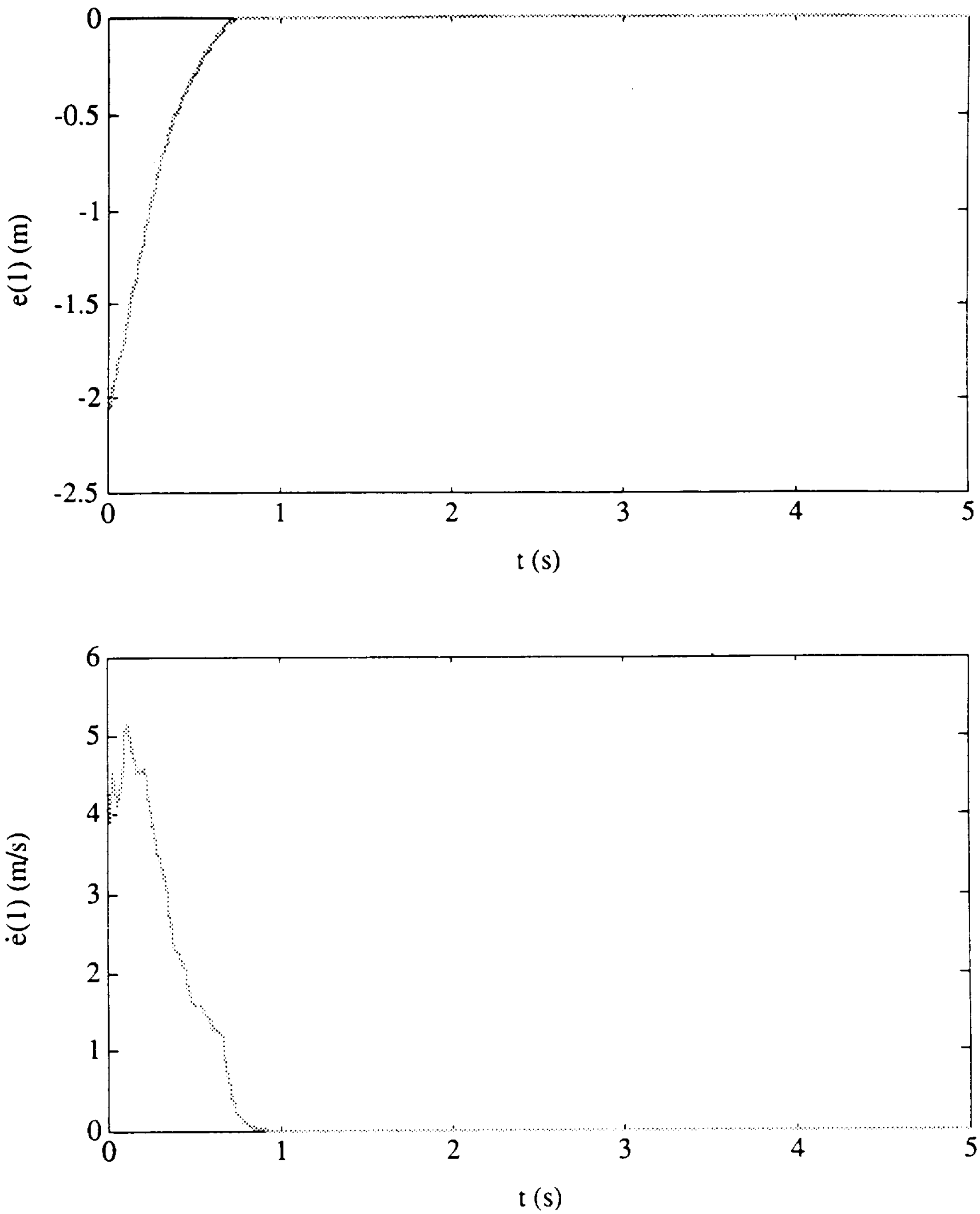

Fig. 5a - Time History of the x-Component of the Tracking Error and Tracking Error Rate - Case 2

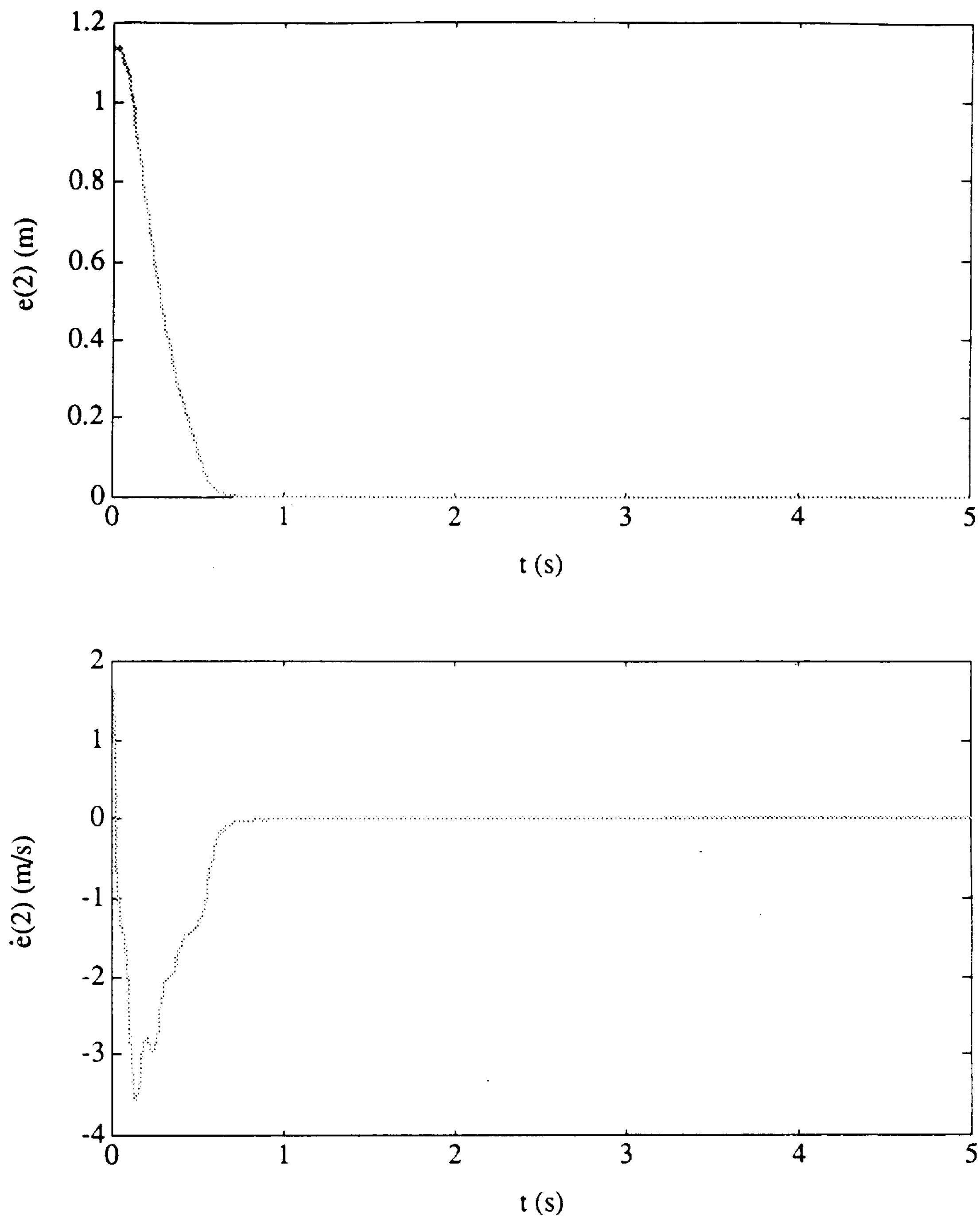

Fig. 5b - Time History of the y-Component of the Tracking Error and Tracking Error Rate - Case 2

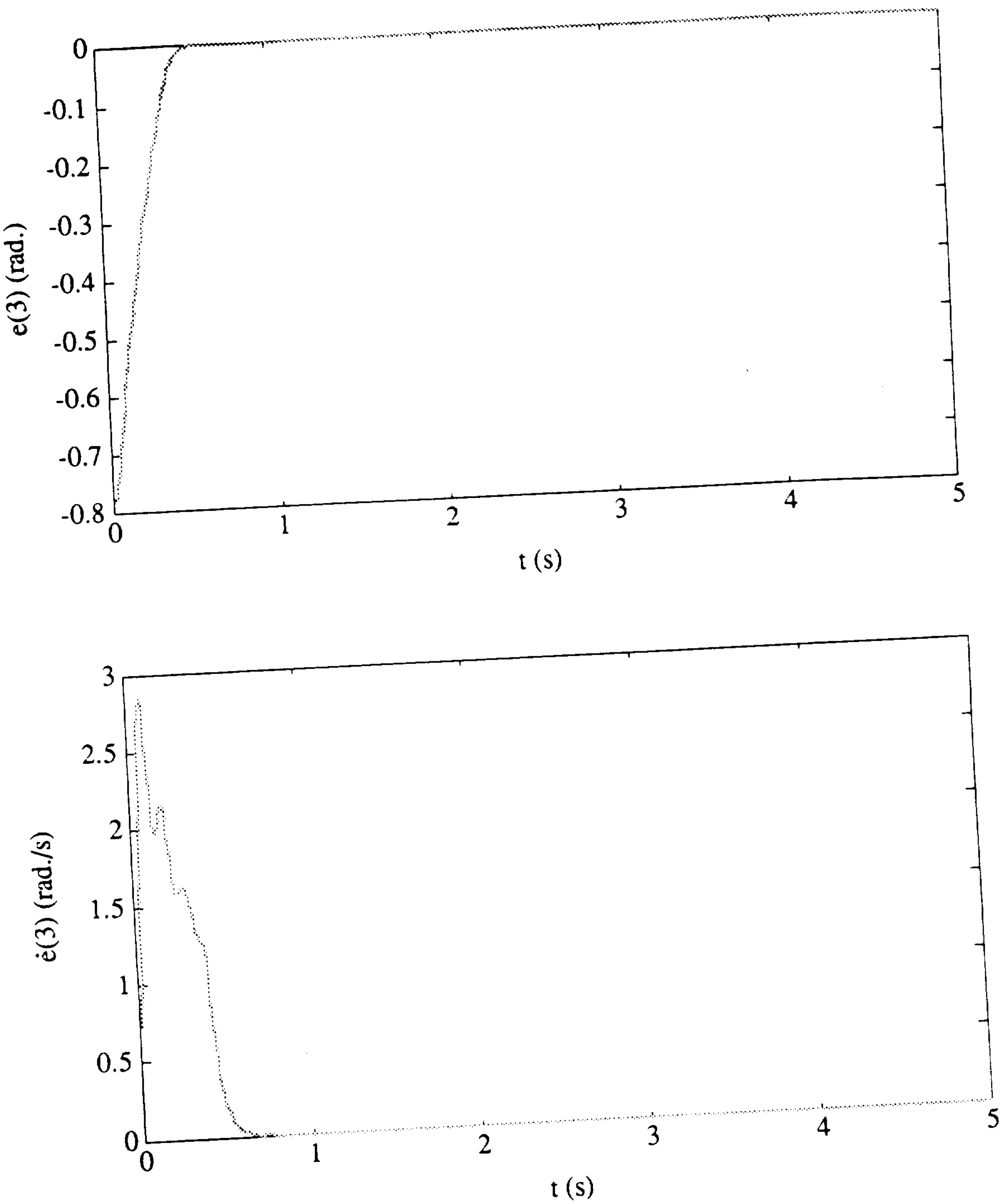

Fig. 5c - Time History of the Orientation Error and Orientation Error Rate - Case 2

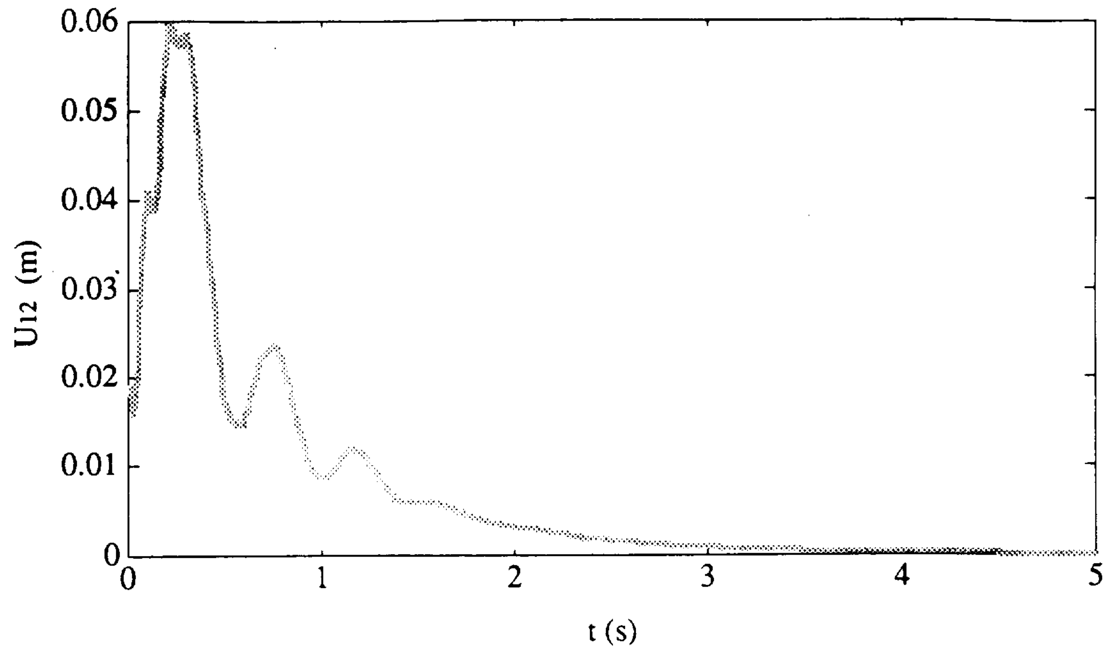

Fig. 6a - Time History of the Tip Elastic Displacement of the First Flexible Body

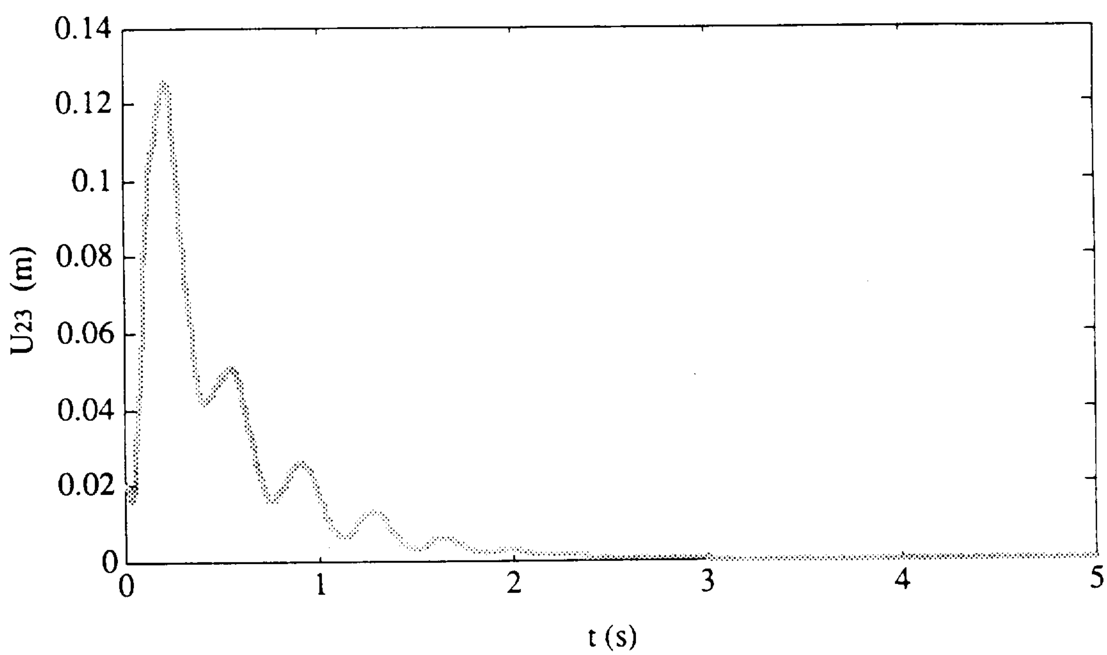

Fig. 6b - Time History of the Tip Elastic Displacement of the Second Flexible Body

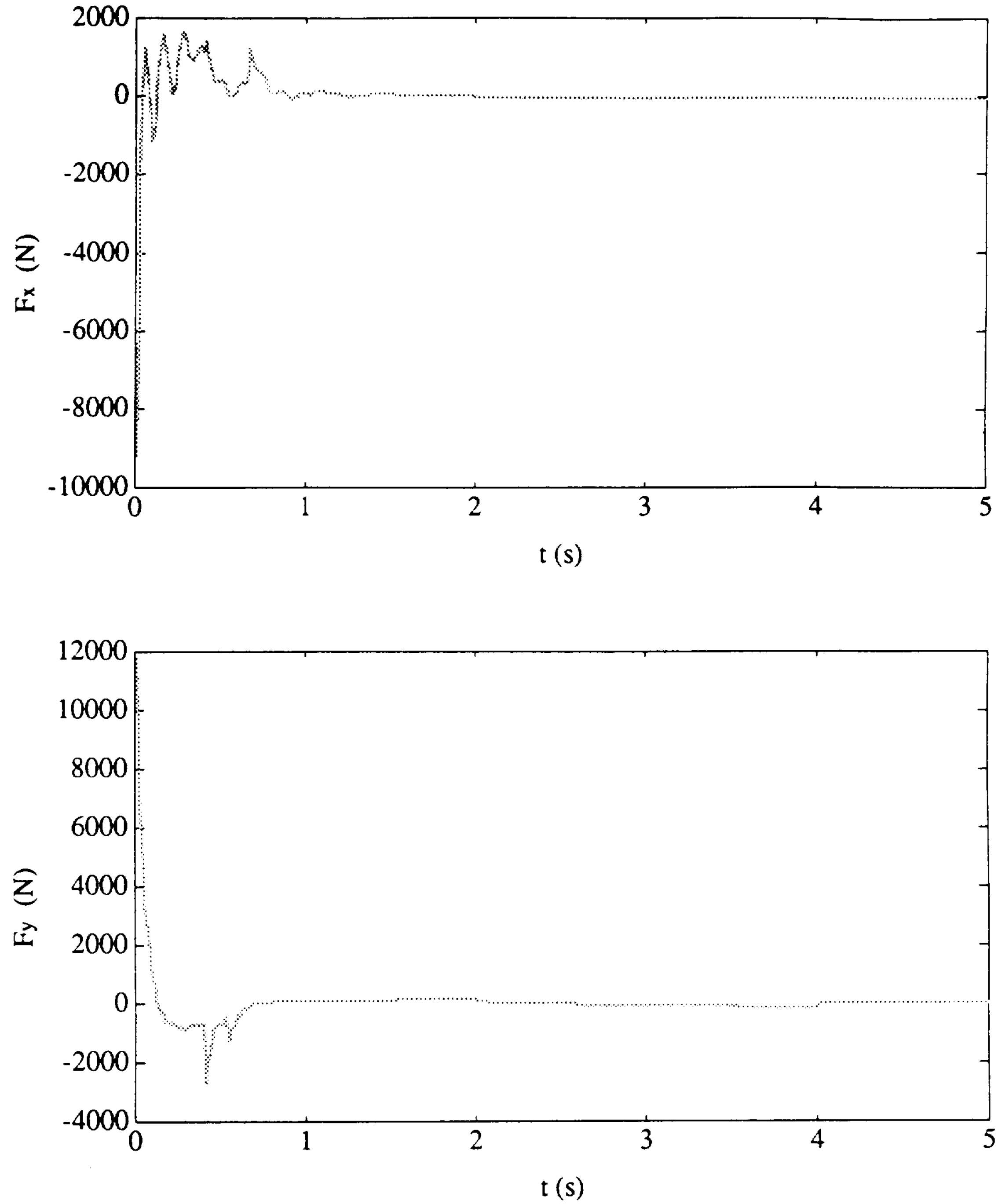

Fig. 7a - Time History of the Control Forces for the Rigid-Body Translation of the Robot Base

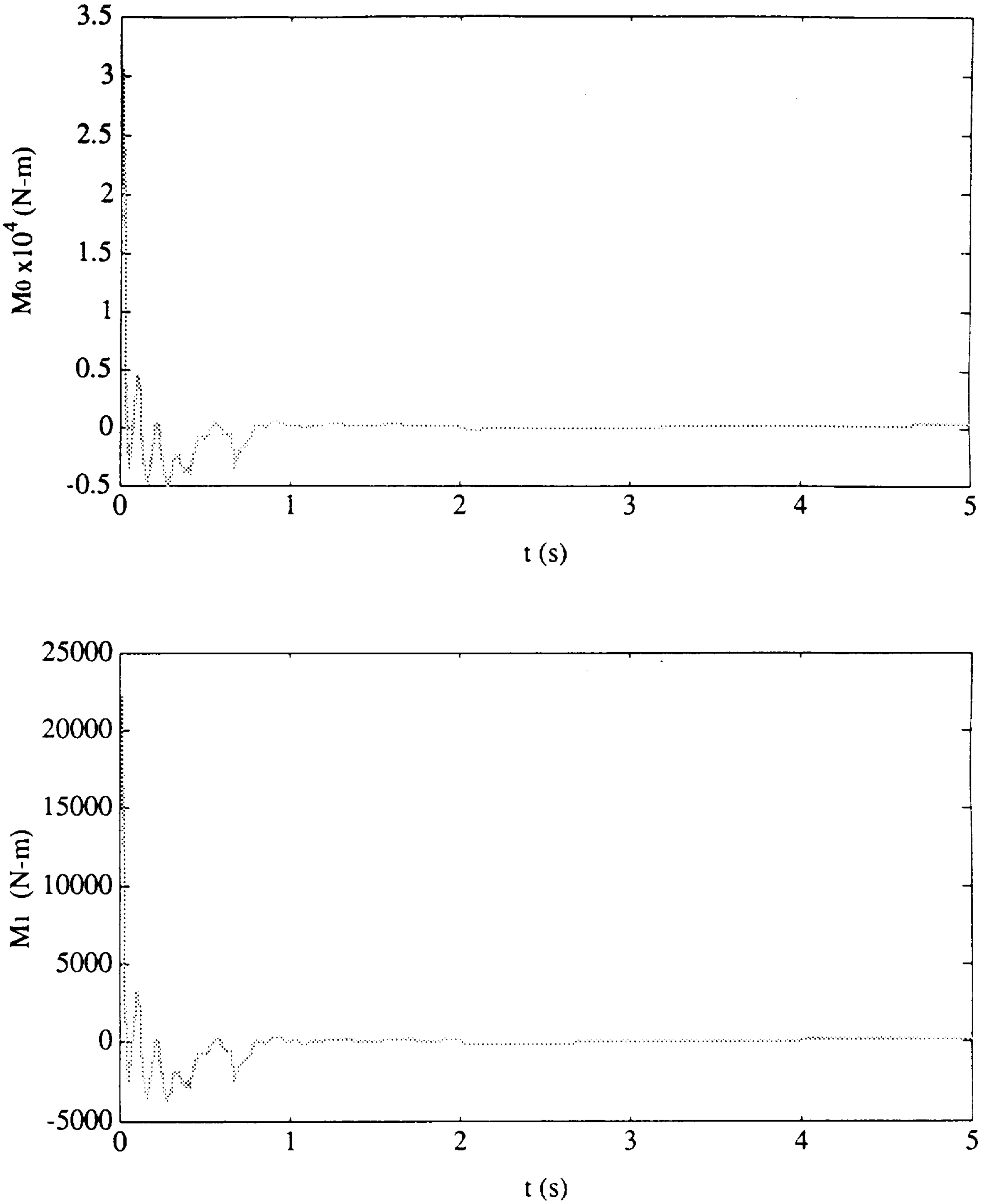

Fig. 7b - Time History of the Control Torques for the Rigid-Body Rotation of the Robot Base and Body 1

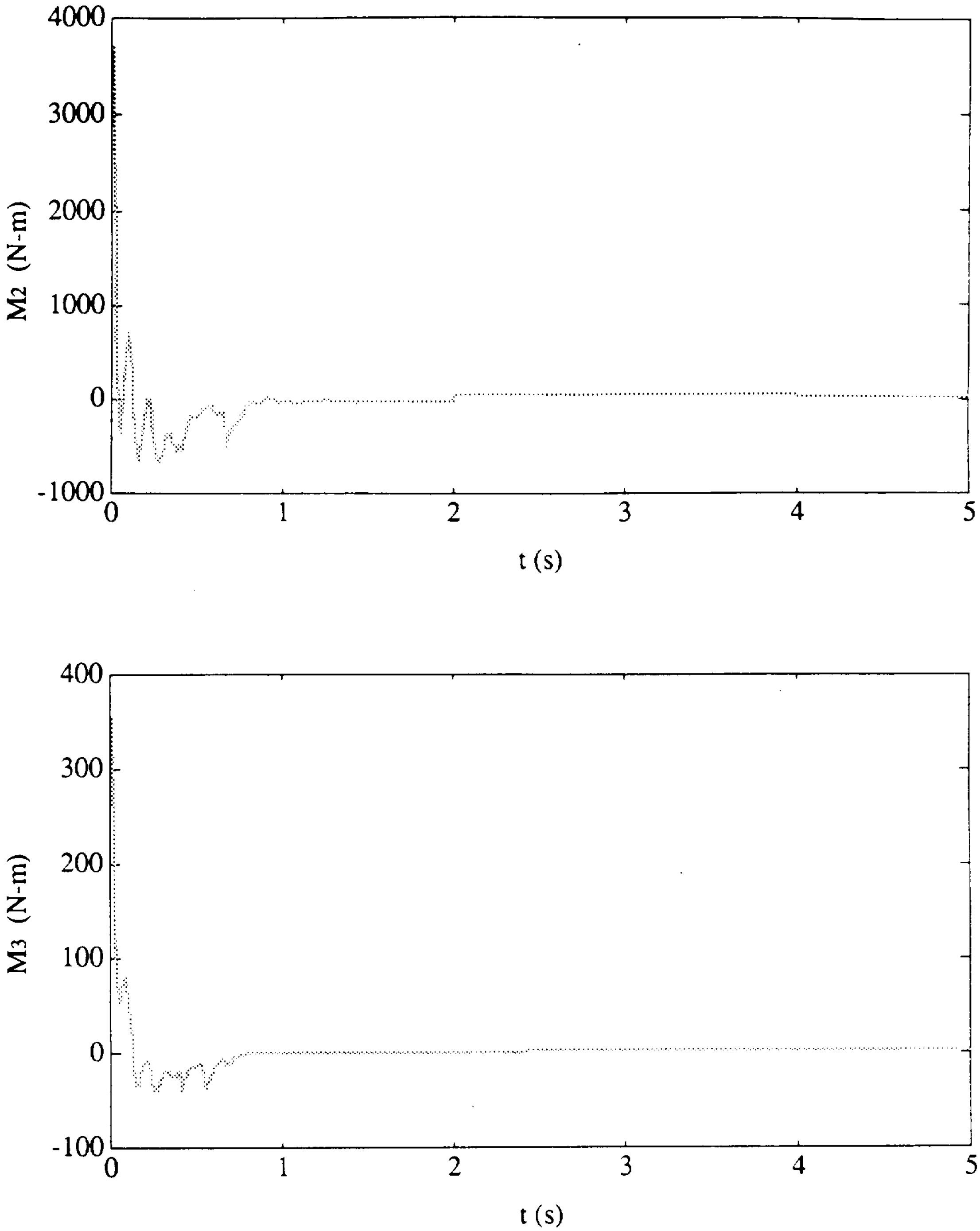

Fig. 7c - Time History of the Control Torques for the Rigid-Body Rotation of Bodies 2 and 3

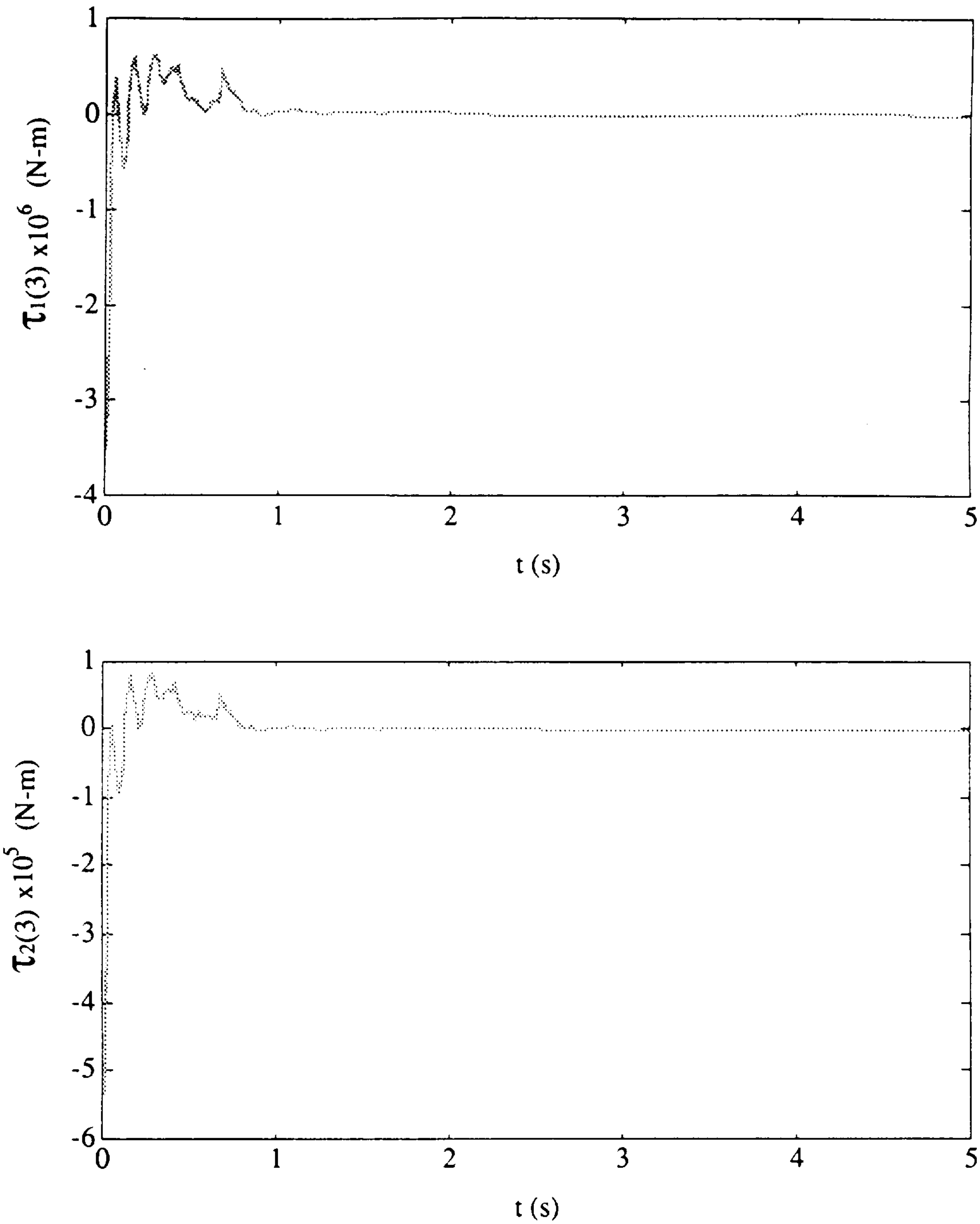

Fig. 8 - Time History of the Control Torques Acting on the Flexible Bodies for Disturbance Rejection

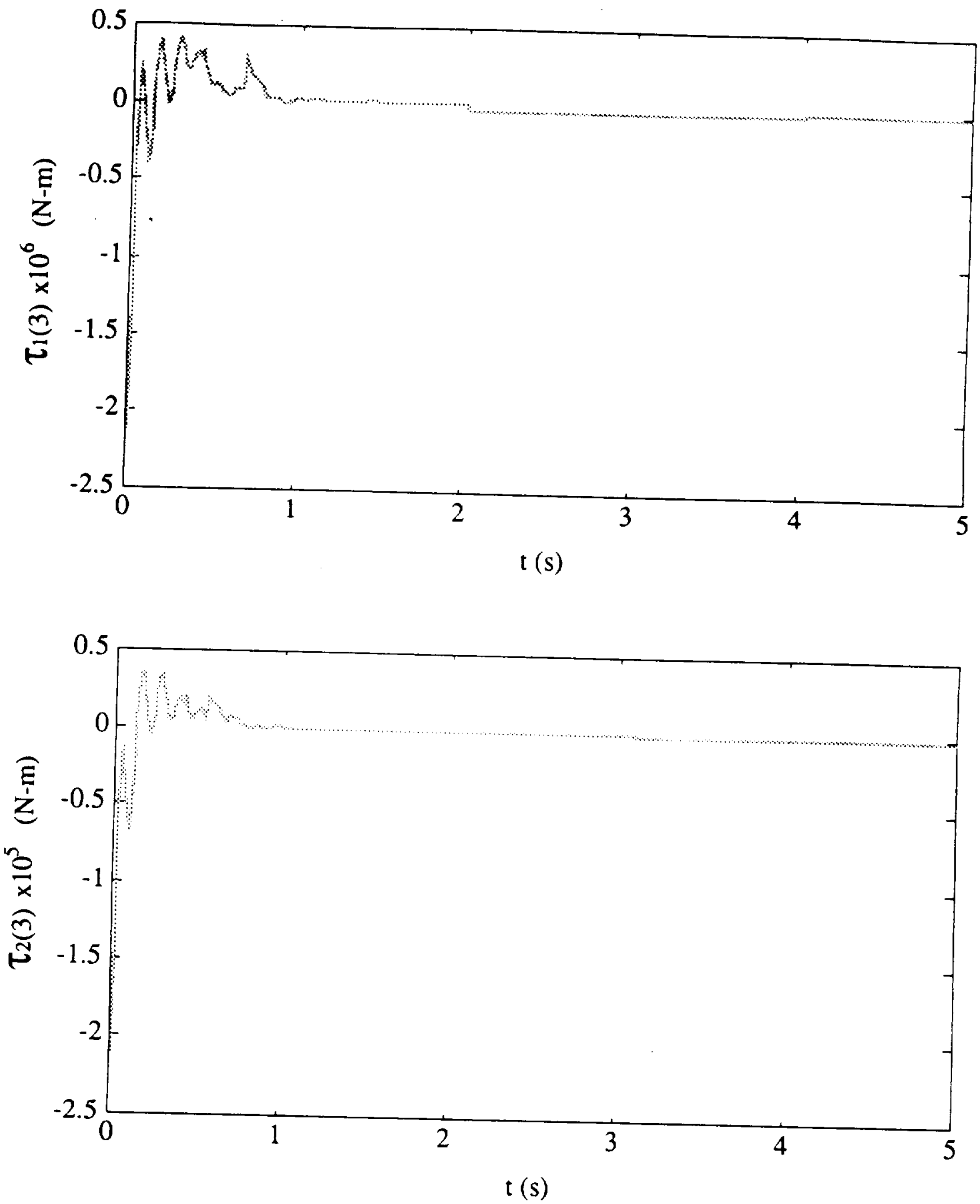

Fig. 9 - Time History of the LQR Control Torques Acting on the Flexible Bodies

HYBRID EQUATIONS OF MOTION FOR FLEXIBLE MULTIBODY SYSTEMS USING QUASI-COORDINATES[†]

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Abstract

A variety of engineering systems, such as automobiles, aircraft, rotorcraft, robots, spacecraft, etc., can be modeled as flexible multibody systems. The individual flexible bodies are in general characterized by distributed parameters. In most earlier investigations they were approximated by some spatial discretization procedure, such as the classical Rayleigh-Ritz method or the finite element method. This paper presents a mathematical formulation for distributed-parameter multibody systems consisting of a set of hybrid (ordinary and partial) differential equations of motion in terms of quasi-coordinates. Moreover, the equations for the elastic motions include rotatory inertia and shear deformation effects. The hybrid set is cast in state form, thus making it suitable for control design.

1. Introduction

A problem of current interest is the dynamics and control of multibody systems. Indeed, a variety of engineering systems, such as automobiles, aircraft, rotorcraft, robots, spacecraft, etc., can be modeled as multibodies. In many engineering applications the bodies can be assumed to be rigid (Refs. 1-12). In many other applications, the flexibility effects have to be included (Refs. 13-24). For the most part, flexible bodies have distributed mass and stiffness properties, which is likely to cause difficulties in producing a solution. As a result, it is common practice to approximate distributed systems by discrete ones through spatial discretization, which can be carried out by means of the classical Rayleigh-Ritz method or the finite element method (Ref. 25). The discretization process amounts to elimination of the spatial coordinates. The equations of motion for the discretized system are derived quite often by the standard Lagrangian approach. For more complex motions, an approach using quasi-coordinates seems to offer many advantages (Refs. 26-29).

Quite recently, there has been some interest in working with distributed models as much as possible, thus avoiding truncation problems arising from spatial discretization. Consistent with this, hybrid (ordinary and partial) differential equations of motion have been derived for flexible multibody systems in Refs. 30 and 31, using the approach of Ref. 25. Hybrid equations of motion in terms of quasi-coordinates have been derived for the first time in Ref. 26 for a spinning rigid body with flexible appendages and generalized later in Ref. 32 for a flexible body undergoing rigid-body and elastic motions. This paper extends the general theory developed in Ref. 32 to systems of flexible multibodies. In addition, the equations for the elastic motions include rotatory inertia and shear deformation effects.

2. Kinematics

We are concerned with structures consisting of a chain of articulated bodies i ($i = 1, 2, \dots, N$), which implies that two adjacent bodies $i-1$ and i are hinged at O_i (Fig. 1). To describe the motion of the system, it will prove convenient to conceive of a set of body axes $x_i y_i z_i$ with the origin at O_i and attached to body i in undeformed state. The bodies are assumed to be slender, with axis x_i coinciding with the long axis of the body. As the body deforms, x_i remains tangent to the body at O_i. At the same time, we consider another set of body axes $x'_i y'_i z'_i$, referred to as intermediate axes, with the origin at O_i and attached to body $i-1$ so that x'_i is along the long axis. We will also find it convenient to introduce an inertial frame of reference XYZ with the origin at O.

We denote the position vector of point O_i relative to the origin O by $\mathbf{R}_{oi} = [X_{oi}\, Y_{oi}\, Z_{oi}]^T$. Then, we denote the position of a typical point $\mathcal{P}_i$ in the undeformed i body relative to O_i by $\mathbf{r}_i$ and the elastic displacement of $\mathcal{P}_i$ by $\mathbf{u}_i$. Hence, the radius vector from O to $\mathcal{P}_i$ in displaced position is simply

$$\mathbf{R}_i = C_i^* \mathbf{R}_{oi} + \mathbf{r}_i + \mathbf{u}_i, \qquad i = 1, 2, \dots, N \qquad (1)$$

where C_i^* is the matrix of direction cosines of axes $x_i y_i z_i$ with respect to axes $x_{i-1} y_{i-1} z_{i-1}$, and note that the vector $\mathbf{R}_{oi}$ is in terms of components along the body axes $x_{i-1} y_{i-1} z_{i-1}$ and the vectors $\mathbf{R}_i$, $\mathbf{r}_i$ and $\mathbf{u}_i$ are in terms of components along the body axes $x_i y_i z_i$.

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We consider here bodies in the form of bars with the long axis x_i passing through O_i and O_{i+1} when the bars are undeformed. We are concerned with bars undergoing torsion about axis x_i and bending about axes y_i and z_i, as well as shearing distortion in the y_i and z_i directions. Then, the vectors $\mathbf{r}_i$ and $\mathbf{u}_i$ can be written in the more explicit form

$$\mathbf{r}_i = [x_i \ \ 0 \ \ 0]^T, \quad \mathbf{u}_i(x_i, t) = [0 \ \ u_{yi}(x_i, t) \ \ u_{zi}(x_i, t)]^T \tag{2a,b}$$

The radius vector $\mathbf{R}_{oi}$ depends on the motion of the preceding $i-1$ bodies in the chain. In particular, we can write the following recursive relation:

$$\mathbf{R}_{oi} = C^*_{i-1}\mathbf{R}_{o,i-1} + \mathbf{r}_{i-1}(\ell_{i-1}) + \mathbf{u}_{i-1}(\ell_{i-1}, t), \quad i = 2, 3, \ldots, N \tag{3}$$

where ℓ_{i-1} is the length of body $i-1$. Note that $\mathbf{R}_{o1} = \mathbf{R}_{o1}(t)$ is simply the radius vector from O to the origin O_1 of the body axes of the first body in the chain.

At this point, we propose to define the rotational motions. In the first place, it will prove convenient to introduce a set of body axes $\xi_i\eta_i\zeta_i$ attached to a typical beam cross section originally in the nominal position x_i and moving with the cross section as body i deforms. In this regard, note that $\xi_{i-1}(\ell_{i-1})\eta_{i-1}(\ell_{i-1})\zeta_{i-1}(\ell_{i-1})$ coincide with $x'_i y'_i z'_i$. Then, denoting the angle of twist by ψ_{xi} and the bending rotation angles by ψ_{yi} and ψ_{zi}, we conclude that axes $\xi_i\eta_i\zeta_i$ experience the angular displacement

$$\boldsymbol{\psi}_i(x_i, t) = [\psi_{xi}(x_i, t) \ \ \psi_{yi}(x_i, t) \ \ \psi_{zi}(x_i, t)]^T \tag{4}$$

with respect to axes $x_i y_i z_i$. On various occasions throughout this paper, we encounter skew symmetric matrices derived from vectors. As an example, if a typical vector $\mathbf{r}$ has components x, y and z, then the associated skew symmetric matrix has the form

$$\tilde{r} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{5}$$

In view of this definition, the matrix of direction cosines of $\xi_i\eta_i\zeta_i$ relative to $x_i y_i z_i$ can be shown to have the expression

$$E_i(x_i, t) = I - \tilde{\psi}_i(x_i, t) \tag{6}$$

in which I is the 3×3 identity matrix, and we note that Eq. (6) follows from the assumption that the components of $\boldsymbol{\psi}_i$ are small. Next, we assume that axes $x_i y_i z_i$ are obtained from axes $x'_i y'_i z'_i$ through the rotations θ_{ij}, where j can take the values 1, or 1,2, or 1,2,3, depending on the nature of the hinge at O_i and denote by $C_i(\boldsymbol{\theta}_i)$ the matrix of direction cosines of $x_i y_i z_i$ relative to $x'_i y'_i z'_i$, where $\boldsymbol{\theta}_i = [\theta_{i1} \ \theta_{i2} \ \theta_{i3}]^T$. Then, the matrix of direction cosines of axes $x_i y_i z_i$ relative to axes $x_{i-1} y_{i-1} z_{i-1}$ is simply

$$C^*_i = C_i E_{i-1}(\ell_{i-1}, t) \tag{7}$$

From kinematics, the velocity vector of the typical point $\mathcal{P}_i$ in displaced position in terms of the rotating body axes $x_i y_i z_i$, has the expression

$$\begin{aligned} \mathbf{V}_i &= \mathbf{V}_{oi} + \tilde{\boldsymbol{\Omega}}_{ri}(\mathbf{r}_i + \mathbf{u}_i) + \mathbf{v}_i \\ &= \mathbf{V}_{oi} + (\tilde{r}_i + \tilde{u}_i)^T \boldsymbol{\Omega}_{ri} + \mathbf{v}_i, \quad i = 1, 2, \ldots, N \end{aligned} \tag{8}$$

where $\mathbf{V}_{oi}$ is the velocity vector of the origin O_i, $\boldsymbol{\Omega}_{ri}$ is the angular velocity vector of axes $x_i y_i z_i$ relative to axes XYZ and

$$\mathbf{v}_i(x_i, t) = \dot{\mathbf{u}}_i(x_i, t) \tag{9}$$

is the elastic velocity vector relative to $x_i y_i z_i$, all in terms of $x_i y_i z_i$ components. We note that the velocity vector of point O_i can be written in the recursive form

$$\begin{aligned} \mathbf{V}_{oi} &= C^*_i \mathbf{V}_{i-1}(\ell_{i-1}, t) \\ &= C^*_i \left\{ \mathbf{V}_{o,i-1} + [\tilde{r}_{i-1}(\ell_{i-1}) + \tilde{u}_{i-1}(\ell_{i-1}, t)]^T \boldsymbol{\Omega}_{r,i-1} \right. \\ &\quad \left. + \mathbf{v}_{i-1}(\ell_{i-1}, t) \right\}, \quad i = 2, 3, \ldots, N \end{aligned} \tag{10}$$

Moreover, introducing the notation

$$\boldsymbol{\Omega}_{ei}(x_i, t) = \dot{\boldsymbol{\psi}}_i(x_i, t), \quad i = 1, 2, \ldots, N \tag{11}$$

the angular velocity vector of the cross-sectional axes $\xi_i\eta_i\zeta_i$ relative to the inertial space is simply

$$\boldsymbol{\Omega}_i = \boldsymbol{\Omega}_{ri} + \boldsymbol{\Omega}_{ei}(x_i, t), \quad i = 1, 2, \ldots, N \tag{12}$$

Finally, letting $\boldsymbol{\omega}_i$ be the angular velocity vector of axes $x_i y_i z_i$ relative to axes $x'_i y'_i z'_i$, in terms of $x_i y_i z_i$ components, the angular velocity vector of $x_i y_i z_i$ is given by the recursive formula

$$\begin{aligned} \boldsymbol{\Omega}_{ri} &= C^*_i \boldsymbol{\Omega}_{i-1}(\ell_{i-1}, t) + \boldsymbol{\omega}_i \\ &= C^*_i \left[\boldsymbol{\Omega}_{r,i-1} + \boldsymbol{\Omega}_{e,i-1}(\ell_{i-1}, t)\right] + \boldsymbol{\omega}_i, \\ &\qquad i = 2, 3, \ldots, N \end{aligned} \tag{13}$$

where the second equality follows from Eq. (12).

3. Standard Lagrange's Equations for Flexible Multibody Systems

The motion of our multibody system is described in terms of rigid-body displacements of sets of body axes and elastic displacements relative to these body axes. As a result, the equations of motion are hybrid, in the sense that they consist of ordinary differential equations for the rigid-body displacements and partial differential equations for the elastic displacements. The equations of motion can be derived by means of the extended Hamiltion's principle (Ref. 33), which can be stated in the form

$$\int_{t_1}^{t_2} (\delta L + \overline{\delta W})\, dt = 0, \qquad \delta \mathbf{q} = 0,\ \delta \mathbf{u}_i = \delta \boldsymbol{\psi}_i = 0,$$

$$i = 1, 2, \ldots, N \quad \text{at} \quad t = t_1, t_2 \tag{14}$$

where

$$L = T - V \tag{15}$$

is the Lagrangian, in which T is the kinetic energy and V is the potential energy, and $\overline{\delta W}$ is the virtual work. Moreover, $\mathbf{q}$ is the rigid-body displacement vector, and $\mathbf{u}_i$, $\boldsymbol{\psi}_i$ ($i = 1, 2, \ldots, N$) are the elastic displacement vectors introduced earlier. Hence, before we can derive equations of motion, we must derive general expressions for T, V and $\overline{\delta W}$.

Taking the x_i-axis to coincide with the centroidal axis of the undeformed beam, the kinetic energy can be shown to consist of two parts, one due to translations and one due to rotations (Ref. 25). Hence, using Eqs. (8) and (12), the kinetic energy can be expressed in the form

$$T = \sum_{i=1}^{N} \int_0^{\ell_i} \hat{T}_i \, dx_i \tag{16}$$

where

$$\begin{aligned}\hat{T}_i =& \frac{1}{2}(\rho_i \mathbf{V}_i^T \mathbf{V}_i + \boldsymbol{\Omega}_i^T \hat{J}_{ci} \boldsymbol{\Omega}_i) = \frac{1}{2}[\rho_i \mathbf{V}_{oi}^T \mathbf{V}_{oi} \\ &+ \boldsymbol{\Omega}_{ri}^T \hat{J}_i \boldsymbol{\Omega}_{ri} + \rho_i \dot{\mathbf{u}}_i^T \dot{\mathbf{u}}_i + 2\mathbf{V}_{oi}^T \bar{\tilde{S}}_i^T \boldsymbol{\Omega}_{ri} \\ &+ 2\rho_i \mathbf{V}_{oi}^T \dot{\mathbf{u}}_i + 2\boldsymbol{\Omega}_{ri}^T \bar{\tilde{S}}_i \dot{\mathbf{u}}_i + \boldsymbol{\Omega}_{ri}^T \hat{J}_{ci} \boldsymbol{\Omega}_{ri} \\ &+ \dot{\boldsymbol{\psi}}_i^T \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i + 2\boldsymbol{\Omega}_{ri}^T \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i] \\ =& \frac{1}{2}[\rho_i \mathbf{V}_{oi}^T \mathbf{V}_{oi} + \boldsymbol{\Omega}_{ri}^T \hat{J}_{ti} \boldsymbol{\Omega}_{ri} \\ &+ \rho_i \dot{\mathbf{u}}_i^T \dot{\mathbf{u}}_i + \dot{\boldsymbol{\psi}}_i^T \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i + 2\mathbf{V}_{oi}^T \bar{\tilde{S}}_i^T \boldsymbol{\Omega}_{ri} \\ &+ 2\rho_i \mathbf{V}_{oi}^T \dot{\mathbf{u}}_i + 2\boldsymbol{\Omega}_{ri}^T (\bar{\tilde{S}}_i \dot{\mathbf{u}}_i + \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i)]\end{aligned} \tag{17}$$

is the kinetic energy density of member i, in which ρ_i is the mass density and

$$\hat{J}_{ti} = \hat{J}_i + \hat{J}_{ci} \tag{18}$$

is the total moment of inertia density matrix, where

$$\begin{aligned}\hat{J}_i =& \rho_i (\tilde{r}_i + \tilde{u}_i)(\tilde{r}_i + \tilde{u}_i)^T \\ =& \rho_i \begin{bmatrix} u_{yi}^2 + u_{zi}^2 & -x_i u_{yi} & -x_i u_{zi} \\ -x_i u_{yi} & x_i^2 + u_{zi}^2 & -u_{yi} u_{zi} \\ -x_i u_{zi} & -u_{yi} u_{zi} & x_i^2 + u_{yi}^2 \end{bmatrix}\end{aligned} \tag{19a}$$

and

$$\hat{J}_{ci} = \mathrm{diag}[\hat{J}_{xixi} \;\; \hat{J}_{yiyi} \;\; \hat{J}_{zizi}] \tag{19b}$$

in which $\hat{J}_{xixi}$, $\hat{J}_{yiyi}$ and $\hat{J}_{zizi}$ are cross-sectional mass moments of inertia densities, and note that, because the elastic deformations are relatively small, they are approximately equal to $\hat{J}_{\xi i \xi i}$, $\hat{J}_{\eta i \eta i}$ and $\hat{J}_{\zeta i \zeta i}$, respectively. Moreover, $\hat{\tilde{S}}_i$ is obtained from

$$\hat{\mathbf{S}}_i = \rho_i (\mathbf{r}_i + \mathbf{u}_i) = \rho_i [x_i \;\; u_{yi} \;\; u_{zi}]^T \tag{20}$$

which is recognized as the first moments of inertia density vector.

Assuming that differential gravity effects are negligibly small, the potential energy reduces to the strain energy. As indicated earlier, the elastic members undergo torsion about x_i and bending about y_i and z_i, as well as shearing distortions in the y_i and z_i directions. Referring to Fig. 2, we conclude that the relations between the bending displacements u_{yi} and u_{zi}, the bending angular displacements ψ_{yi} and ψ_{zi} and the shearing distortion angles β_{yi} and β_{zi} are

$$u'_{yi} = \psi_{zi} + \beta_{zi}, \quad u'_{zi} = -\psi_{yi} - \beta_{yi} \tag{21a,b}$$

where primes denote partial derivatives with respect to x_i. From mechanics of materials, the relation between the twisting moment M_{xi} and the twist angle ψ_{xi} is simply

$$M_{xi} = k_{xi} G_i I_{xi} \psi'_{xi} \tag{22}$$

where k_{xi} is a factor depending on the shape of the cross section and $G_i I_{xi}$ is the torsional rigidity, in which G_i is the shear modulus and I_{xi} is the polar area moment of inertia about axis x_i. Moreover, the bending moments are related to the bending rotational displacements by

$$M_{yi} = E_i I_{yi} \psi'_{yi}, \quad M_{zi} = E_i I_{zi} \psi'_{zi} \tag{23a,b}$$

in which E_i is Young's modulus and I_{yi} and I_{zi} are area moments of inertia about axes parallel to y_i and z_i, respectively, and passing through the center of the cross-sectional area, and the shearing forces are related to the shearing distortion angles according to

$$Q_{yi} = k_{zi} G_i A_i \beta_{zi}, \quad Q_{zi} = -k_{yi} G_i A_i \beta_{yi} \tag{24a,b}$$

where k_{yi} and k_{zi} are factors depending on the shape of the cross sectional area, G_i is the shear modulus and A_i is the cross-sectional area.

The strain energy can be expressed as

$$V = \sum_{i=1}^{N} \int_0^{\ell_i} \hat{V}_i \, dx_i \tag{25}$$

where, using Eqs. (21)-(24),

$$\begin{aligned}\hat{V}_i =& \frac{1}{2}\left(M_{xi}\psi'_{xi} + M_{yi}\psi'_{yi} + M_{zi}\psi'_{zi} + Q_{yi}\beta_{zi} - Q_{zi}\beta_{yi}\right) \\ =& \frac{1}{2}\left[k_{xi} G_i I_{xi} (\psi'_{xi})^2 + E_i I_{yi} (\psi'_{yi})^2 + E_i I_{zi} (\psi'_{zi})^2 \right. \\ &\left. + k_{yi} G_i A_i (u'_{yi} - \psi_{zi})^2 + k_{zi} G_i A_i (u'_{zi} + \psi_{yi})^2\right]\end{aligned} \tag{26}$$

is the potential energy density for member i.

Next, we wish to develop an expression for the virtual work due to nonconservative actuator forces and torques. Using the analogy with Eqs. (8) and (12), the virtual work can be written in the form

$$\begin{aligned}\overline{\delta W} &= \sum_{i=1}^{N}\left[\int_0^{\ell_i}(\mathbf{f}_i^T\delta\mathbf{R}_i^* + \mathbf{m}_i^T\delta\mathbf{\Theta}_i^*)\,dx_i\right] + \sum_{i=2}^{N}\mathbf{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \\ &= \sum_{i=1}^{N}\left\{\int_0^{\ell_i}[\mathbf{f}_i^T(\delta\mathbf{R}_{oi}^* + \tilde{r}_i^T\delta\mathbf{\Theta}_{ri}^* + \delta\mathbf{u}_i) \right. \\ &\qquad \left. + \mathbf{m}_i^T(\delta\mathbf{\Theta}_{ri}^* + \delta\boldsymbol{\psi}_i)]\,dx_i\right\} + \sum_{i=2}^{N}\mathbf{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \\ &= \sum_{i=1}^{N}\left[\mathbf{F}_{ri}^{*T}\delta\mathbf{R}_{oi}^* + \mathbf{M}_{ri}^{*T}\delta\mathbf{\Theta}_{ri}^* \right. \\ &\qquad \left. + \int_0^{\ell_i}(\mathbf{f}_i^T\delta\mathbf{u}_i + \mathbf{m}_i^T\delta\boldsymbol{\psi}_i)\,dx_i\right] + \sum_{i=2}^{N}\mathbf{M}_{oi}^{*T}\delta\boldsymbol{\theta}_i^* \end{aligned} \quad (27)$$

in which $\mathbf{f}_i$ and $\mathbf{m}_i$ are distributed actuator forces and torques acting over the domain i, $\mathbf{M}_{oi}^*$ are torque actuators located at points O_i and acting on both members $i-1$ and i, for $i = 2, 3, \ldots, N$, $\delta\mathbf{R}_i^*$ is the virtual displacement vector of point $\mathcal{P}_i$, $\delta\mathbf{\Theta}_i^*$ is the virtual rotation vector of axes $\xi_i\eta_i\zeta_i$, $\delta\boldsymbol{\theta}_i^*$ is the virtual rotation vector of axes $x_iy_iz_i$ relative to axes $x_i'y_i'z_i'$, $\delta\mathbf{R}_{oi}^*$ is the virtual displacement vector of point O_i and $\delta\mathbf{\Theta}_{ri}^*$ is the virtual rotation vector of axes $x_iy_iz_i$ relative to axes XYZ, where all of these vectors are in terms of components along axes $x_iy_iz_i$, and asterisks indicate quasi-coordinates (Ref. 33) and associated forces and torques. Note that the term $\mathbf{f}_i^T\tilde{u}_i^T\delta\mathbf{\Theta}_{ri}^*$ was omitted from $\delta\mathbf{R}_i^*$ on the basis that it is second-order in magnitude. Moreover,

$$\mathbf{F}_{ri}^* = \int_0^{\ell_i}\mathbf{f}_i\,dx_i, \quad \mathbf{M}_{ri}^* = \int_0^{\ell_i}(\tilde{r}_i\mathbf{f}_i + \mathbf{m}_i)\,dx_i \quad (28a,b)$$

are, respectively, resultant forces and torques acting on member i.

Before proceeding with the derivation of Lagrange's equations by means of the extended Hamilton's principle, Eq. (14), it is advisable to identify a set of generalized coordinates capable of describing the motion of the system fully. From Eqs. (3), we conclude that the motion of only one of the points O_i is independent. We choose this point as O_1, so that we retain only $\mathbf{R}_{o1}(t)$ for inclusion in the set of generalized coordinates. On the other hand, because O_i represent hinge points, the rigid-body rotation vectors $\boldsymbol{\theta}_i(t)$ $(i = 1, 2, \ldots, N)$ are all independent. Similarly, the nonzero components of the elastic displacement and rotation vectors, $\mathbf{u}_i(x_i, t)$ and $\boldsymbol{\psi}_i(x_i, t)$ $(i = 1, 2, \ldots, N)$, respectively, are also all independent. It will prove convenient to introduce the rigid-body motion vector

$$\mathbf{q}(t) = [\mathbf{R}_{o1}^T(t)\ \boldsymbol{\theta}_1^T(t)\ \boldsymbol{\theta}_2^T(t)\ldots\boldsymbol{\theta}_N^T(t)]^T \quad (29)$$

so that we propose to derive a vector Lagrange ordinary differential equation for $\mathbf{q}(t)$ and N pairs of vector Lagrange partial differential equations for $\mathbf{u}_i(x_i, t)$ and $\boldsymbol{\psi}_i(x_i, t)$ $(i = 1, 2, \ldots, N)$. To this end, we wish to express the Lagrangian in general functional form, and we note that the Lagrangian contains not only $\mathbf{q}$, $\mathbf{u}_i$ and $\boldsymbol{\psi}_i$ but also time and spatial derivatives of these vectors. Moreover, we observe from Eqs. (3), (7), (10) and (13) that the Lagrangian contains terms involving $\mathbf{u}_i(\ell_i, t)$, $\dot{\mathbf{u}}_i(\ell_i, t)$, $\boldsymbol{\psi}_i(\ell_i, t)$ and $\dot{\boldsymbol{\psi}}_i(\ell_i, t)$. Such terms will contribute to the dynamic boundary conditions accompanying the partial differential equations for $\mathbf{u}_i(x_i, t)$ and $\boldsymbol{\psi}_i(x_i, t)$. In view of this, we express the Lagrangian in the general form

$$\begin{aligned}L = &L[\mathbf{q}, \dot{\mathbf{q}}, \mathbf{u}_i, \mathbf{u}_i', \dot{\mathbf{u}}_i, \boldsymbol{\psi}_i, \boldsymbol{\psi}_i', \dot{\boldsymbol{\psi}}_i, \mathbf{u}_i(\ell_i, t), \\ &\dot{\mathbf{u}}_i(\ell_i, t), \boldsymbol{\psi}_i(\ell_i, t), \dot{\boldsymbol{\psi}}_i(\ell_i, t)]\end{aligned} \quad (30)$$

The extended Hamilton's principle, Eq. (14), calls for the variation of the Lagrangian, which can be expressed symbolically as

$$\begin{aligned}\delta L = &\left(\frac{\partial L}{\partial \mathbf{q}}\right)^T\delta\mathbf{q} + \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T\delta\dot{\mathbf{q}} \\ &+ \sum_{i=1}^{N}\int_0^{\ell_i}\left[\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i}\right)^T\delta\mathbf{u}_i + \left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right)^T\delta\mathbf{u}_i' + \cdots \right. \\ &\left. + \left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right)^T\delta\dot{\boldsymbol{\psi}}_i\right]dx_i + \sum_{i=1}^{N}\left\{\left[\frac{\partial L}{\partial \mathbf{u}_i(\ell_i, t)}\right]^T\delta\mathbf{u}_i(\ell_i, t) \right. \\ &+ \left[\frac{\partial L}{\partial \dot{\mathbf{u}}_i(\ell_i, t)}\right]^T\delta\dot{\mathbf{u}}_i(\ell_i, t) + \left[\frac{\partial L}{\partial \boldsymbol{\psi}_i(\ell_i, t)}\right]^T\delta\boldsymbol{\psi}_i(\ell_i, t) \\ &\left. + \left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(\ell_i, t)}\right]^T\delta\dot{\boldsymbol{\psi}}_i(\ell_i, t)\right\}\end{aligned} \quad (31)$$

where $\hat{L}_i = \hat{T}_i - \hat{V}_i$ is the Lagrangian density for body i. Moreover, $(\partial L/\partial\mathbf{q})^T$ represents the row matrix $[\partial L/\partial q_1\ \partial L/\partial q_2 \cdots \partial L/\partial q_{N_R}]$, etc., where N_R is the total number of independent rigid-body degrees of freedom. Consistent with the generalized coordinates used, the virtual work has the form

$$\begin{aligned}\overline{\delta W} = &\mathbf{Q}^T\delta\mathbf{q} + \sum_{i=1}^{N}\int_0^{\ell_i}(\mathbf{f}_i^T\delta\mathbf{u}_i + \mathbf{m}_i^T\delta\boldsymbol{\psi}_i)\,dx_i \\ &+ \sum_{i=1}^{N}[\mathbf{U}_i^T\delta\mathbf{u}_i(\ell_i, t) + \mathbf{\Psi}_i^T\delta\boldsymbol{\psi}_i(\ell_i, t)]\end{aligned} \quad (32a)$$

where we write the generalised force vector $\mathbf{Q}$ in the form

$$\mathbf{Q} = [\mathbf{F}_1^T \; \mathbf{M}_1^T \; \mathbf{M}_2^T \cdots \mathbf{M}_N^T]^T \tag{32b}$$

and note that $\mathbf{F}_1$ is a generalised force and $\mathbf{M}_1, \ldots, \mathbf{M}_N$ are generalised torques. They can all be related to the actuator forces and moments, but we postpone further discussion of this subject, and the derivation of specific formulas for $\mathbf{U}_i$ and $\boldsymbol{\Psi}_i$ until later.

Introducing Eqs. (31) and (32) into Eq. (14), carrying out the usual integrations by parts and recalling that the virtual displacements vanish at $t = t_1, t_2$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) + \mathbf{Q} \right]^T \delta \mathbf{q} \right. \\
&+ \sum_{i=1}^{N} \left\langle \int_0^{\ell_i} \left\{ \left[\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right) \right. \right. \right. \\
&\left. - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\right) + \mathbf{f}_i \right]^T \delta \mathbf{u}_i + \left[\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right) \right. \\
&\left. \left. - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right) + \mathbf{m}_i \right]^T \delta \boldsymbol{\psi}_i \right\} dx_i \\
&\left. + \left[\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right)^T \delta \mathbf{u}_i + \left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right)^T \delta \boldsymbol{\psi}_i \right] \Bigg|_0^{\ell_i} \right\rangle \\
&+ \sum_{i=1}^{N-1} \left\langle \left\{ \frac{\partial L}{\partial \mathbf{u}_i(\ell_i,t)} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}_i(\ell_i,t)}\right] \right. \right. \\
&\left. + \mathbf{U}_i \right\}^T \delta \mathbf{u}_i(\ell_i,t) + \left\{ \frac{\partial L}{\partial \boldsymbol{\psi}_i(\ell_i,t)} \right. \\
&\left. \left. \left. - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(\ell_i,t)}\right] + \boldsymbol{\Psi}_i \right\}^T \delta \boldsymbol{\psi}_i(\ell_i,t) \right\rangle \right\} dt = 0
\end{aligned} \tag{33}$$

Then, invoking the arbitrariness of the virtual displacements, we obtain the system Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q} \tag{34a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\mathbf{u}}_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right) - \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i} = \mathbf{f}_i, \quad i = 1, 2, \ldots, N; \; 0 < x_i < \ell_i \tag{34b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{\boldsymbol{\psi}}_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right) - \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i} = \mathbf{m}_i, \quad i = 1, 2, \ldots, N; \; 0 < x_i < \ell_i \tag{34c}$$

where $\mathbf{u}_i$ and $\boldsymbol{\psi}_i$ must be such that the equations

$$\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'}\right)^T \delta \mathbf{u}_i \Bigg|_{x_i=0} = 0, \quad \left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'}\right)^T \delta \boldsymbol{\psi}_i \Bigg|_{x_i=0} = 0, \quad i = 1, 2, \ldots, N \tag{35a,b}$$

$$\left(\frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'} \Bigg|_{x_i=\ell_i} + \mathbf{U}_i - \left\{ \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}_i(\ell_i,t)}\right] - \frac{\partial L}{\partial \mathbf{u}_i(\ell_i,t)} \right\} \right)^T \delta \mathbf{u}_i(\ell_i,t) = 0, \quad i = 1, 2, \ldots, N-1 \tag{35c}$$

$$\left(\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'} \Bigg|_{x_i=\ell_i} + \boldsymbol{\Psi}_i - \left\{ \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(\ell_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(\ell_i,t)} \right\} \right)^T \delta \boldsymbol{\psi}_i(\ell_i,t) = 0, \quad i = 1, 2, \ldots, N-1 \tag{35d}$$

$$\frac{\partial \hat{L}_N}{\partial \mathbf{u}_N'} \delta \mathbf{u}_N(x_N,t) \Bigg|_{x_N=\ell_N} = 0 \tag{35e}$$

$$\frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}_N'} \delta \boldsymbol{\psi}_N(x_N,t) \Bigg|_{x_N=\ell_N} = 0 \tag{35f}$$

must be satisfied. Recalling that the body axes $x_i y_i z_i$ are embedded in the body at $x_i = 0$, we conclude that satisfaction of Eqs. (35) is guaranteed if

$$\mathbf{u}_i(0,t) = \mathbf{0}, \; \boldsymbol{\psi}_i(0,t) = \mathbf{0}, \quad i = 1, 2, \ldots, N \tag{36a,b}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\mathbf{u}}_i(\ell_i,t)}\right] - \frac{\partial L}{\partial \mathbf{u}_i(\ell_i,t)} = \frac{\partial \hat{L}_i}{\partial \mathbf{u}_i'} \Bigg|_{x_i=\ell_i} + \mathbf{U}_i, \quad i = 1, 2, \ldots, N-1 \tag{36c}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{\psi}}_i(\ell_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(\ell_i,t)} = \frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}_i'} \Bigg|_{x_i=\ell_i} + \boldsymbol{\Psi}_i, \quad i = 1, 2, \ldots, N-1 \tag{36d}$$

$$\frac{\partial \hat{L}_N}{\partial \mathbf{u}_N'} \Bigg|_{x_N=\ell_N} = \mathbf{0}, \quad \frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}_N'} \Bigg|_{x_N=\ell_N} = \mathbf{0} \tag{36e,f}$$

Equations (34a) represent ordinary differential equations for the rigid-body motion and Eqs. (34b) and (34c) represent partial differential equations for the elastic motions. Moreover, Eqs. (36) are recognized as the boundary conditions accompanying the partial differential equations. Although Eqs. (34a), Eqs. (34b), (36a), (36c) and (36e) on the one hand and Eqs. (34c), (36b), (36d) and (36f) on the other hand have the appearance of independent sets of equations, they are in fact simultaneous. They constitute a hybrid (ordinary and partial) set of differential equations governing the motion of the multibody system shown in Fig. 1.

4. Lagrange's Equations for Flexible Multibody Systems in Terms of Quasi-Coordinates

Equations (34) seem very simple, but they are not. The reason for this is that the kinetic energy is only an implicit function of $\mathbf{q}$ and $\dot{\mathbf{q}}$ and not an explicit one. The kinetic energy is an explicit function of $\mathbf{V}_{oi}$ and $\boldsymbol{\omega}_i$, which are commonly known as derivatives of quasi-coordinates (Ref. 33). Actually, the kinetic energy is an explicit function of $\boldsymbol{\Omega}_i$, but $\boldsymbol{\Omega}_i$ is related directly to $\boldsymbol{\omega}_i$, as can be seen from Eq. (13). As shown in Ref. 32 for a single flexible body, hybrid Lagrange's equations of motion in terms of quasi-coordinates are considerably simpler than the standard Lagrange's equations. We propose to show in this paper that the same is true for multibodies.

Recalling definition (29) of the rigid-body displacement vector $\mathbf{q}(t)$, we can rewrite Eq. (34a) in the more detailed form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}_{o1}}\right) - \frac{\partial L}{\partial \mathbf{R}_{o1}} = \mathbf{F}_1 \qquad (37a)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_i}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}_i} = \mathbf{M}_i, \; i = 1, 2, \ldots, N \qquad (37b)$$

The vectors $\mathbf{R}_{o1}$, $\dot{\mathbf{R}}_{o1}$ and $\mathbf{F}_1$ are in terms of components along the inertial axes XYZ. Moreover, the components of the symbolic vector $\boldsymbol{\theta}_i$ represent rotations about nonorthogonal axes leading from $x_i'y_i'z_i'$ to $x_iy_iz_i$ and the components of $\mathbf{M}_i$ are associated moments. An example of such rotations are Euler's angles (Ref. 33). As the quasi-velocity counterpart of the generalized velocity vector $\dot{\mathbf{q}}(t)$, we choose

$$\mathbf{w} = [\mathbf{V}_{o1}^T \;\; \boldsymbol{\omega}_1^T \;\; \boldsymbol{\omega}_2^T \cdots \boldsymbol{\omega}_N^T]^T \qquad (38)$$

and we note that $\mathbf{w}$ does not equal the time derivative $\dot{\mathbf{q}}$ of the displacements. We also note that every three-dimensional vector entering into $\mathbf{w}$ is in terms of the corresponding orthogonal body axes $x_iy_iz_i$. The relation between the velocity vector $\mathbf{V}_{o1}$ in terms of body axes and the velocity vector $\dot{\mathbf{R}}_{o1}$ in terms of inertial axes is simply

$$\mathbf{V}_{o1} = C_1 \dot{\mathbf{R}}_{o1} \qquad (39)$$

where C_1 is the matrix of direction cosines first introduced in Sec. 2, and that between the velocity vector $\boldsymbol{\omega}_i$ in terms of body axes and the Eulerian-type velocities $\dot{\boldsymbol{\theta}}_i$ can be written as

$$\boldsymbol{\omega}_i = D_i \dot{\boldsymbol{\theta}}_i, \quad i = 1, 2, \ldots, N \qquad (40)$$

where D_i is a given transformation matrix (Ref. 33). Equations (39) and (40) and their reciprocal relations can be expressed in the compact form

$$\mathbf{w} = A^T(\mathbf{q})\dot{\mathbf{q}}, \quad \dot{\mathbf{q}} = B(\mathbf{q})\mathbf{w} \qquad (41a,b)$$

where

$$A = \text{block-diag}[C_1^T \; D_1^T \; D_2^T \cdots D_N^T] \qquad (42a)$$

$$B = \text{block-diag}[C_1^T \; D_1^{-1} \; D_2^{-1} \cdots D_N^{-1}] \qquad (42b)$$

Equations (37) postulate a Lagrangian in terms of generalized coordinates and velocities, Eq. (30), when in fact the Lagrangian defined by Eqs. (15), (16), (17), (25) and (26) is in terms of generalized coordinates and quasi-velocities. To distinguish between the two forms, we define

$$L^* = L^*[\mathbf{q}, \mathbf{w}, \mathbf{u}_i, \mathbf{u}_i', \dot{\mathbf{u}}_i, \boldsymbol{\psi}_i, \boldsymbol{\psi}_i', \dot{\boldsymbol{\psi}}_i, \mathbf{u}_i(\ell_i, t), \dot{\mathbf{u}}_i(\ell_i, t), \boldsymbol{\psi}_i(\ell_i, t), \dot{\boldsymbol{\psi}}_i(\ell_i, t)] \qquad (43)$$

We propose to obtain Lagrange's equations in terms of quasi-coordinates by transforming Eqs. (37). To this end, we use the chain rule for derivatives with respect to vectors and consider Eq. (39) to obtain

$$\frac{\partial L}{\partial \dot{\mathbf{R}}_{o1}} = \frac{\partial (C_1 \dot{\mathbf{R}}_{o1})^T}{\partial \dot{\mathbf{R}}_{o1}} \frac{\partial L^*}{\partial \mathbf{V}_{o1}} = C_1^T \frac{\partial L^*}{\partial \mathbf{V}_{o1}} \qquad (44a)$$

$$\frac{\partial L}{\partial \mathbf{R}_{o1}} = \frac{\partial L^*}{\partial \mathbf{R}_{o1}} \qquad (44b)$$

But, it is shown in the Appendix that the matrix of direction cosines C_i and quasi-velocity vector $\boldsymbol{\omega}_i$ satisfy the relation

$$\dot{C}_i = \tilde{\omega}_i^T C_i \qquad (45)$$

so that differentiating Eq. (44a) with respect to time, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}_{o1}}\right) = \frac{d}{dt}\left(C_1^T \frac{\partial L^*}{\partial \mathbf{V}_{o1}}\right) = C_1^T \tilde{\omega}_1 \frac{\partial L^*}{\partial \mathbf{V}_{o1}} + C_1^T \frac{d}{dt}\left(\frac{\partial L^*}{\partial \mathbf{V}_{o1}}\right) \qquad (46)$$

Then, inserting Eqs. (44b) and (46) into Eq. (37a) and premultiplying by C_1, we obtain the translational Lagrange's equations in terms of quasi-coordinates

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \mathbf{V}_{o1}}\right) + \tilde{\omega}_1 \frac{\partial L^*}{\partial \mathbf{V}_{o1}} - C_1 \frac{\partial L^*}{\partial \mathbf{R}_{o1}} = \mathbf{F}_1^* \qquad (47)$$

where

$$\mathbf{F}_1^* = C_1 \mathbf{F}_1 \tag{48}$$

is the resultant force acting on body 1 in terms of body-axes components.

As far as the rotational motion is concerned, we consider first the equations for body 1. Using the chain rule for derivatives with respect to vectors once again and using Eq. (40), we obtain

$$\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_1} = \frac{\partial (D_1 \dot{\boldsymbol{\theta}}_1)^T}{\partial \dot{\boldsymbol{\theta}}_1} \frac{\partial L^*}{\partial \boldsymbol{\omega}_1} = D_1^T \frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \tag{49a}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} + \frac{\partial (C_1 \dot{\mathbf{R}}_{o1})^T}{\partial \boldsymbol{\theta}_1} \frac{\partial L^*}{\partial \mathbf{V}_{o1}} + \frac{\partial (D_1 \dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1} \frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \tag{49b}$$

Moreover, Eq. (A-29) from the Appendix, with a replaced by $\dot{\mathbf{R}}_{o1}$ yields the relation

$$\frac{\partial (C_1 \dot{\mathbf{R}}_{o1})^T}{\partial \boldsymbol{\theta}_1} = -D_1^T \tilde{V}_{o1} \tag{50}$$

and Eq. (A-27) shows that

$$\dot{D}_1^T = \frac{\partial (D_1 \dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1} + D_1^T \tilde{\omega}_1 \tag{51}$$

Hence, using Eqs. (49)-(51), we can write

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}_1}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}_1} &= \left[\dot{D}_1^T - \frac{\partial (D_1 \dot{\boldsymbol{\theta}}_1)^T}{\partial \boldsymbol{\theta}_1}\right] \frac{\partial L^*}{\partial \boldsymbol{\omega}_1} \\ &+ D_1^T \frac{d}{dt}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + D_1^T \tilde{V}_{o1} \frac{\partial L^*}{\partial \mathbf{V}_{o1}} - \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} \\ &= D_1^T \left[\frac{d}{dt}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + \tilde{V}_{o1} \frac{\partial L^*}{\partial \mathbf{V}_{o1}} + \tilde{\omega}_1 \frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right] - \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} \end{aligned} \tag{52}$$

Inserting Eq. (52) into Eq. (37b) and premultiplying the result by D_1^{-T}, where the superscript $-T$ denotes the inverse of the transposed matrix, we obtain the rotational Lagrange's equations for the first body in terms of quasi-coordinates

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_1}\right) + \tilde{V}_{o1} \frac{\partial L^*}{\partial \mathbf{V}_{o1}} + \tilde{\omega}_1 \frac{\partial L^*}{\partial \boldsymbol{\omega}_1} - D_1^{-T} \frac{\partial L^*}{\partial \boldsymbol{\theta}_1} = \mathbf{M}_1^* \tag{53}$$

where

$$\mathbf{M}_1^* = D_1^{-T} \mathbf{M}_1 \tag{54}$$

is the resultant torque acting on body 1 in terms of body-axes components. The equations of motion for the remaining bodies can be obtained in the same manner, except that $\mathbf{V}_{oi}$ $(i = 2, 3, \ldots, N)$ are not independent, as can be concluded from Eqs. (10). Hence, from Eq. (53), the remaining rotational Lagrange's equations in terms of quasi-coordinates are

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \boldsymbol{\omega}_i}\right) + \tilde{\omega}_i \frac{\partial L^*}{\partial \boldsymbol{\omega}_i} - D_i^{-T} \frac{\partial L^*}{\partial \boldsymbol{\theta}_i} = \mathbf{M}_i^*, \quad i = 2, 3, \ldots, N \tag{55}$$

where

$$\mathbf{M}_i^* = D_i^{-T} \mathbf{M}_i, \quad i = 2, 3, \ldots, N \tag{56}$$

Equations (47), (53) and (55) can be cast in a single matrix equation. Indeed, recalling Eqs. (29), (38), (41b) and (42b), the rigid-body Lagrange's equations of motion in terms of quasi-coordinates can be written in the compact form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{w}}\right) + H \frac{\partial L}{\partial \mathbf{w}} - B^T \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}^* \tag{57}$$

where the asterisk in L^* was dropped for convenience. Moreover,

$$H = \begin{bmatrix} \tilde{\omega}_1 & 0 & 0 & \cdots & 0 \\ \tilde{V}_{o1} & \tilde{\omega}_1 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{\omega}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{\omega}_N \end{bmatrix} \tag{58}$$

and

$$\mathbf{Q}^* = B^T \mathbf{Q} = [\mathbf{F}_1^{*T} \ \mathbf{M}_1^{*T} \ \mathbf{M}_2^{*T} \cdots \mathbf{M}_N^{*T}]^T \tag{59}$$

The hybrid set of equations of motion is completed by adjoining to Eq. (57) the partial differential equations for the elastic motions, Eqs. (34b) and (34c), and the associated boundary conditions, Eqs. (36).

5. Explicit Hybrid Equations of Motion for Flexible Multibody Systems

Using Eqs. (16) and (17), we can write the kinetic energy in the form

$$\begin{aligned} T &= \sum_{i=1}^{N} \int_0^{\ell_i} \hat{T}_i \, dx_i \\ &= \frac{1}{2} \sum_{i=1}^{N} \Big[m_i \mathbf{V}_{oi}^T \mathbf{V}_{oi} + \mathbf{\Omega}_{ri}^T J_{ii} \mathbf{\Omega}_{ri} + \int_0^{\ell_i} \rho_i \dot{\mathbf{u}}_i^T \dot{\mathbf{u}}_i \, dx_i \\ &+ \int_0^{\ell_i} \dot{\boldsymbol{\psi}}_i^T \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i \, dx_i + 2\mathbf{V}_{oi}^T (\tilde{S}_i^T \mathbf{\Omega}_{ri} + \int_0^{\ell_i} \rho_i \dot{\mathbf{u}}_i \, dx_i) \\ &+ 2\mathbf{\Omega}_{ri}^T \int_0^{\ell_i} \left(\tilde{\hat{S}}_i \dot{\mathbf{u}}_i + \hat{J}_{ci} \dot{\boldsymbol{\psi}}_i \right) dx_i \Big] \end{aligned} \tag{60}$$

and we observe that T does not depend explicitly on the quasi-velocities $\mathbf{V}_{o1}$ and $\boldsymbol{\omega}_i$ $(i = 1, 2, \ldots, N)$, but on $\mathbf{V}_{oi}$ and $\mathbf{\Omega}_{ri}$ $(i = 1, 2, \ldots, N)$. To resolve this inconvenience, we make use of the discrete step function γ_i, defined by

$$\gamma_i = \begin{cases} 0, & \text{if } i = -1, -2, -3, \ldots \\ 1, & \text{if } i = 0, 1, 2, 3, \ldots \end{cases} \tag{61}$$

and then make repeated use of Eqs. (10) and (13) to establish the relations

$$\Omega_{ri} = \sum_{j=1}^{N} C^*_{ij} \left[\gamma_{i-j}\omega_j + \gamma_{i-j-1}\Omega_{ej}(\ell_j, t)\right] \tag{62a}$$

$$\begin{aligned} V_{oi} =& C^*_{i1}V_{o1} + \sum_{j=1}^{N} \gamma_{i-j-1} C^*_{ij} \left[\tilde{u}^T_{cj}\Omega_{rj} + v_j(\ell_j, t)\right] \\ =& C^*_{i1}V_{o1} + \sum_{j=1}^{N} \{\Gamma_{ij}\omega_j + \Gamma_{i,j+1}\Omega_{ej}(\ell_j, t) \\ & + \gamma_{i-j-1}C^*_{ij}v_j(\ell_j, t)\} \end{aligned} \tag{62b}$$

$$\dot{\Omega}_{ri} = \sum_{j=1}^{N} C^*_{ij} \left[\gamma_{i-j}\dot{\omega}_j + \gamma_{i-j-1}\dot{\Omega}_{ej}(\ell_j, t)\right] + d_{\Omega i} \tag{63a}$$

$$\begin{aligned} \dot{V}_{oi} =& C^*_{i1}\dot{V}_{o1} + \sum_{j=1}^{N} \left\{\Gamma_{ij}\dot{\omega}_j + \Gamma_{i,j+1}\dot{\Omega}_{ej}(\ell_j, t) \right. \\ & \left. + \gamma_{i-j-1}C^*_{ij}\dot{v}_j(\ell_j, t)\right\} + d_{Vi} \end{aligned} \tag{63b}$$

$$\delta\Theta^*_{ri} = \sum_{j=1}^{N} C^*_{ij} \left[\gamma_{i-j}\delta\theta^*_j + \gamma_{i-j-1}\delta\psi_j(\ell_j, t)\right] \tag{63c}$$

$$\begin{aligned} \delta R^*_{oi} =& C^*_{i1}\delta R^*_{o1} + \sum_{j=1}^{N} \left[\Gamma_{ij}\delta\theta^*_j + \Gamma_{i,j+1}\delta\psi_j(\ell_j, t) \right. \\ & \left. + \gamma_{i-j-1}C^*_{ij}\delta u_j(\ell_j, t)\right] \end{aligned} \tag{63d}$$

in which C^*_{ij} is simply the matrix of direction cosines of axes $x_iy_iz_i$ with respect to axes $x_jy_jz_j$, defined for all indices i, j between 1 and N, and consequently

$$C^*_{ij} = \prod_{k=j+1}^{i} C^*_k, \quad 1 \le j < i \le N \tag{64a}$$

$$C^*_{ii} = I, \quad 1 \le i \le N \tag{64b}$$

$$(C^*_{ij})^T = C^*_{ji}, \quad C^*_{ik}C^*_{kj} = C^*_{ij}, \quad 1 \le i,j,k \le N \tag{64c,d}$$

The other quantities appearing explicitly or implicitly in Eqs. (62) and (63) are given by

$$u_{ci} = [\ell_i \; u_{yi}(\ell_i, t) \; u_{zi}(\ell_i, t)]^T \tag{65a}$$

$$\Gamma_{ij} = \sum_{k=j}^{i-1} C^*_{ik}\tilde{u}^T_{ck}C^*_{kj} \tag{65b}$$

$$d_{\Omega i} = \sum_{j=1}^{N} \dot{C}^*_{ij} \left[\gamma_{i-j}\omega_j + \gamma_{i-j-1}\Omega_{ej}(\ell_j, t)\right] \tag{65c}$$

$$\begin{aligned} d_{Vi} =& \dot{C}^*_{i1}V_{o1} + \sum_{j=1}^{i-1} \left\{\dot{C}^*_{ij}[\tilde{u}^T_{cj}\Omega_{rj} + v_j(\ell_j, t)] \right. \\ & \left. + C^*_{ij}\left[\tilde{\Omega}_{rj}v_j(\ell_j, t) + \tilde{u}^T_{cj}d_{\Omega j}\right]\right\} \end{aligned} \tag{65d}$$

$$\dot{C}^*_{ij} = \left(\widetilde{-\Omega_{ri} + C^*_{ij}\Omega_{rj}}\right) C^*_{ij} \tag{65e}$$

We also note that C^*_{ij} depends only on θ_k, for $\min(i,j) < k \le \max(i,j)$, and on $\psi_k(\ell_k, t)$, for values of k satisfying $\min(i,j) \le k < \max(i,j)$. Hence, using Eqs. (A-29) and (A-30), we can derive the relations

$$\frac{\partial(C^*_{ij}a)^T}{\partial\theta_k} = (\gamma_{j-k} - \gamma_{i-k})D^T_k C^*_{kj}\tilde{a}C^*_{ji} \tag{66a}$$

provided a does not depend on θ_k, and

$$\frac{\partial(C^*_{ij}a)^T}{\partial\psi_k(\ell_k, t)} = (\gamma_{j-k-1} - \gamma_{i-k-1})E_k(\ell_k)C^*_{kj}\tilde{a}C^*_{ji} \tag{66b}$$

provided a does not depend on $\psi_k(\ell_k, t)$. Some other relations that will prove useful are as follows:

$$\frac{\partial\Omega^T_{ri}}{\partial R_{o1}} = 0 \tag{67a}$$

$$\begin{aligned} \frac{\partial\Omega^T_{ri}}{\partial\theta_k} =& D^T_k \sum_{j=1}^{N}(\gamma_{j-k} - \gamma_{i-k})C^*_{kj}[\gamma_{i-j}\tilde{\omega}_j \\ & + \gamma_{i-j-1}\tilde{\Omega}_{ej}(\ell_j, t)]C^*_{ji} \end{aligned} \tag{67b}$$

$$\frac{\partial\Omega^T_{ri}}{\partial u_k(\ell_k, t)} = 0 \tag{67c}$$

$$\begin{aligned} \frac{\partial\Omega^T_{ri}}{\partial\psi_k(\ell_k, t)} =& E_k(\ell_k, t) \sum_{j=1}^{N}(\gamma_{j-k-1} - \gamma_{i-k-1})C^*_{kj}[\gamma_{i-j}\tilde{\omega}_j \\ & + \gamma_{i-j-1}\tilde{\Omega}_{ej}(\ell_j, t)]C^*_{ji} \end{aligned} \tag{67d}$$

$$\frac{\partial\Omega^T_{ri}}{\partial V_{o1}} = 0, \quad \frac{\partial\Omega^T_{ri}}{\partial\omega_k} = \gamma_{i-k}C^*_{ik} \tag{67e,f}$$

$$\frac{\partial\Omega^T_{ri}}{\partial v_k(\ell_k, t)} = 0, \quad \frac{\partial\Omega^T_{ri}}{\partial\Omega_{ek}(\ell_k, t)} = \gamma_{i-k-1}C^*_{ik} \tag{67g,h}$$

$$\frac{\partial V^T_{oi}}{\partial R_{o1}} = 0 \tag{68a}$$

$$\begin{aligned} \frac{\partial V^T_{oi}}{\partial\theta_k} =& D^T_k \left[(\gamma_{1-k} - \gamma_{i-k})C^*_{k1}\tilde{V}_{o1}C^*_{1i} \right. \\ & + \sum_{j=1}^{i-1}\left\{(\gamma_{j-k} - \gamma_{i-k})C^*_{kj}\left[\widetilde{\tilde{u}^T_{cj}\Omega_{rj} + v_j(\ell_j, t)}\right] \right. \\ & \left.\left. + D^{-T}_k \frac{\partial\Omega^T_{rj}}{\partial\theta_k}\tilde{u}_{cj}\right\}C^*_{ji}\right] \end{aligned} \tag{68b}$$

$$\frac{\partial V^T_{oi}}{\partial u_k(\ell_k, t)} = -\sum_{j=1}^{i-1}\tilde{\Omega}_{rj}C^*_{ji} \tag{68c}$$

$$\begin{aligned} \frac{\partial V^T_{oi}}{\partial\psi_k(\ell_k, t)} =& E_k(\ell_k, t)\left[-\gamma_{i-k-1}C^*_{k1}\tilde{V}_{o1}C^*_{1i} \right. \\ & + \sum_{j=1}^{i-1}\left\{(\gamma_{j-k-1} - \gamma_{i-k-1})C^*_{kj}\left[\widetilde{\tilde{u}^T_{cj}\Omega_{rj} + v_j(\ell_j, t)}\right]\right. \end{aligned}$$

$$+ E_k^{-1}(\ell_k,t)\frac{\partial \Omega_{rj}^T}{\partial \psi_k(\ell_k,t)}\tilde{\mathbf{u}}_{cj}\Big\} C_{ji}^*\Big] \tag{68d}$$

$$\frac{\partial \mathbf{V}_{oi}^T}{\partial \mathbf{V}_{o1}} = C_{1i}^*,\ \frac{\partial \mathbf{V}_{oi}^T}{\partial \boldsymbol{\omega}_k} = \Gamma_{ik}^T \tag{68e,f}$$

$$\frac{\partial \mathbf{V}_{oi}^T}{\partial \mathbf{v}_k(\ell_k,t)} = \gamma_{i-k-1}C_{ki}^*,\ \frac{\partial \mathbf{V}_{oi}^T}{\partial \boldsymbol{\Omega}_{ok}(\ell_k,t)} = \Gamma_{i,k+1}^T \tag{68g,h}$$

Then, using the chain rule for vectors when needed, we obtain the momenta

$$\mathbf{p}_{Vo1} = \frac{\partial L}{\partial \mathbf{V}_{o1}} = \sum_{i=1}^{N} C_{1i}^* \frac{\partial L_i}{\partial \mathbf{V}_{oi}} \tag{69a}$$

$$\mathbf{p}_{\omega j} = \frac{\partial L}{\partial \boldsymbol{\omega}_j} = \sum_{i=1}^{N}\left(\Gamma_{ij}^T\frac{\partial L_i}{\partial \mathbf{V}_{oi}} + \gamma_{i-j}C_{ij}^*\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}}\right) \tag{69b}$$

where

$$\frac{\partial L_i}{\partial \mathbf{V}_{oi}} = m_i\mathbf{V}_{oi} + \tilde{S}_i^T\boldsymbol{\Omega}_{ri} + \int_0^{\ell_i}\rho_i\dot{\mathbf{u}}_i dx_i \tag{70a}$$

$$\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}} = J_{ti}\boldsymbol{\Omega}_{ri} + \tilde{S}_i\mathbf{V}_{oi} + \int_0^{\ell_i}\left(\tilde{\hat{S}}_i\dot{\mathbf{u}}_i + \hat{J}_{ci}\dot{\psi}\right)dx_i \tag{70b}$$

For future reference, we also indicate that

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \mathbf{V}_{oi}}\right) = m_i\dot{\mathbf{V}}_{oi} + \tilde{S}_i^T\dot{\boldsymbol{\Omega}}_{ri} + \int_0^{\ell_i}\rho_i\ddot{\mathbf{u}}_i dx_i + \mathbf{d}_{tVi} \tag{71a}$$

$$\frac{d}{dt}\left(\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}}\right) = J_{ti}\dot{\boldsymbol{\Omega}}_{ri} + \tilde{S}_i\dot{\mathbf{V}}_{oi} + \int_0^{\ell_i}\left(\tilde{\hat{S}}_i\ddot{\mathbf{u}}_i + \hat{J}_{ci}\ddot{\psi}_i\right)dx_i + \mathbf{d}_{t\Omega i} \tag{71b}$$

where

$$\mathbf{d}_{tVi} = \tilde{\Omega}_{ri}\int_0^{\ell_i}\rho_i\dot{\mathbf{u}}_i dx_i \tag{72a}$$

$$\mathbf{d}_{t\Omega i} = \dot{J}_{ti}\boldsymbol{\Omega}_{ri} - \tilde{V}_{oi}\int_0^{\ell_i}\rho_i\dot{\mathbf{u}}_i dx_i \tag{72b}$$

$$\dot{J}_{ti} = \int_0^{\ell_i}\rho_i\left\{\tilde{\dot{\mathbf{u}}}_i(x_i\tilde{e}_1 + \tilde{\mathbf{u}}_i)^T + (x_i\tilde{e}_1 + \tilde{\mathbf{u}}_i)\tilde{\dot{\mathbf{u}}}_i^T\right\}dx_i \tag{72c}$$

and

$$\begin{aligned}\dot{\mathbf{p}}_{Vo1} = &\left(\sum_{i=1}^{N} m_i\right)\dot{\mathbf{V}}_{o1} + \sum_{j=1}^{N}\left[\sum_{i=1}^{N}\left(m_iC_{1i}^*\Gamma_{ij} + \gamma_{i-j}C_{1i}^*\tilde{S}_i^TC_{ij}^*\right)\right]\dot{\boldsymbol{\omega}}_j \\ &+ \sum_{j=1}^{N}\left[\sum_{i=1}^{N}\gamma_{i-j-1}m_iC_{1j}^*\right]\dot{\mathbf{v}}_j(\ell_j,t) \\ &+ \sum_{j=1}^{N}\left[\sum_{i=1}^{N}\left(m_iC_{1i}^*\Gamma_{i,j+1} + \gamma_{i-j-1}C_{1i}^*\tilde{S}_i^TC_{ij}^*\right)\right]\dot{\boldsymbol{\Omega}}_{oj}(\ell_j,t) \\ &+ \sum_{i=1}^{N}\left(C_{1i}^*\int_0^{\ell_i}\rho_i\ddot{\mathbf{u}}_i dx_i\right) \\ &+ \sum_{i=1}^{N}\left[\dot{C}_{1i}^*\frac{\partial L_i}{\partial \mathbf{V}_{oi}} + C_{1i}^*\left(m_i\mathbf{d}_{Vi} + \tilde{S}_i^T\mathbf{d}_{\Omega i} + \mathbf{d}_{tVi}\right)\right]\end{aligned} \tag{73a}$$

$$\begin{aligned}\dot{\mathbf{p}}_{\omega j} = &\left[\sum_{i=1}^{N}\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)C_{i1}^*\right]\dot{\mathbf{V}}_{o1} \\ &+ \sum_{k=1}^{N}\left\{\sum_{i=1}^{N}\left[\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)\Gamma_{ik} + \gamma_{i-k}\left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)C_{ik}^*\right]\right\}\dot{\boldsymbol{\omega}}_k \\ &+ \sum_{k=1}^{N}\left[\sum_{i=1}^{N}\gamma_{i-k-1}\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)C_{ik}^*\right]\dot{\mathbf{v}}_k(\ell_k,t) \\ &+ \sum_{k=1}^{N}\left\{\sum_{i=1}^{N}\left[\left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)\Gamma_{i,k+1} + \gamma_{i-k-1}\left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)C_{ik}^*\right]\right\}\dot{\boldsymbol{\Omega}}_{ok}(\ell_k,t) \\ &+ \sum_{i=1}^{N}\left[\int_0^{\ell_i}\left(\rho_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{\hat{S}}_i\right)\ddot{\mathbf{u}}_i dx_i\right] \\ &+ \sum_{i=1}^{N}\left[\gamma_{i-j}C_{ij}^*\int_0^{\ell_i}\hat{J}_{ci}\ddot{\psi}_i dx_i\right] + \sum_{i=1}^{N}\left[\dot{\Gamma}_{ij}^T\frac{\partial L_i}{\partial \mathbf{V}_{oi}} + \gamma_{i-j}\dot{C}_{ij}^*\frac{\partial L_i}{\partial \boldsymbol{\Omega}_{ri}} + \Gamma_{ij}^T\mathbf{d}_{tVi} + \gamma_{i-j}C_{ij}^*\mathbf{d}_{t\Omega i} \right. \\ &\left. + \left(m_i\Gamma_{ij}^T + \gamma_{i-j}C_{ij}^*\tilde{S}_i\right)\mathbf{d}_{Vi} + \left(\Gamma_{ij}^T\tilde{S}_i^T + \gamma_{i-j}C_{ij}^*J_{ti}\right)\mathbf{d}_{\Omega i}\right]\end{aligned} \tag{73b}$$

We also define equivalent forces and moments

$$\mathbf{F}_{p1}^* = C_1\frac{\partial L}{\partial \mathbf{R}_{o1}} = 0 \tag{74a}$$

$$\mathbf{M}_{pj}^* = D_j^{-T}\frac{\partial L}{\partial \boldsymbol{\theta}_j} = D_j^{-T}\sum_{i=1}^{N}\left(\frac{\partial \mathbf{V}_{oi}^T}{\partial \boldsymbol{\theta}_j}\frac{\partial L}{\partial \mathbf{V}_{oi}} + \frac{\partial \boldsymbol{\Omega}_{ri}^T}{\partial \boldsymbol{\theta}_j}\frac{\partial L}{\partial \boldsymbol{\Omega}_{ri}}\right) \tag{74b}$$

and the remaining pertinent terms

$$\frac{\partial L}{\partial \mathbf{u}_j(\ell_j,t)} = \sum_{i=1}^{N}\frac{\partial \mathbf{V}_{oi}^T}{\partial \mathbf{u}_j(\ell_j,t)}\frac{\partial L_i}{\partial \mathbf{V}_{oi}} \tag{75a}$$

$$\frac{\partial L}{\partial \mathbf{v}_j(\ell_j,t)} = \sum_{i=1}^{N} \frac{\partial \mathbf{V}_{oi}^T}{\partial \mathbf{v}_j(\ell_j,t)} \frac{\partial L_i}{\partial \mathbf{V}_{oi}} \tag{75b}$$

$$\frac{\partial L}{\partial \boldsymbol{\psi}_j(\ell_j,t)} = \sum_{i=1}^{N} \left(\frac{\partial \mathbf{V}_{oi}^T}{\partial \boldsymbol{\psi}_j(\ell_j,t)} \frac{\partial L}{\partial \mathbf{V}_{oi}} + \frac{\partial \Omega_{ri}^T}{\partial \boldsymbol{\psi}_j(\ell_j,t)} \frac{\partial L}{\partial \Omega_{ri}} \right) \tag{75c}$$

$$\frac{\partial L}{\partial \Omega_{ej}(\ell_j,t)} = \sum_{i=1}^{N} \left(\frac{\partial \mathbf{V}_{oi}^T}{\partial \Omega_{ej}(\ell_j,t)} \frac{\partial L}{\partial \mathbf{V}_{oi}} + \frac{\partial \Omega_{ri}^T}{\partial \Omega_{ej}(\ell_j,t)} \frac{\partial L}{\partial \Omega_{ri}} \right) \tag{75d}$$

in which some of the partial derivatives are given by Eqs. (67).

Finally, adjoining the kinematic relations expressed by Eqs. (9), (11), (39) and (40) and inserting Eqs. (68)-(70) into Eqs. (34b), (34c) and (57), we obtain the hybrid state equations in terms of quasi-coordinates

$$\dot{\mathbf{R}}_{o1} = C_1^T \mathbf{V}_{o1}, \quad \dot{\theta}_i = D_i^{-1}\omega_i, \quad i = 1,2,\ldots,N \tag{76a,b}$$

$$\dot{\mathbf{u}}_i(x_i,t) = \mathbf{v}_i(x_i,t),\ \dot{\boldsymbol{\psi}}_i(x_i,t) = \Omega_{ei}(x_i,t), \quad i = 1,2,\ldots,N \tag{76c,d}$$

$$\dot{\mathbf{p}}_{Vo1} = -\tilde{\omega}_1 \mathbf{p}_{Vo1} + \mathbf{F}_1^* \tag{76e}$$

$$\dot{\mathbf{p}}_{\omega 1} = -\tilde{V}_{o1}\mathbf{p}_{Vo1} - \tilde{\omega}_1 \mathbf{p}_{\omega 1} + \mathbf{M}_{p1}^* + \mathbf{M}_{o1}^* \tag{76f}$$

$$\dot{\mathbf{p}}_{\omega i} = -\tilde{\omega}_i \mathbf{p}_{\omega i} + \mathbf{M}_{pi}^* + \mathbf{M}_{oi}^*, \quad i = 2,3,\ldots,N \tag{76g}$$

$$\begin{aligned} \rho_i[\dot{v}_{yi} + \dot{V}_{oyi} + x_i\dot{\Omega}_{rzi} - u_{zi}\dot{\Omega}_{rxi} - 2\Omega_{rxi}v_{zi} + \Omega_{rzi}V_{oxi} \\ - \Omega_{rxi}V_{ozi} + x_i\Omega_{rxi}\Omega_{ryi} - (\Omega_{rxi}^2 + \Omega_{rzi}^2)u_{yi} \\ + \Omega_{ryi}\Omega_{rzi}u_{zi}] - [k_{yi}G_iA_i(u'_{yi} - \psi_{zi})]' = f_{yi} \end{aligned} \tag{76h}$$

$$\begin{aligned} \rho_i[\dot{v}_{zi} + \dot{V}_{ozi} - x_i\dot{\Omega}_{ryi} + u_{yi}\dot{\Omega}_{rxi} + 2\Omega_{rxi}v_{yi} + \Omega_{rxi}V_{oyi} \\ - \Omega_{ryi}V_{oxi} + x_i\Omega_{rxi}\Omega_{rzi} - (\Omega_{rxi}^2 + \Omega_{ryi}^2)u_{zi} \\ + \Omega_{ryi}\Omega_{rzi}u_{yi}] - [k_{zi}G_iA_i(u'_{zi} + \psi_{yi})]' = f_{zi} \end{aligned} \tag{76i}$$

$$\hat{J}_{xixi}(\dot{\Omega}_{exi} + \dot{\Omega}_{rxi}) - (k_{xi}G_iI_{xi}\psi'_{xi})' = m_{xi} \tag{76j}$$

$$\hat{J}_{yiyi}(\dot{\Omega}_{eyi} + \dot{\Omega}_{ryi}) + k_{zi}G_iA_i(u'_{zi} + \psi_{yi}) - (E_iI_{yi}\psi'_{yi})' = m_{yi} \tag{76k}$$

$$\hat{J}_{zizi}(\dot{\Omega}_{ezi} + \dot{\Omega}_{rzi}) - k_{yi}G_iA_i(u'_{yi} - \psi_{zi}) - (E_iI_{zi}\psi'_{zi})' = m_{zi} \tag{76k}$$

The associated boundary conditions, Eqs. (36), are given by

$$\mathbf{u}_i(0,t) = \mathbf{0},\ \boldsymbol{\psi}_i(0,t) = \mathbf{0}, \quad i = 1,2,\ldots,N \tag{77a,b}$$

$$\left.\frac{\partial \hat{L}_i}{\partial \mathbf{u}'_i}\right|_{x_i=\ell_i} - \left\{ \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \mathbf{v}_i(\ell_i,t)}\right] - \frac{\partial L}{\partial \mathbf{u}_i(\ell_i,t)} \right\} = \mathbf{U}_i, \quad i = 1,2,\ldots,N-1 \tag{77c}$$

$$\left.\frac{\partial \hat{L}_i}{\partial \boldsymbol{\psi}'_i}\right|_{x_i=\ell_i} - \left\{ \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \Omega_{e,i}(\ell_i,t)}\right] - \frac{\partial L}{\partial \boldsymbol{\psi}_i(\ell_i,t)} \right\} = \boldsymbol{\Psi}_i, \quad i = 1,2,\ldots,N-1 \tag{77d}$$

$$\left.\frac{\partial \hat{L}_N}{\partial \mathbf{u}'_N}\right|_{x_N=\ell_N} = \mathbf{0}, \quad \left.\frac{\partial \hat{L}_N}{\partial \boldsymbol{\psi}'_N}\right|_{x_N=\ell_N} = \mathbf{0} \tag{77e,f}$$

and the generalized forces and torques are given by

$$\mathbf{F}_1^* = \sum_{i=1}^{N} C_{1i}^* \mathbf{F}_{ri}^* \tag{78a}$$

$$\mathbf{M}_1^* = \sum_{i=1}^{N} (\Gamma_{i1}^T \mathbf{F}_{ri}^* + C_{1i}^* \mathbf{M}_{ri}^*) \tag{78b}$$

$$\mathbf{M}_i^* = \mathbf{M}_{oi}^* + \sum_{j=1}^{N} (\Gamma_{ji}^T \mathbf{F}_{rj}^* + \gamma_{j-i} C_{ij}^* \mathbf{M}_{rj}^*), \quad i = 2,3,\ldots,N \tag{78c}$$

$$\mathbf{U}_i = \sum_{j=1}^{N} \gamma_{j-i-1} C_{ij}^* \mathbf{F}_{rj}^*, \quad i = 1,2,\ldots,N-1 \tag{78d}$$

$$\boldsymbol{\Psi}_i = \sum_{j=1}^{N} (\Gamma_{j,i+1}^T \mathbf{F}_{rj}^* + \gamma_{j-i-1} C_{ij}^* \mathbf{M}_{rj}^*), \quad i = 1,2,\ldots,N-1 \tag{78e}$$

where we have made use of Eqs. (27), (32a) and (63c,d).

6. Summary and Conclusions

In recent years, there has been an increasing interest in deriving the equations of motion for flexible multibody systems by treating the mass and stiffness of the bodies as distributed parameters. The equations of motion are generally derived by means of the extended Hamilton's principle, leading to a hybrid set of equations, where hybrid is to be taken in the sense that the rigid-body translations and rotations of the bodies are described by ordinary differential equations and the elastic motions are described by partial differential equations with appropriate boundary conditions. In earlier investigations, the rigid-body rotations were described by Eulerian-type angles, which tend to complicate unduly the equations of motion, unless the motion remains planar.

This paper presents a mathematical formulation for flexible multibodies in terms of quasi-coordinates, which permits the derivation of the equations for general rigid-body motions with considerably more ease than be using Eulerian-type angles. As an added feature, the equations for the elastic motions include rotatory inertia and shear deformation effects. The equations of motion are cast in state form, making them suitable for control design.

7. Bibliography

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Appendix

i. Derivative rules

If $A = [A_{ij}]$ is an $m \times n$ matrix, then we define the partial derivative of A with respect to a scalar τ to be the $m \times n$ matrix $\partial A/\partial \tau = [\partial A_{ij}/\partial \tau]$. If A is a function of time t, then the derivative of A with respect to t is denoted by $\dot{A} = dA/dt = [dA_{ij}/dt]$. Let $B = [B_{ij}]$ be an $M \times N$ matrix. Then, the derivative of a matrix with respect to a matrix, $\partial A/\partial B$, is the $mM \times nN$ matrix defined by

$$\frac{\partial A}{\partial B} = \begin{bmatrix} \frac{\partial A}{\partial B_{11}} & \frac{\partial A}{\partial B_{12}} & \cdots & \frac{\partial A}{\partial B_{1N}} \\ \frac{\partial A}{\partial B_{21}} & \frac{\partial A}{\partial B_{22}} & \cdots & \frac{\partial A}{\partial B_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial A}{\partial B_{M1}} & \frac{\partial A}{\partial B_{M2}} & \cdots & \frac{\partial A}{\partial B_{MN}} \end{bmatrix} \qquad (A-1)$$

Furthermore, let L be a scalar and $\mathbf{f} = [f_1 \cdots f_m]^T$, $\mathbf{q} = [q_1 \cdots q_n]^T$, $\mathbf{s} = [s_1 \cdots s_r]^T$ be column matrices. Then $\partial L/\partial \mathbf{q}$ is a column matrix, $\partial \mathbf{f}^T/\partial \mathbf{q}$ is an $n \times m$ matrix and $\partial \mathbf{f}/\partial \mathbf{q}^T = (\partial \mathbf{f}^T/\partial \mathbf{q})^T$. The chain rules for differentiation have the form

$$\frac{\partial \mathbf{f}^T}{\partial \mathbf{s}} = \frac{\partial \mathbf{q}^T}{\partial \mathbf{s}} \frac{\partial \mathbf{f}^T}{\partial \mathbf{q}} \quad \text{or} \quad \frac{\partial \mathbf{f}}{\partial \mathbf{s}^T} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}^T} \frac{\partial \mathbf{q}}{\partial \mathbf{s}^T} \qquad (A-2)$$

$$\frac{\partial L}{\partial \mathbf{s}} = \frac{\partial \mathbf{q}^T}{\partial \mathbf{s}} \frac{\partial L}{\partial \mathbf{q}} \quad \text{or} \quad \frac{\partial L}{\partial \mathbf{s}^T} = \frac{\partial L}{\partial \mathbf{q}^T} \frac{\partial \mathbf{q}}{\partial \mathbf{s}^T} \qquad (A-3)$$

Moreover,

$$\dot{\mathbf{f}} = \frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}^T} \dot{\mathbf{q}} \qquad (A-4)$$

$$\frac{\partial (A\mathbf{q})}{\partial \mathbf{q}^T} = A \quad \text{or} \quad \frac{\partial (A\mathbf{q})^T}{\partial \mathbf{q}} = A^T \qquad (A-5)$$

$$\frac{\partial \left(\frac{1}{2}\mathbf{q}^T A \mathbf{q}\right)}{\partial \mathbf{q}} = A\mathbf{q} \qquad (A-6)$$

provided A does not depend on $\mathbf{q}$.

ii. Proper orthogonal matrices

Throughout this paper, we encounter proper orthogonal matrices C, which are functions of three independent coordinates $\theta = [\theta_1\ \theta_2\ \theta_3]^T$. These matrices can be identified as matrices of direction cosines of one coordinate system $\xi_1\xi_2\xi_3$, with corresponding unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, with respect to another coordinate system $x_1x_2x_3$, with corresponding unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Hence, letting $C = [C_{ij}]$, the entries C_{ij} can be expressed as

$$C_{ij} = \mathbf{b}_i \cdot \mathbf{n}_j, \quad i,j = 1,2,3 \qquad (A-10)$$

which implies that

$$\mathbf{n}_j = \sum_{k=1}^{3} (\mathbf{n}_j \cdot \mathbf{b}_k)\mathbf{b}_k = \sum_{k=1}^{3} C_{kj}\mathbf{b}_k, \quad j = 1,2,3 \qquad (A-11)$$

At this point we wish to establish a relation between the body axes components of the angular velocity ω of coordinate system $\xi_1\xi_2\xi_3$ with respect to coordinate system $x_1x_2x_3$ and the time derivative of C_{ij} with respect to coordinate system $x_1x_2x_3$. First, recall (Ref. 33) that ω is uniquely characterized by

$$\dot{\mathbf{b}}_i = \omega \times \mathbf{b}_i, \quad i = 1,2,3 \qquad (A-12)$$

where in this case the "dot" requires holding $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ constant. Then, taking the time derivative of Eq. (A-10), using Eqs. (A-11) and (A-12), and some identity involving scalar and vector products, we obtain

$$\begin{aligned} \dot{C}_{ij} &= \dot{\mathbf{b}}_i \cdot \mathbf{n}_j = (\omega \times \mathbf{b}_i) \cdot \mathbf{n}_j = (\mathbf{b}_i \times \mathbf{n}_j) \cdot \omega \\ &= (\mathbf{b}_i \times \sum_{k=1}^{3} C_{kj}\mathbf{b}_k) \cdot \omega = \sum_{k=1}^{3} C_{kj}(\mathbf{b}_i \times \mathbf{b}_k) \cdot \omega \end{aligned} \qquad (A-13)$$

Now we observe that $(\mathbf{b}_i \times \mathbf{b}_k) \cdot \omega$, where $i, k = 1, 2, 3$, are merely the entries of the 3×3 matrix

$$[\omega_{ik}] = \begin{bmatrix} 0 & \mathbf{b}_3 \cdot \omega & -\mathbf{b}_2 \cdot \omega \\ -\mathbf{b}_3 \cdot \omega & 0 & \mathbf{b}_1 \cdot \omega \\ \mathbf{b}_2 \cdot \omega & -\mathbf{b}_1 \cdot \omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = \tilde{\omega}^T \quad (A-14)$$

where $[\omega_1 \ \ \omega_2 \ \ \omega_3]^T$ are the $\xi_1\xi_2\xi_3$ components of ω, and we have used the fact that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ form a right-handed set of unit vectors. Inserting Eqs. (A-14) into Eq. (A-13), we obtain

$$\dot{C}_{ij} = \sum_{k=1}^{3} \omega_{ik} C_{kj} \quad (A-15)$$

which can be expressed in the matrix form

$$\dot{C} = \tilde{\omega}^T C \quad (A-16)$$

The relationship between ω and $\dot{\theta}$ has the form

$$\omega = D(\theta)\dot{\theta} \quad (A-17)$$

We now propose to derive some relation between D and C. In the first place, taking the partial derivative of $CC^T = I$ with respect to θ_i, we obtain

$$C\frac{\partial C^T}{\partial \theta_i} + \frac{\partial C}{\partial \theta_i}C^T = C\frac{\partial C^T}{\partial \theta_i} + \left(C\frac{\partial C^T}{\partial \theta_i}\right)^T = 0, \quad i = 1, 2, 3 \quad (A-18)$$

from which we conclude that the 3×3 matrix $C(\partial C^T/\partial \theta_i)$ is skew symmetric. We denote the matrix by

$$\tilde{S}_i = C\frac{\partial C^T}{\partial \theta_i}, \quad i = 1, 2, 3 \quad (A-19)$$

where $\tilde{S}_i$ is obtained from the column matrix $\mathbf{S}_i = [S_{1i} \ S_{2i} \ S_{3i}]^T$ in the usual manner. We now calculate the time derivative of C^T in the form

$$\dot{C}^T = \sum_{i=1}^{3} \frac{\partial C^T}{\partial \theta_i}\dot{\theta}_i = C^T \sum_{i=1}^{3} \left(C\frac{\partial C^T}{\partial \theta_i}\right)\dot{\theta}_i = C^T \sum_{i=1}^{3} \tilde{S}_i\dot{\theta}_i$$
$$= C^T\left(\sum_{i=1}^{3} \mathbf{S}_i\dot{\theta}_i\right) = C^T\left([\mathbf{S}_1 \ \mathbf{S}_2 \ \mathbf{S}_3]\,\dot{\theta}\right) = C^T(\widetilde{S\dot{\theta}}) \quad (A-20)$$

Comparing Eqs. (A-16), (A-17) and (A-19), we conclude that

$$S = [\mathbf{S}_1 \ \ \mathbf{S}_2 \ \ \mathbf{S}_3] = D \quad (A-21)$$

Equation (A-20) relates C and D in an implicit manner.

Next, we wish to derive an expression for $\dot{D}$. Taking the partial derivative of Eq. (A-18) with respect to θ_j and replacing $\tilde{S}_i$ by $\tilde{D}_i$, we obtain

$$\frac{\partial \tilde{D}_i}{\partial \theta_j} = \frac{\partial C}{\partial \theta_j}\frac{\partial C^T}{\partial \theta_i} + C\frac{\partial^2 C^T}{\partial \theta_j \partial \theta_i} = \left(C\frac{\partial C^T}{\partial \theta_j}\right)^T\left(C\frac{\partial C^T}{\partial \theta_i}\right)$$
$$+ C\frac{\partial^2 C^T}{\partial \theta_j \partial \theta_i} = \tilde{D}_j^T\tilde{D}_i + C\frac{\partial^2 C^T}{\partial \theta_j \partial \theta_i} = -\tilde{D}_j\tilde{D}_i + C\frac{\partial^2 C^T}{\partial \theta_j \partial \theta_i} \quad (A-22)$$

Interchanging i and j in Eq. (A-22), we have

$$\frac{\partial \tilde{D}_j}{\partial \theta_i} = -\tilde{D}_i\tilde{D}_j + C\frac{\partial^2 C^T}{\partial \theta_i \partial \theta_j} \quad (A-23)$$

Then, subtracting Eq. (A-23) from Eq. (A-22), we can write

$$\frac{\partial \tilde{D}_i}{\partial \theta_j} - \frac{\partial \tilde{D}_j}{\partial \theta_i} = \tilde{D}_i\tilde{D}_j - \tilde{D}_j\tilde{D}_i = (\widetilde{\tilde{D}_i\mathbf{D}_j}) \quad (A-24)$$

which implies that

$$\frac{\partial \mathbf{D}_i}{\partial \theta_j} - \frac{\partial \mathbf{D}_j}{\partial \theta_i} = \tilde{D}_i\mathbf{D}_j \quad (A-25)$$

This formula can be used in turn to derive an expression for $\dot{D}$. First, we recall Eq. (A-17) and write

$$\dot{\mathbf{D}}_i = \sum_{j=1}^{3} \frac{\partial \mathbf{D}_i}{\partial \theta_j}\dot{\theta}_j = \sum_{j=1}^{3}\left(\frac{\partial \mathbf{D}_j}{\partial \theta_i}\dot{\theta}_j + \tilde{D}_i\mathbf{D}_j\dot{\theta}_j\right)$$
$$= \frac{\partial\left(\sum_{j=1}^{3}\mathbf{D}_j\dot{\theta}_j\right)}{\partial \theta_i} + \tilde{D}_i\left(\sum_{j=1}^{3}\mathbf{D}_j\dot{\theta}_j\right)$$
$$= \frac{\partial(D\dot{\theta})}{\partial \theta_i} + \tilde{D}_i\omega = \frac{\partial(D\dot{\theta})}{\partial \theta_i} + \tilde{\omega}^T\mathbf{D}_i \quad (A-26)$$

This implies that

$$\dot{D}^T = \begin{bmatrix} \dot{\mathbf{D}}_1^T \\ \dot{\mathbf{D}}_2^T \\ \dot{\mathbf{D}}_3^T \end{bmatrix} = \frac{\partial(D\dot{\theta})^T}{\partial \theta} + D^T\tilde{\omega} \quad (A-27)$$

Next, we consider the partial derivative of $(C\mathbf{a})^T$ with respect to θ, where $\mathbf{a}$ does not depend on θ. First, we recall Eqs. (A-19) and (A-21) and write

$$\frac{\partial(C\mathbf{a})^T}{\partial \theta_i} = \mathbf{a}^T\frac{\partial C^T}{\partial \theta_i} = \mathbf{a}^T C^T\left(C\frac{\partial C^T}{\partial \theta_i}\right) = (C\mathbf{a})^T\tilde{D}_i$$
$$= -(C\mathbf{a})^T\tilde{D}_i^T = (\widetilde{C\mathbf{a}}\mathbf{D}_i)^T = -(\tilde{D}_iC\mathbf{a})^T$$
$$= \mathbf{D}_i^T(\widetilde{C\mathbf{a}})^T \quad (A-28)$$

which implies that

$$\frac{\partial(C\mathbf{a})^T}{\partial \theta} = \begin{bmatrix} -\mathbf{D}_1^T(\widetilde{C\mathbf{a}}) \\ -\mathbf{D}_2^T(\widetilde{C\mathbf{a}}) \\ -\mathbf{D}_3^T(\widetilde{C\mathbf{a}}) \end{bmatrix} = -D^T(\widetilde{C\mathbf{a}}) \quad (A-29)$$

The companion formula

$$\frac{\partial(C^T\mathbf{a})^T}{\partial \theta} = D^T\tilde{\mathbf{a}}C \quad (A-30)$$

can be derived in a similar manner.

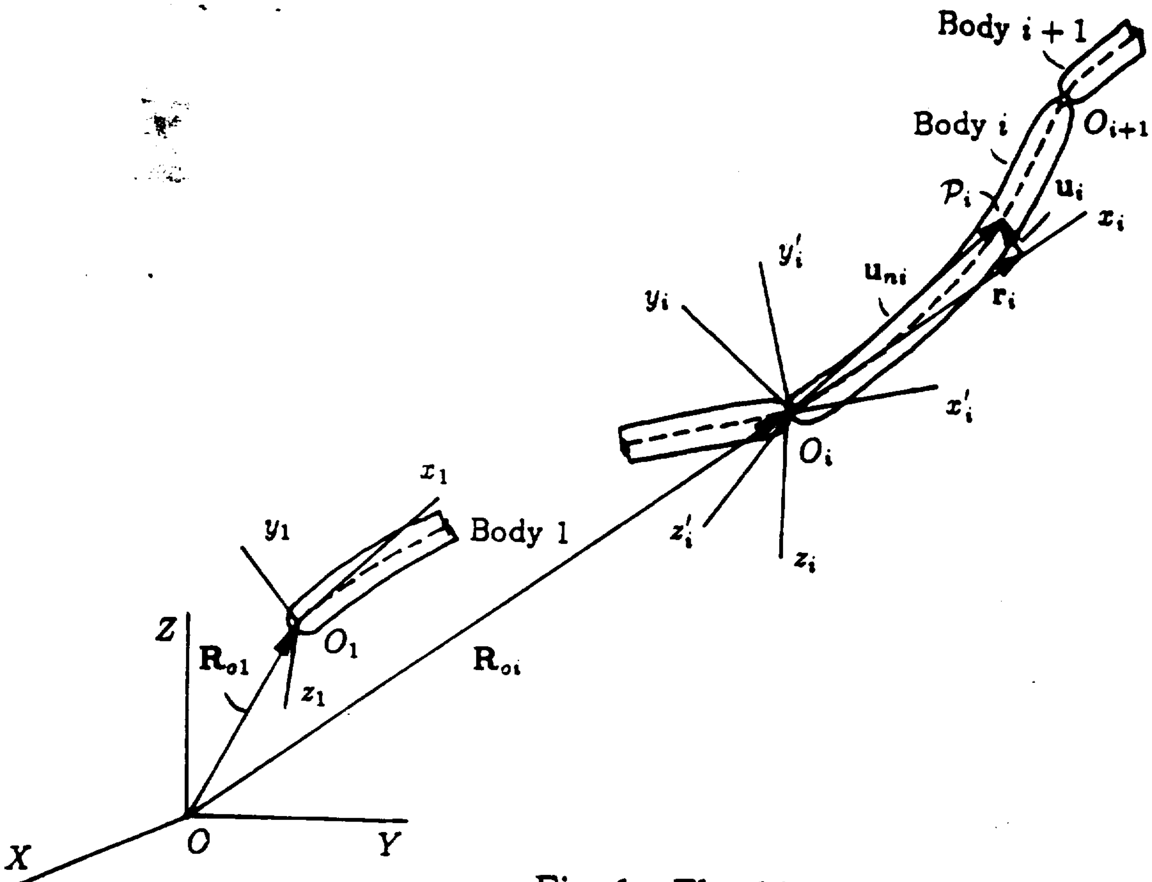

Fig. 1 - Flexible Multibody System

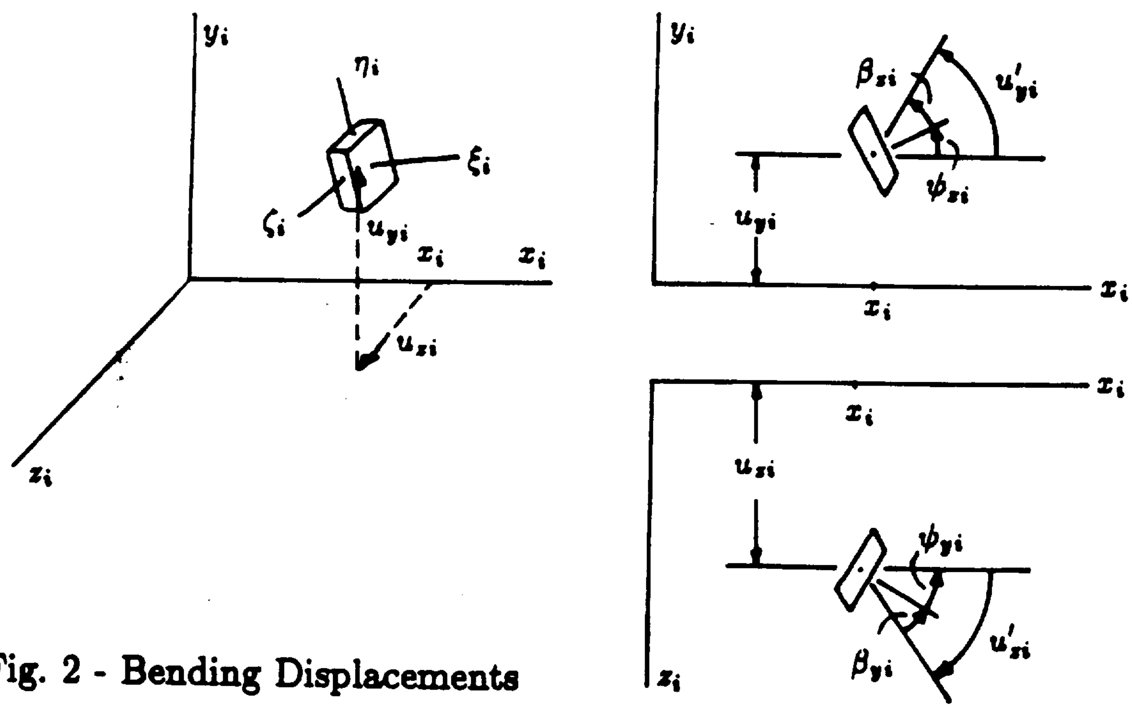

Fig. 2 - Bending Displacements