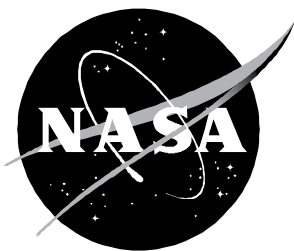# Introduction to Generalized Functions With Applications in Aerodynamics and Aeroacoustics

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ADDENDUM

F. Farassat: The Integration of $\delta'(f)$ in a Multidimensional Space,
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Contents

Symbols

$A(x)$	coefficient of second order term of linear ordinary differential equation
$A(\alpha)$	lower limit of integral in Leibniz rule depending on parameter α
a	constant
BC, BC_1, BC_2	boundary conditions
$B(x)$	coefficient of first order term in second order linear ordinary differential equation
$B(\alpha)$	upper limit of integral in Leibniz rule depending on parameter α
b	constant
C, C_1, C_2	constants
$C(x)$	coefficient of zero order term (the unknown function) in second order linear ordinary differential equation
c	constant, also speed of sound
D	space of infinitely differentiable functions with bounded support (test functions)
D'	space of generalized functions based on D
E_1, E_2	expressions in integrands of Kirchhoff formula for moving surfaces
$E(\alpha)$	function defined by equation (3.70)
E_h	shift operator $E_h f(x) = f(x + h)$
$\mathbf{E}_{ij}$	viscous stress tensor
F	in $F[\phi]$, defines linear functional on test function space; generalized function
$F(\mathbf{y};\mathbf{x},t)$	$= [f(\mathbf{y},\tau)]_{\mathrm{ret}} = f(\mathbf{y}, t - \frac{r}{c})$
$\widetilde{F}(\mathbf{y};\mathbf{x},t)$	$= [\widetilde{f}(\mathbf{y},\tau)]_{\mathrm{ret}} = \widetilde{f}(\mathbf{y}, t - \frac{r}{c})$
$f(x), f(\mathbf{x})$	arbitrary ordinary functions
$f_1(x)$	arbitrary function
$f_i(\tau)$	components of moving compact force, $i = 1$ to 3
$f(\mathbf{x},t)$	equation of moving surface defined as $f(\mathbf{x},t) = 0$, $f > 0$ outside surface
$\widetilde{f}(\mathbf{x},t)$	moving surface defined by $\widetilde{f}(\mathbf{x},t) = 0$ intersection of which with $f(\mathbf{x},t) = 0$ defines edge of open surface $f = 0$, $\widetilde{f} > 0$
$g(x,y), g(\mathbf{x},\mathbf{y})$	Green's function
g	$= \tau - t + \frac{r}{c}$
$g_1(x,y), g_2(x,y)$	define Green's function for $x < y$ and $x > y$, respectively
$g_{(2)}$	determinant of coefficients of first fundamental form of surface
$g(x), g(\mathbf{x})$	arbitrary functions
H	in $H[\phi]$, linear functional $\int_0^\infty \phi(x)\, dx$ based on Heaviside function
H_f	local mean curvature of surface $f = 0$
$H(x,\alpha)$	function defined by equation (3.71)
h	constant
$h(x)$	Heaviside function

t_1	in $\frac{\partial}{\partial t_1}$, directional derivative in direction of $\mathbf{t}_1$
u_i	components of fluid velocity, $i = 1$ to 3
u_n	local fluid normal velocity
u^i	curvilinear coordinate variables, $i = 1$ to 3
v_n	local outward normal velocity of surface
$v_{n'}$	local inward normal velocity of surface
$\mathbf{x}$	observer variable; (x_1, x_2, x_3)
$\mathbf{y}$	source variable; (y_1, y_2, y_3)
α	constant, parameter
α_f	constant depending on shape of surface $f = 0$
β	constant
Γ	in $d\Gamma$, length parameter along curve of intersection of surfaces $f = 0$ and $g = 0$
$\boldsymbol{\Gamma}$	strength of vorticity
γ	height of cylinder
Δ	jump in function at discontinuity
$\delta(x), \delta(\mathbf{x}), \delta(f)$	Dirac delta function
$\delta[\phi]$	linear functional representing Dirac delta function
$\boldsymbol{\delta}_{ij}$	Kronecker delta $\boldsymbol{\delta}_{ij} = 0$ if $i \neq j$, $\boldsymbol{\delta}_{ii} = 1$
ε	small parameter
$\boldsymbol{\eta}$	Lagrangian variable
θ	angle between ∇f and ∇g; angle between radiation direction $\widehat{\mathbf{r}}$ and local normal to surface $\mathbf{n}$
θ_1	angle between $\widehat{\mathbf{r}}$ and $\mathbf{n}_1$
θ'	angle between $\mathbf{N}$ and $\widetilde{\mathbf{N}}$
Λ	$= |\nabla F|$, $F = [f]_{\mathrm{ret}}$, $|\nabla f| = 1$
$\widetilde{\Lambda}$	$= |\nabla \widetilde{F}|$, $\widetilde{F} = [\widetilde{f}]_{\mathrm{ret}}$, $|\nabla \widetilde{f}| = 1$
Λ_0	$= |\nabla F \times \nabla \widetilde{F}|$, F, and $\widetilde{F}$ as defined above
$\boldsymbol{\nu}$	unit inward geodesic normal
ξ	variable of Fourier transform
ρ	density
ρ_0	density of undisturbed medium
Σ	surface $F(\mathbf{y}; \mathbf{x}, t) = 0$
ϕ	test function, arbitrary function
ϕ_1, ϕ_2	test functions
ϕ_n	sequence of test functions; component of vector field ϕ normal to surface
$\phi^{(k)}$	kth derivative of ϕ
$\Phi(\mathbf{x}, t)$	unknown function of inhomogeneous wave equation

$\tilde{\phi}$	extension of function ϕ to unbounded space
$\phi_{1,i}$	components of vector function $\boldsymbol{\phi}_1$, $i = 1$ to 3
τ	source time
Ω	open interval or region of space; $\partial\Omega$ boundary of Ω
$\Omega(\tau)$	sphere $r = c(t - \tau)$, $(\mathbf{x}, t, \tau)$ kept fixed

Subscripts:

h	in E_h, shift of function by amount h to right or left
n, n'	component of vector field in direction of local normal $\mathbf{n}$ or $\mathbf{n}'$
n	index of sequence such as ϕ_n
0	in ρ_0, indicates condition of undisturbed medium
ret	retarded time
x	in ℓ_x, indicates that derivatives in ℓ act on variable x in $\ell_x g(x, y)$
ε	continuous index in function such as $h_\varepsilon(x)$

Superscripts:

k	in $\phi^{(k)}$, kth derivative of ϕ
n	in $\phi^{(n)}$, nth derivative of ϕ

Notation:

$\square^2$	D'Alembertian, wave operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$
[ ]	in $F[\phi]$, indicates functional evaluated for ϕ, ϕ a test function
supp	support of function
$\sim$	in $\underset{\sim}{\phi}$, indicates restriction of ϕ to support of delta function
$\wedge$	in $\widehat{\psi}$, indicates Fourier transform
$*$	in τ^*, indicates emission time
∇	gradient operator
∇_2	surface gradient operator
∇_y	gradient operator acting on variable $\mathbf{y}$
$-$	over derivative such as $\bar{f}'(x)$, indicates generalized differentiation
∂	in $\partial\Omega$, indicates boundary of region Ω

Summary

Since the early 1950's, when Schwartz published his theory of distributions, generalized functions have found many applications in various fields of science and engineering. One of the most useful aspects of this theory in applications is that discontinuous functions can be handled as easily as continuous or differentiable functions. This provides a powerful tool in formulating and solving many problems of aerodynamics and acoustics. Furthermore, generalized function theory elucidates and unifies many ad hoc mathematical approaches used by engineers and scientists in these two fields. In this paper, we define generalized functions as continuous linear functionals on the space of infinitely differentiable functions with compact support, then introduce the concept of generalized differentiation. Generalized differentiation is the most important concept in generalized function theory and the applications we present utilize mainly this concept. First, some results of classical analysis, such as Leibniz rule of differentiation under the integral sign and the divergence theorem, are derived with the generalized function theory. The divergence theorem is shown to remain valid for discontinuous vector fields provided that all the derivatives are viewed as generalized derivatives. An implication of this is that all conservation laws of fluid mechanics are valid, as they stand for discontinuous fields with all derivatives treated as generalized derivatives. When the derivatives are written as the sum of ordinary derivatives and the jump in the field parameters across discontinuities times a delta function, the jump conditions can be easily found. For example, the unsteady shock jump conditions can be derived from mass and momentum conservation laws. Generalized function theory makes this derivation very easy. Other applications of the generalized function theory in aerodynamics discussed here are the derivations of general transport theorems for deriving governing equations of fluid mechanics, the interpretation of the finite part of divergent integrals, the derivation of the Oswatitsch integral equation of transonic flow, and the analysis of velocity field discontinuities as sources of vorticity. Applications in aeroacoustics presented here include the derivation of the Kirchhoff formula for moving surfaces, the noise from moving surfaces, and shock noise source strength based on the Ffowcs Williams–Hawkings equation.

1. Introduction

In the early 1950's, Schwartz published his theory of distributions that we call generalized functions. (See ref. 1.) Earlier, Dirac had introduced the delta function $\delta(x)$ by the sifting property

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)\,dx = \phi(0) \tag{1.1}$$

Dirac recognized that no ordinary function could have the sifting property. Nevertheless, he thought of $\delta(x)$ as a useful mathematical object in algebraic manipulations that could be viewed as the limit of a sequence of ordinary functions. The Dirac delta function is a generalized function in the theory of distributions. Schwartz established rigorously the properties of generalized functions. His theory has had an enormous impact on many areas of mathematics, particularly on partial differential equations. Generalized function theory has been used in many fields of science and engineering.

To include mathematical objects such as the Dirac delta function into analysis, we must somehow extend the concept of a function. The process we use to introduce new objects is familiar in mathematics. We extended natural numbers to integers, integers to rationals, and rationals to real numbers. We also extended real numbers to complex numbers. In each extension, new objects were introduced in the number system while most properties of the old number system were retained. Furthermore, for each extension, we had to think of the new number system in a different way from the old system. For example, in going from integers to

rationals, we view numbers as ordered pairs of integers (a, b), where $b \neq 0$. We identify ordered pairs $(a, 1)$ with integer a. The new number system (the rationals) includes the old number system (the integers). We must now think of numbers as ordered pairs (a, b), which we usually write as a/b, instead of as a single number a for integers. Similarly, to extend the concept of function to include the Dirac delta function, we must think of functions differently.

We explain in section 2 how to think of functions as functional (i.e., the mapping of a suitable function space into scalars). In this way, the Dirac delta function can naturally be included in the extended space of functions that we call distributions or generalized functions. The usefulness of this theory stems from the powerful operational properties of generalized functions. In addition, solutions with discontinuities can be handled easily in the differential equation or by using the Green's function approach. Many ad hoc mathematical methods used by engineers and scientists are unified and elucidated by generalized function theory. In fluid dynamics, the derivations of transport theorems, conservation laws, and jump conditions are facilitated by that theory. Geometric identities for curves, surfaces, and volumes, particularly when in motion and deformation, can be derived easily with generalized function theory. In section 2 we also define generalized functions as continuous linear functionals on some space of test functions. Some operations on generalized functions are defined in this section, as are various approaches to introduce generalized functions in mathematics.

In section 3 we present some definitions and results for generalized functions as well as some important results for generalized derivatives, multidimensional delta functions, and the finite part of divergent integrals. In section 4 we present various aerodynamic applications including derivation of two transport theorems—the interpretation of velocity discontinuity as a vortex sheet and the derivation of the Oswatitsch integral equation of transonic flow. The aeroacoustic applications include the derivation of the solution of the wave equation with various inhomogeneous source terms, the Kirchhoff equation for moving surfaces, the Ffowcs Williams–Hawkings equations, and shock noise source strength. All these applications depend on the concept of generalized differentiation. Concluding remarks are in section 5 and the references follow.

Many articles and books have been published on the topic of generalized function theory. Most of these works have extremely abstract presentations. In particular, multidimensional generalized functions, which are most useful in applications, are often treated cursorily in applied mathematics and physics books. Of course, some exceptions are available. (See refs. 2–7.) Multidimensional generalized functions are relatively easy to learn and use if the theory is stripped of some abstraction. To work with multidimensional generalized functions, some knowledge of differential geometry and of tensor analysis is required. (See also refs. 8 and 9.) In this paper, we present the rudiments of generalized function theory for engineers and scientists with emphasis on applications in aerodynamics and aeroacoustics. The presentation is expository. The intent is to interest readers in the subject and to reveal the power of the generalized function theory. Some illustrative mathematical examples are given here to help in the understanding of the abstract concepts inherent in generalized functions.

2. What Are Generalized Functions?

2.1. Schwartz Functional Approach

It can be shown from classical Lebesgue integration theory that the Dirac delta function cannot be an ordinary function. By an ordinary function we mean a locally Lebesgue integrable function (i.e., one that has a finite integral over any bounded region). To include the Dirac delta function in mathematics, we must change the way we think of an ordinary function $f(x)$.

Conventionally, we think of this function as a table of ordered pairs $(x, f(x))$. Of course, often this table has an uncountably infinite number of ordered pairs. We show this table as a curve representing the function in a plane. In generalized function theory, we also describe $f(x)$ by a table of numbers. These numbers are produced by the relation

$$F[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx \tag{2.1}$$

where the function $\phi(x)$ comes from a given space of functions called the test function space. For a fixed function $f(x)$, equation (2.1) is a mapping of the test function space into real or complex numbers. Such a mapping is called a functional. We use square brackets to denote functional (e.g., $F[\phi]$ and $\delta[\phi]$). Therefore, a function $f(x)$ is now described by a table of its functional values over a given space of test functions. We must first, however, specify the test function space.

The test function space that we use here is the space D of all infinitely differentiable functions with bounded support. The support $supp\ \phi(x)$ of a function $\phi(x)$ is the closure of the set on which $\phi(x) \neq 0$. For an ordinary function $f(x)$, the functional $F[\phi]$ is linear in that, if ϕ_1 and ϕ_2 are in D and if α and β are two constants, then

$$F[\alpha\phi_1 + \beta\phi_2] = \alpha F[\phi_1] + \beta F[\phi_2] \tag{2.2}$$

The functional $F[\phi]$ is also continuous in the following sense. Take a sequence of functions $\{\phi_n\}$ in D and let this sequence have the following two properties:

1. There exists a bounded interval I such that for all n, $supp\ \phi_n \subset I$.

2. $\lim_{n\to\infty} \phi_n^{(k)}(x) = 0$ uniformly for all $k = 0, 1, 2, \ldots..$.

Such a sequence is said to go to 0 in D and is written $\phi_n \xrightarrow{D} 0$. Here $supp\ \phi_n$ stands for support of ϕ_n. We then say that the functional $F[\phi]$ is continuous if $F[\phi_n] \to 0$ for $\phi_n \xrightarrow{D} 0$. We will have more to say in this section about the space D and why we require the two conditions above in the definition of $\phi_n \xrightarrow{D} 0$.

As an important example of a function $\phi(x)$ in D, for a given finite $a > 0$, we define

$$\phi(x; a) = \begin{cases} \exp\left(\frac{a^2}{x^2 - a^2}\right) & (|x| < a) \\ 0 & (|x| \geq a) \end{cases} \tag{2.3}$$

Note that $supp\ \phi(x; a) = [-a, a]$ and is bounded. We can show that $\phi(x; a)$ is infinitely differentiable. Therefore, $\phi(x; a) \in D$. The proof of infinite differentiability at $x = \pm a$ is somewhat messy and algebraically complicated and we will not belabor this point here. We can show that from any continuous function $g(x)$, we can construct another function $\psi(x)$ in D from the relation

$$\psi(x) = \int_b^c g(t)\phi(x - t; a)\,dt \tag{2.4}$$

where the interval $[b, c]$ is finite. The support of $\psi(x)$ is $[b - a, c + a]$, which is bounded. The infinite differentiability of $\psi(x)$ follows from infinite differentiability of $\phi(x; a)$. Therefore, $\psi(x) \in D$. There exists an uncountably infinite number of continuous functions. (Consider the family of continuous functions $\sin(\alpha x)$, α, $\varepsilon(0, 1)$. This family has an uncountable number of members.) It follows from the above argument that there exists an uncountably infinite number of functions in space D, so our table constructed from $F[\phi]$ by equation (2.1) representing

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the ordinary function $f(x)$ has an uncountably infinite number of elements. This fact has an important consequence. Two ordinary functions f and g that are not equal in the Lebesgue sense (i.e., two functions that are not equal on a set with nonzero measure) generate tables by equation (2.1) that differ in some entries. Thus, the space D is so large that the functionals on D generated by equation (2.1) can distinguish different ordinary functions.

We now give an example of a sequence $\{\phi_n\}$ in D such that $\phi_n \xrightarrow{D} 0$. Using the function $\phi(x; a)$ in equation (2.3), we define

$$\phi_n(x) = \frac{1}{n}\phi(x; a) \tag{2.5}$$

This sequence can easily be shown to satisfy the two conditions required for $\phi_n \xrightarrow{D} 0$. We note in particular that $supp\,\phi_n = [-a, a]$ for all n.

Now we define distributions or generalized functions of Schwartz. First, we note that for an ordinary function $f(x)$ (i.e., a locally Lebesgue integrable function), the functional $F[\phi]$ given by equation (2.1) is linear and continuous. The proof of linearity is obvious. The proof of continuity requires only that $\phi_n \to 0$ uniformly, which already follows from $\phi_n \xrightarrow{D} 0$. Remembering that we are now looking at functions by their table of functional values over the space D and that this functional is linear and continuous, we ask if all the continuous linear functionals on space D are generated by ordinary functions through the relation given in equation (2.1). We find they are not! Some continuous linear functionals on space D are not generated by ordinary functions. For example,

$$\delta[\phi] = \phi(0) \tag{2.6}$$

Proof of linearity is obvious. Continuity follows again from $\phi_n \to 0$ uniformly. However, this functional has the sifting property that the Dirac delta function requires. As we stated earlier, no ordinary function has the sifting property. Therefore, this approach introduces the delta function rigorously into mathematics. We define generalized functions as continuous linear functionals on space D. The space of generalized functions on D is denoted D'. Figure 1 shows schematically how we extended the space of ordinary functions to generalized functions. We call ordinary functions regular generalized functions, whereas all other generalized functions (such as the Dirac delta function) are called singular generalized functions.

For algebraic manipulations, we retain the notation of ordinary functions for generalized functions for convenience. We symbolically introduce the notation $\delta(x)$ for the Dirac delta function by the relation

$$\delta[\phi] = \phi(0) = \int \delta(x)\phi(x)\,dx \tag{2.7}$$

Note that the integral on the right side of equation (2.7) does not stand for conventional integration of a function. Rather, it stands for $\phi(0)$. We can now use $\delta(x)$ in mathematical expressions as if it were an ordinary function. However, we must remember that singular generalized functions are not, in general, defined pointwise; they define a functional (i.e., a function from our new point of view) when they are multiplied by a test function and appear under an integral sign. Thus, when a singular generalized function appears in an expression, it is always in an intermediate stage in the solution of a real physical problem.

More facts about space D in multidimensions, convergence to 0 in D, and the concept of continuity of a functional are appropriate now. The multidimensional test function space D is
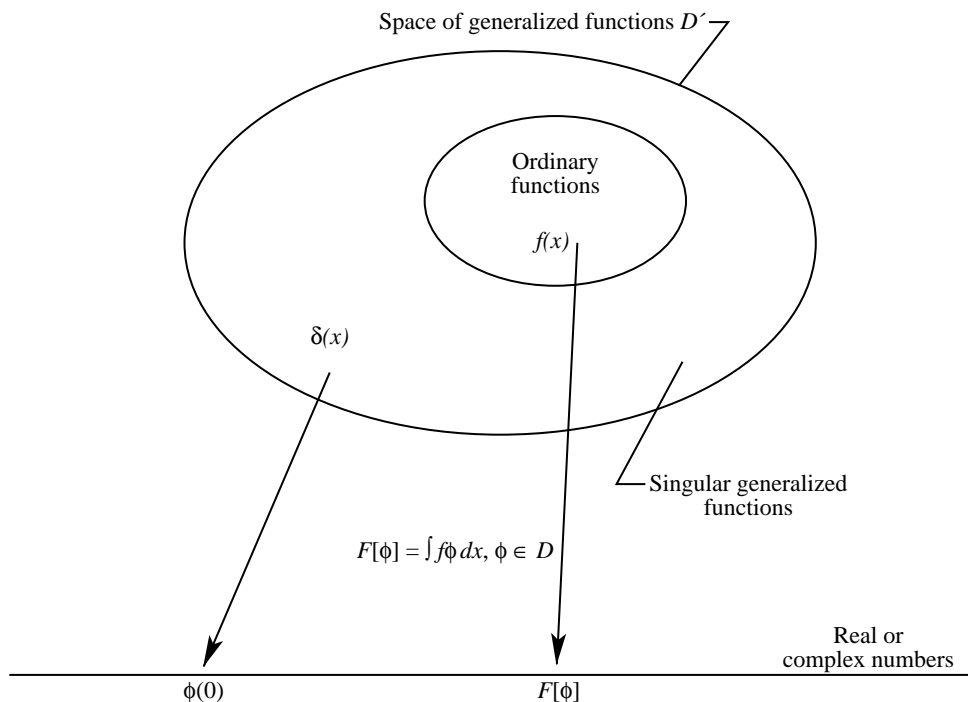
Figure 1. Generalized functions are continuous linear functionals on space D of test functions.

defined as the space of infinitely differentiable functions with bounded support. For example, for $a > 0$,

$$\phi(\mathbf{x}; a) = \begin{cases} \exp\left(\frac{a^2}{|\mathbf{x}|^2 - a^2}\right) & (|\mathbf{x}| < a) \\ 0 & (|\mathbf{x}| \geq a) \end{cases} \tag{2.8}$$

where $|\mathbf{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ is the Euclidean norm. Other functions in this space can be constructed by using any continuous function $g(\mathbf{x})$ and the convolution relation

$$\psi(\mathbf{x}) = \int_{\Omega} g(\mathbf{t}) \phi(\mathbf{x} - \mathbf{t}; a) \, d\mathbf{t} \tag{2.9}$$

where Ω is a bounded region. The multidimensional generalized function space D' is defined as the space of continuous linear functionals on the space D. In the multidimensional case, a number of important singular generalized functions of the delta function type appear in applications. In one dimension, the support of $\delta(x)$ consists of one point, $x = 0$. We define the support of a generalized function later. In the multidimensional case, in addition to $\delta(\mathbf{x})$, which has the support $\mathbf{x} = 0$, there is also $\delta(f)$ with support on the surface $f(\mathbf{x}) = 0$. Section 2.2 contains a detailed explanation of $\delta(f)$.

We now discuss the definition of continuity of linear functionals on the space D. Continuity is a topological property. Space D is a linear or vector space. It is made into a topological vector space by defining the neighborhood of $\phi(x) = 0$ by a sequence of seminorms. The two conditions required above in the definition of $\phi_n \xrightarrow{D} 0$ follow from the conditions used to define the neighborhood of $\phi(x) = 0$. (See refs. 10 and 11.) The definition of continuity of linear functionals on space D can be based on the weak or strong topologies of space D'. (See ref. 11.) It so happens that the definitions of continuity based on these topologies are equivalent to the

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earlier definition of $F[\phi_n] \to 0$ if and only if $\phi_n \xrightarrow{D} 0$. Furthermore, we note that because D is a linear space, we can define $\phi_n \xrightarrow{D} \phi$ if $\phi_n - \phi \xrightarrow{D} 0$ and because $F[\phi]$ is linear, we can also say that $F[\phi]$ is continuous if when $\phi_n \xrightarrow{D} \phi$, we have $F[\phi_n] \to F[\phi]$.

We conclude this discussion with one more important fact. Although in this paper we confine ourselves to the test function space D, in many applications we should use a different test function space. For example, to define Fourier transformation, we should use a test function space S of infinitely differentiable functions that go to 0 at infinity faster than $|x|^{-n}$ for any $n > 0$ (the space of rapidly decreasing functions). Other test function spaces are defined in Carmichael and Mitrović (ref. 10) and in references 12 and 13. Generalized functions on these spaces are defined as continuous linear functionals after a suitable definition of convergence to 0 in the test function space is given to get a topological vector space. Note that in all these spaces of generalized functions, the important singular generalized functions (such as the Dirac delta function) are retained with properties essentially similar to those we study below in space D'. It can be shown that if $A \subset B$ and if A and B are two test function spaces used to define generalized function spaces A' and B', respectively, then we have $A \subset B \subset B' \subset A'$ (i.e., the space of generalized functions A' is larger than B'). In particular, $D \subset S \subset S' \subset D'$, where S is the space of rapidly decreasing functions defined above.

2.2. How Can Generalized Functions Be Introduced in Mathematics?

Although Schwartz developed the theory of distributions, like many great ideas in mathematics and science, the subject has a long history. Synowiec (ref. 14) has stated that evolution of the concepts of distribution theory followed a familiar pattern in mathematics of multiple and simultaneous discoveries because the appropriate ideas were 'in the air.' Several good sources on the history of theory of distributions are available. (See refs. 15 and 16.) Therefore, we will not present a detailed history here. Also, many different approaches in mathematics can be used to introduce and develop systematically generalized function theory. We mention several of them here.

2.2.1. Functional approach to generalized functions. In the functional approach, generalized functions are defined as continuous linear functionals. This approach (which we use here) was originally introduced by Schwartz (ref. 1) and is the most popular and direct method of studying generalized functions. (See refs. 3, 4, and 7.) The operations on ordinary functions such as differentiation and Fourier transformation are extended by first writing these operations in the language of functionals for ordinary functions, then by using them to define the operations for all generalized functions. After the rules of these operations are obtained, the usual notation of ordinary functions can be used for all generalized functions. A working knowledge is relatively easy to develop with this notation without confusion. Some elementary knowledge of functional analysis is needed in this approach.

2.2.2. Sequential approach to generalized functions. The sequential approach is essentially based on the original idea of Dirac in defining a delta function as the limit of a sequence of ordinary functions. The approach was originated by Mikusiński from a theorem in distribution theory that the space of generalized functions is complete. Therefore, singular generalized functions such as the delta function can be defined as the limit of ordinary (i.e., regular) functions much like defining irrational numbers as limits of a Cauchy sequence of rational numbers. Many good books have been published on this subject. (See refs. 2, 6, and 17.) To define a generalized function, the analyst is required to construct and work with a sequence of infinitely differentiable functions. Although mathematics only to the level of advanced calculus is involved, the algebraic manipulations are technical and laborious. An extension to the multidimensional case also appears more difficult than with the functional approach.

2.2.3. Bremermann approach to generalized functions. In the Bremermann approach, generalized functions of real variables are viewed as the boundary values of analytic functions on the real axis. (See refs. 12 and 13.) The Bremermann approach has its basis in earlier works on Fourier transformation in the complex plane to define Fourier transforms of polynomials. This approach employs some of the powerful results of analytic function theory and is particularly useful in Fourier analysis and partial differential equations. A recent book on the subject is by Carmichael and Mitrović. (See ref. 10.)

2.2.4. Mikusiński approach to generalized functions. The Mikusiński approach is based on ideas from abstract algebra. A commutative ring is constructed from functions with support on a semi-infinite axis by defining the operations of addition and multiplication as ordinary addition and convolution of functions, respectively. This commutative ring has no zero divisors by a theorem of Titchmarsh. (See refs. 18 and 20.) Therefore, it can be extended to a field by the addition of a multiplicative identity and the multiplicative inverses of all functions. This multiplicative identity turns out to be the Dirac delta function. The Mikusiński approach gives a rigorous explanation of the Heaviside operational calculus and solves other problems such as the solution of recursion relations. One limitation of this approach is that the supports of the functions are confined to semi-axis or half-space in the multidimensional case. Good sources for this approach are Mikusiński (ref. 18), Mikusiński and Boehme (ref. 19), and an excellent expository book by Erdélyi (ref. 20)

2.2.5. Other approaches to generalized functions. Several other important approaches introduce generalized functions in mathematics. One approach is based on the nonstandard analysis of Robinson. (See ref. 21.) Nonstandard analysis uses formal logic theory to extend the real line by the rigorous inclusion of Leibniz infinitesimals. Many interesting applications of this theory, particularly in dynamical systems, are now available. Another more recent approach is presented in Colombeau (refs. 22 and 23) and Rosinger (ref. 24). This approach uses advanced algebraic and topological concepts to develop a theory of generalized functions in which multiplication of arbitrary functions is allowed and it is gaining popularity at present. Applications to nonlinear partial differential equations are given by Rosinger (ref. 24) and Oberguggenberger (ref. 25).

3. Some Definitions and Results

3.1. Introduction

In this section, some important definitions and results used later are presented. Then, the generalized derivative, the multidimensional delta functions, and the finite part of divergent integrals are discussed. This paper is application oriented so we are selective about the material presented here. We also freely refer to a generalized function by a symbolic or a functional notation.

3.1.1. Multiplication of a generalized function by an infinitely differentiable function. Let $f(x)$ be a generalized function in D' defined by the functional $F[\phi]$ and let $a(x)$ be an infinitely differentiable function. Then, $a(x)f(x)$ is defined by the rule

$$aF[\phi] = F[a\phi] \tag{3.1}$$

Note that aF stands for the functional that defines af. Also, because ϕ is in D, so is $a\phi$. We can use this definition to define $a(x)\delta(x)$. Let $\delta[\phi]$ be the Dirac function given by equation (2.6); then,

$$a\delta[\phi] = \delta[a\phi] = a(0)\phi(0) \tag{3.2}$$

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Symbolically, this equation is interpreted as

$$a(x)\delta(x) = a(0)\delta(x) \tag{3.3}$$

In the space D', multiplication of two arbitrary generalized functions is not defined; however, this statement needs clarification. Obviously, ordinary functions are also generalized functions and any two ordinary functions can be multiplied; thus, they can also be multiplied in the sense of distributions. However, multiplication of a regular and a singular generalized function or two singular generalized functions may not always be defined. For example, the multiplication of $\delta(x)$ by itself (i.e., $\delta^2(x)$) is not defined, neither is $f(x)\delta(x)$, where $f(x)$ has a jump discontinuity or a singularity at $x = 0$. In applications, experience or inconsistencies in the results occasionally show that some multiplications of two generalized functions are not allowed. Sometimes this problem can be removed by rewriting the expression such that the troublesome multiplication is avoided. For example, difficulties with multiplication of distributions appear if we use the mass continuity and momentum equations in nonconservative forms to find shock jump conditions (section 4.2). These difficulties can be removed by using the conservation laws in conservative forms. To overcome the problem of multiplication of distributions in space D', new spaces of generalized functions have been defined. (See refs. 23–27.) Colombeau (ref. 27, chapters 1–3) gives an intuitive description of the problem of multiplication of distributions and shows how to remedy this problem.

3.1.2. Shift operator. Let $f(x)$ be an ordinary function and define the shift operator as $E_h f(x) = f(x + h)$. Then, if $F[\phi]$ is the functional representing $f(x)$ by equation (2.1) and if the shifted function $E_h f(x)$ is represented by $E_h F[\phi]$, we have

$$\begin{aligned}
E_h F[\phi] &= \int f(x + h)\phi(x)\,dx \\
&= \int f(x)\phi(x - h)\,dx \\
&= F[E_{-h}\phi] \tag{3.4}
\end{aligned}$$

This rule can now be used for all generalized functions in D' because $E_{-h}\phi$ is in D. For example, $E_h \delta(x) = \delta(x + h)$ has the property

$$\begin{aligned}
E_h \delta[\phi] &= \int \delta(x + h)\phi(x)\,dx \\
&= \delta[\phi(x - h)] \\
&= \phi(-h) \tag{3.5}
\end{aligned}$$

Note that the integral in equation (3.5) is meaningless and stands for the functional $E_h \delta[\phi]$, which in turn is given by $\delta[E_{-h}\phi]$.

3.1.3. Equality of two generalized functions $f(x)$ and $g(x)$ on an Ω open set. Two generalized functions f and g in D' given by functionals $F[\phi]$ and $G[\phi]$ on D, respectively, are equal on an Ω open set if $F[\phi] = G[\phi]$ for all ϕ such that $supp\ \phi \subset \Omega$. For example, $\delta(x) = 0$ on open sets $(0, \infty)$ and $(-\infty, 0)$. Note that generalized functions are compared only on open intervals.

3.1.4. Support of a generalized function. The support of a generalized function $f(x)$ is the complement with respect to the real line of the open set on which $f(x) = 0$. For example, the support of $\delta(x)$ is the set $\{0\}$; that is, the point $x = 0$.

3.1.5. Sequence of generalized functions. A sequence of generalized functions $F_n[\phi]$ is convergent if the sequence of numbers $\{F_n[\phi]\}$ is convergent for all ϕ in D. For example, let

$$\delta_n(x) = \begin{cases} n^2\left(\frac{1}{n} - |x|\right) & \left(|x| \le \frac{1}{n}\right) \\ 0 & \left(|x| > \frac{1}{n}\right) \end{cases} \tag{3.6}$$

This function is shown in figure 2 and is, of course, an ordinary function. It can be shown that

$$\lim_{n\to\infty} \delta_n(x) = \delta(x) \tag{3.7}$$

Thus, for the functional $\delta_n[\phi]$ representing $\delta_n(x)$,

$$\delta_n[\phi] = \int \delta_n(x)\phi(x)\,dx \tag{3.8}$$

when ϕ is in D, we have

$$\lim_{n\to\infty} \delta_n[\phi] = \phi(0)$$
$$= \delta[\phi] \tag{3.9}$$

The index in the definition of convergence can be a continuous variable. For example, $F_\varepsilon[\phi]$ is convergent as $\varepsilon \to 0$ if $\lim_{\varepsilon\to 0} F_\varepsilon[\phi]$ exists for all ϕ in D.

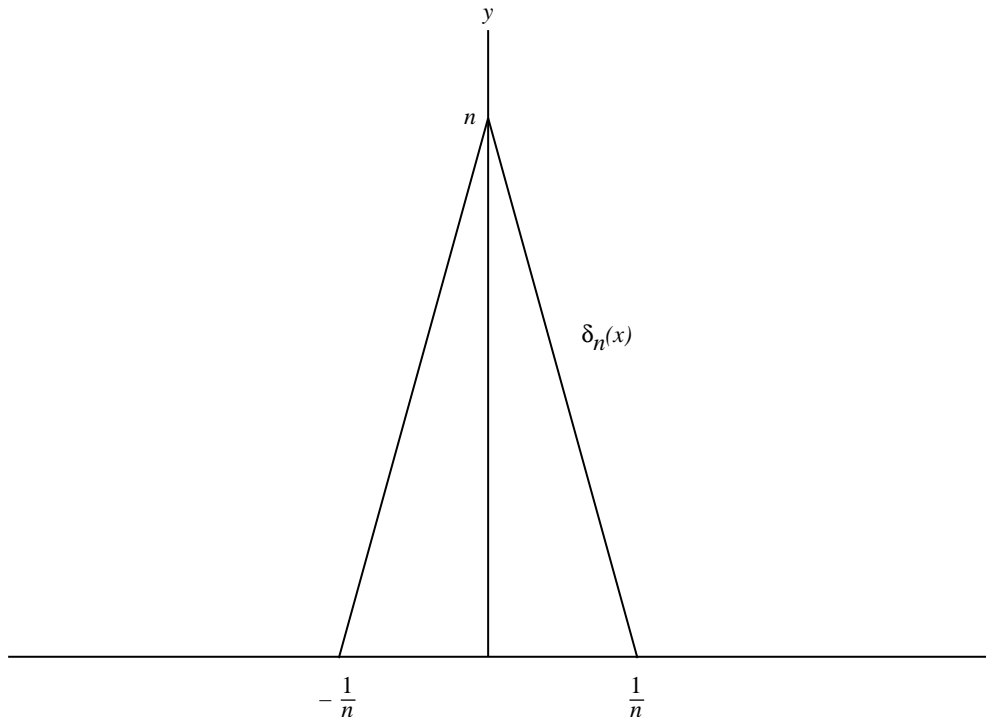The following important theorem characterizes D' and has significant applications. (See ref. 7.)

Figure 2. Example of δ sequence.

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Theorem: The space D' is complete.

This theorem implies that a convergent sequence of generalized functions in D' always converges to a generalized function in D'.

We use this theorem later in this section when we discuss the finite part of divergent integrals.

3.1.6. Odd and even generalized functions. A generalized function $F[\phi]$ is even if $F[\phi(-x)] = F[\phi(x)]$ and odd if $F[\phi(-x)] = -F[\phi(x)]$. For example $\delta(x)$ is even and x is odd.

3.1.7. Derivative of a generalized function. The derivative of a generalized function is the most important operation used in this paper. Let $f(x)$ be an ordinary function with a continuous first derivative (i.e., f is a C^1 function). If $f(x)$ is represented by the functional $F[\phi]$ in equation (2.1), then we naturally identify its derivative $f'(x)$ with $F'[\phi]$ given by the functional

$$F'[\phi] = \int f'\phi\,dx \tag{3.10}$$

Now we integrate by parts and use the fact that ϕ has compact support to get

$$F'[\phi] = -\int f\phi'\,dx$$
$$= -F[\phi'] \tag{3.11}$$

Because $\phi \in D$, then $\phi' \in D$. Thus, $F[\phi']$ is a functional on D. We now use equation (3.11) to define the derivative of all generalized functions in D'. We can keep taking higher order derivatives and obtain the following result:

$$F^{(n)}[\phi] = (-1)^n F\big[\phi^{(n)}\big] \qquad\qquad (n = 1, 2, ...) \tag{3.12}$$

We have thus arrived at the following important theorem.

Theorem: Generalized functions have derivatives of all orders.

We have obtained a very surprising result. Even locally Lebesgue integrable functions that are discontinuous are infinitely differentiable as generalized functions. What are the implications of this theorem in applications? We address this question about generalized derivatives and their applications in section 3.2. First, some examples would be helpful.

Example 1. The derivative of the delta function $\delta'(x)$ has the property,

$$\delta'[\phi] = -\delta[\phi']$$
$$= -\phi'(0) \tag{3.13}$$

Symbolically, we can write

$$\int \delta'(x)\phi(x)\,dx = -\phi'(0) \tag{3.14}$$

Note that $\delta'(x)$ is an odd generalized function.

Example 2. The Heaviside function is defined as

$$h(x) = \begin{cases} 1 & (x > 0) \\ 0 & (x < 0) \end{cases} \tag{3.15}$$

or in functional notation,

$$H[\phi] = \int_0^\infty \phi(x)\,dx \tag{3.16}$$

This function is discontinuous at $x = 0$. To define the generalized derivative, we use equation (3.11) as follows:

$$\begin{aligned}
H'[\phi] &= -H[\phi'] \\
&= -\int_0^\infty \phi'\,dx \\
&= \phi(0) \\
&= \delta[\phi]
\end{aligned} \tag{3.17}$$

Symbolically, we write

$$\bar{h}'(x) = \delta(x) \tag{3.18}$$

Note the use of the bar over h' to signify generalized differentiation because $h'(x) = 0$ where now h' stands for the ordinary derivative.

We give one more important characterization of space D' (ref. 7) known as the structure theorem of distribution theory.

Theorem: Generalized functions in D' are generalized derivatives of a finite order of continuous functions.

For example, we note that the Dirac delta function is the second generalized derivative of the continuous function

$$f(x) = \begin{cases} x & (x \geq 0) \\ 0 & (x < 0) \end{cases} \tag{3.19}$$

3.1.8. Fourier transforms of generalized functions. We now work with the space of rapidly decreasing test functions S. (See sec. 2.1, the last paragraph.) In this space the Fourier transform of each test function is again in S. (See refs. 2, 4, 6, and 7.) We define the Fourier transform of an ordinary function $\psi(x)$ as

$$\widehat{\psi}(\xi) = \int_{-\infty}^\infty \psi(x)e^{2\pi i x\xi}\,dx \tag{3.20}$$

Let $f(x)$ be an ordinary function that has the Fourier transform $\hat{f}(\xi)$ (e.g., let f be square integrable on $(-\infty, \infty)$). Then for $\psi(x)$ in S, the Parseval relation is

$$\int_{-\infty}^\infty \hat{f}(x)\psi(x)\,dx = \int_{-\infty}^\infty f(x)\widehat{\psi}(x)\,dx \tag{3.21}$$

If now $F[\psi]$ is identified with $f(x)$, then we should identify $\widehat{F}[\psi]$ with $\hat{f}(\xi)$. However, equation (3.21) is actually the relation

$$\widehat{F}[\psi] = F[\widehat{\psi}] \tag{3.22}$$

We use this relation as the definition of the Fourier transform of generalized functions in space S'. For example,

$$\widehat{\delta}[\psi] = \delta[\widehat{\psi}] = \widehat{\psi}(0) = \int_{-\infty}^\infty \psi(x)\,dx \tag{3.23}$$

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The last integral is the functional generated by the function 1 so that

$$\widehat{\delta}(\xi) = 1 \tag{3.24}$$

Thus, the Fourier transform of the Dirac delta function is the constant function 1.

We will not discuss this subject further because we do not use Fourier transforms extensively in this paper. We note, however, that if ψ is in D, then $\widehat{\psi}$ is not necessarily in D and equation (3.22) is meaningless in D'. Therefore, we must change the test function space from D to S. Another method of fixing this problem is to use the Fourier transforms of functions in space D as a new test function space $\widehat{D}$. The Fourier transformations of functions in D' are now continuous linear functionals on space $\widehat{D}$. These generalized functions are called ultradistributions. (See ref. 28.)

3.1.9. Exchange of limit processes. One of the most powerful results in generalized function theory is that the limit processes can be exchanged. For example, all the following exchanges are permissible:

$$\frac{\bar{d}}{dx} \int_\Omega \cdots = \int_\Omega \frac{\bar{d}}{dx} \cdots \tag{3.25a}$$

$$\frac{\bar{d}}{dx} \sum_n \cdots = \sum_n \frac{\bar{d}}{dx} \cdots \tag{3.25b}$$

$$\sum_n \int_\Omega \cdots = \int_\Omega \sum_n \cdots \tag{3.25c}$$

$$\lim_{n \to \infty} \int_\Omega \cdots = \int_\Omega \lim_{n \to \infty} \cdots \tag{3.25d}$$

$$\frac{\bar{d}}{dx} \lim_{n \to \infty} \cdots = \lim_{n \to \infty} \frac{\bar{d}}{dx} \cdots \tag{3.25e}$$

$$\lim_{n \to \infty} \sum_m \cdots = \sum_m \lim_{n \to \infty} \cdots \tag{3.25f}$$

$$\frac{\bar{\partial}^2}{\partial x_i \partial x_j} \cdots = \frac{\bar{\partial}^2}{\partial x_j \partial x_i} \cdots \tag{3.25g}$$

Here, as before, a bar over the derivative indicates generalized differentiation. For example, let us consider the Fourier series of the simple periodic function with period 2π

$$f(x) = \begin{cases} 1 & (0 < x < \pi) \\ -1 & (-\pi < x < 0) \end{cases} \tag{3.26}$$

which is

$$f(x) = \sum_{m=0}^\infty \frac{4}{(2m+1)\pi} \sin(2m+1)x \tag{3.27}$$

This function is shown in figure 3. The function $f(x)$ has a jump of $2(-1)^n$ at $x = n\pi$ for $n = 0, \pm 1, \pm 2$. By a result given in section 3.2.1 (eq. (3.43)),

$$\frac{\bar{d}f}{dx} = 2 \sum_{n=-\infty}^\infty (-1)^n \delta(x - n\pi) \tag{3.28}$$
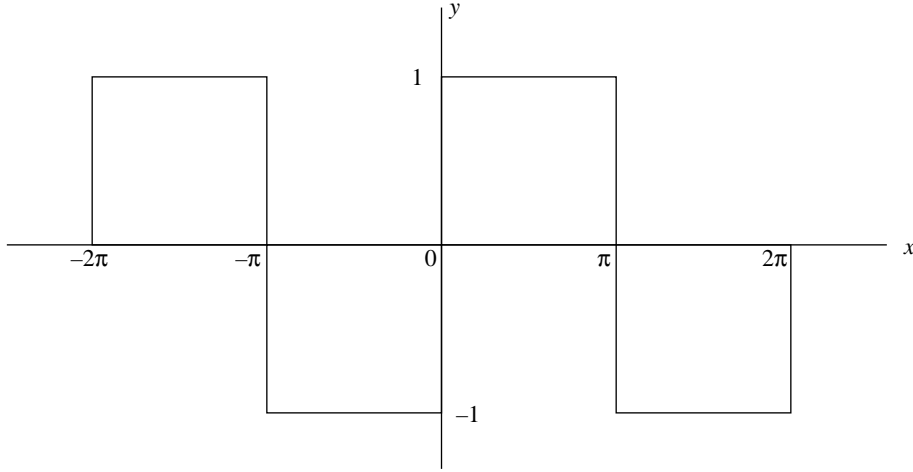
Figure 3. Periodic function with jump discontinuity of $2(-1)^n$ at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.

Also, by equation (3.25b), we have

$$\frac{\overline{df}}{dx} = \frac{\overline{d}}{dx} \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin(2m+1)x$$

$$= \sum_{m=0}^{\infty} \frac{\overline{d}}{dx} \frac{4}{(2m+1)\pi} \sin(2m+1)x$$

$$= \sum_{m=0}^{\infty} \frac{4}{\pi} \cos(2m+1)x \tag{3.29}$$

From equations (3.28) and (3.29), we conclude that

$$\frac{2}{\pi} \sum_{m=0}^{\infty} \cos(2m+1)x = \sum_{n=-\infty}^{\infty} (-1)^n \delta(x - n\pi) \tag{3.30}$$

The series on the left is divergent in the classical sense. Nevertheless, such a result is often useful in signal analysis. Another important application of exchange of limit processes is in obtaining the finite part of a divergent integral. (See sec. 3.4.)

3.1.10. Integration of generalized functions. We say that $G[\phi]$ is an integral of $F[\phi]$ if

$$G'[\phi] = F[\phi] \tag{3.31}$$

For example, we can easily show that the Heaviside function is an integral of the Dirac delta function because

$$H'[\phi] = -H[\phi']$$

$$= \phi(0)$$

$$= \delta[\phi] \tag{3.32}$$

Let $K[\phi]$ be a generalized function such that

$$K'[\phi] = 0 \tag{3.33}$$

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for all $\phi \in D$. Then, if $G[\phi]$ is an integral of $F[\phi]$, it follows that $(G + K)[\phi] = G[\phi] + K[\phi]$ is also an integral of $F[\phi]$. References 7 and 29 show that the only solution of equation (3.33) in D' is

$$K[\phi] = \int c\phi(x)\,dx \tag{3.34}$$

where c is an arbitrary constant (i.e., $K[\phi]$ is a constant distribution). This result corresponds to the classical indefinite integration of a function

$$\int f(x)\,dx = g(x) + c \tag{3.35}$$

We use the same notation symbolically for all generalized functions. For example, we write

$$\int \delta(x)\,dx = h(x) + c \tag{3.36}$$

where $h(x)$ is the Heaviside function. Note that the integral on the left of equation (3.36) is meaningless in terms of the classical integration theories.

3.2. Generalized Derivative

The generalized differentiation concept is quite important in generalized function theory; this section focuses on it and gives some very useful results for applications. Indeed, the results themselves, rather than the mathematical rigor used in deriving them, are of interest in this paper. As before, a bar over the differentiation symbol denotes generalized derivatives if there is an ambiguity in interpretation. For example, we use $\bar{d}f/dx$, $\bar{f}'(x)$, $\overline{\partial}f/\partial x_i$, and $\overline{\partial}^2 f/\partial x_i \partial x_j$ to denote generalized derivatives of ordinary functions, but we do not use a bar over $\delta'(x)$ and $\partial\delta(f)/\partial x_i$ because it is obvious that these derivatives can only be generalized derivatives because $\delta(x)$ and $\delta(f)$ are singular generalized functions.

3.2.1. Functions with discontinuities in one dimension. Let $f(x)$ be a piecewise smooth function with one discontinuity at x_0 with a jump at this point defined by the relation

$$\Delta f = f(x_{0+}) - f(x_{0-}) \tag{3.37}$$

We want to find the generalized derivative of $f(x)$. Let ϕ be in D and let x_0 be in the support of $\phi(x)$. Then if $F[\phi]$ is the functional representing $f(x)$ by equation (2.1), we have for $supp\ \phi = [a, b]$, the result

$$F'[\phi] = -F\left[\phi'\right]$$

$$= -\int_a^b f(x)\phi'(x)\,dx$$

$$= -\left[\int_a^{x_{0-}} + \int_{x_{0+}}^b f(x)\phi'(x)\,dx\right]$$

$$= \int_a^b f'(x)\phi(x)\,dx + [f(x_{0+}) - f(x_{0-})]\phi(x_0)$$

$$= \int_a^b f'(x)\phi(x)\,dx + \Delta f\,\phi(x_0) \tag{3.38}$$

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We have performed an integration by parts to get to the last step. We have also used the fact that $\phi(a) = \phi(b) = 0$ in the integration by parts. Noting that $\phi(x_0) = \delta[\phi(x + x_0)] = E_{-x_0}\delta[\phi(x)]$, where E_{-x_0} is the shift operator, we write equation (3.38) symbolically as

$$\bar{f}'(x) = f'(x) + \Delta f\, \delta(x - x_0) \tag{3.39}$$

One question is the use of $\bar{f}'(x)$ compared with the ordinary derivative $f'(x)$. Let us study equation (3.38). The functional $F'[\phi]$ corresponding to $\bar{f}'(x)$ indeed has retained the memory of the jump Δf on the right side of the equation. Symbolically, $\bar{f}'(x)$ can be integrated over $[c, x]$, where $c < x_0 < x$, to give the result

$$f(x) = \int_c^x f'(x)\, dx + f(c) + \Delta f h(x - x_o) \tag{3.40}$$

Thus, we have recovered the original discontinuous function. We note, however, that

$$f(x) \neq \int_c^x f'(x)\, dx + f(c) \tag{3.41}$$

because the memory of the jump Δf is not retained in $f'(x)$ but is retained in $\bar{f}'(x)$. If a function $f(x)$ has n discontinuities at x_i, $i = 1 - n$ with the jump Δf_i at x_i defined by

$$\Delta f_i = f(x_{i+}) - f(x_{i-}) \tag{3.42}$$

then

$$\bar{f}'(x) = f'(x) + \sum_{i=1}^{n} \Delta f_i\, \delta(x - x_i) \tag{3.43}$$

This equation is the first indication that when we work with discontinuous functions in applications, the proper setting for the problem is in the space of generalized functions. In particular, if an integral method, such as the approach that uses the Green's function, is used to find the solution, essentially no significant changes to algebraic manipulations are needed in finding discontinuous solutions provided we stay in the space of generalized functions. Again, we will have more to say about this later. (See sec. 3.2.3.)

3.2.2. Functions with discontinuities in multidimensions. Let us now consider the function $f(\mathbf{x})$, which is discontinuous across the surface $g(\mathbf{x}) = 0$. Let us define the jump Δf across $g = 0$ by the relation

$$\Delta f = f(g = 0+) - f(g = 0-) \tag{3.44}$$

Note that $g = 0+$ is on the side of the surface $g = 0$ into which ∇g points. We would like to find $\overline{\partial} f / \partial x_i$. To do this we use the results from section 3.2.1 as follows. Let us put a surface coordinate system (u^1, u^2) on $g = 0$ and extend the coordinates to the space in the vicinity of this surface along normals. Let $u^3 = g$ be the third coordinate variable that is well defined by the function g in the vicinity of this same surface. We note that f in variables u^1 and u^2 is continuous, but it is discontinuous in variable u^3. Therefore, we have

$$\frac{\overline{\partial} f}{\partial u^i} = \frac{\partial f}{\partial u^i} \qquad (i = 1, 2) \tag{3.45a}$$

$$\frac{\overline{\partial} f}{\partial u^3} = \frac{\partial f}{\partial u^3} + \Delta f\, \delta(u^3) \tag{3.45b}$$

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In equation (3.45b), we used equation (3.39). Thus, using the summation convention on index j, we get

$$\frac{\overline{\partial} f}{\partial x^i} = \frac{\overline{\partial} f}{\partial u^j} \frac{\partial u^j}{\partial x_i}$$

$$= \frac{\partial f}{\partial u^j} \frac{\partial u^j}{\partial x_i} + \Delta f \frac{\partial u^3}{\partial x_i} \delta(u^3)$$

$$= \frac{\partial f}{\partial x_i} + \Delta f \frac{\partial g}{\partial x_i} \delta(g) \tag{3.46}$$

We can write this in vector notation as

$$\overline{\nabla} f = \nabla f + \Delta f \, \nabla g \, \delta(g) \tag{3.47}$$

In section 3.2.3, we discuss how to interpret $\delta(g)$ when $g = 0$ is a surface. We can similarly define generalized divergence and curl as follows:

$$\overline{\nabla} \cdot \mathbf{f} = \nabla \cdot \mathbf{f} + \nabla g \cdot \Delta \mathbf{f} \, \delta(g) \tag{3.48a}$$

$$\overline{\nabla} \times \mathbf{f} = \nabla \times \mathbf{f} + \nabla g \times \Delta \mathbf{f} \, \delta(g) \tag{3.48b}$$

The rigorous derivation of both these results requires some knowledge of the invariant definition of divergence and curl in general curvilinear coordinate systems. (See refs. 8 and 9.) We can combine the above three results by using $*$ for the three operations such that

$$\overline{\nabla} * \mathbf{f} = \nabla * \mathbf{f} + \nabla g * \Delta \mathbf{f} \, \delta(g) \tag{3.49}$$

3.2.3. Ordinary differential equations and Green's function. We give a few simple results here. One important question discussed in connection with integrals of generalized functions is the solution of

$$\overline{f}'(x) = 0 \tag{3.50}$$

in D'. It can be shown easily that the only solution of this equation is the classical one (refs. 7 and 29)

$$f(x) = C \tag{3.51}$$

where C is a constant. However, the solution of the equation

$$x f(x) = 0 \tag{3.52}$$

which is not a differential equation, is

$$f(x) = C \delta(x) \tag{3.53}$$

To get this solution, some simple results from the generalized Fourier transform are used. (See ref. 29.) Taking the Fourier transform of both sides of equation (3.52), we get

$$\frac{\overline{d}}{d\xi} \hat{f}(\xi) = 0 \tag{3.54}$$

Therefore, after integration of equation (3.54), we have

$$\hat{f}(\xi) = C \tag{3.55}$$

By taking the inverse Fourier transform of both sides of equation (3.55), we get equation (3.53).

From this result, the solution of

$$x\bar{f}'(x) = 1 \tag{3.56}$$

is found as

$$f(x) = \ln|x| + C_1 + C_2 h(x) \tag{3.57}$$

where C_1 and C_2 are constants and $h(x)$ is the Heaviside function. The solution $C_2 h(x)$ comes from the fact that the generalized function $\bar{f}'(x)$ satisfying the equation

$$x\bar{f}'(x) = 0 \tag{3.58}$$

is, from equation (3.53),

$$\bar{f}'(x) = C_2 \delta(x) \tag{3.59}$$

Thus, the solution of the homogeneous equation (3.58) is the integral of this function

$$f(x) = C_1 + C_2 h(x) \tag{3.60}$$

Let us now consider a second order linear ordinary differential equation with two linear and homogeneous boundary conditions (BC) as follows:

$$\left. \begin{array}{l} \ell u = A(x)u'' + B(x)u' + C(x) = f(x) \\ \text{BC}_1[u] = 0 \\ \text{BC}_2[u] = 0 \end{array} \right\} \qquad (x \in [0,1]) \tag{3.61}$$

Let us also assume that we know u is a C^1 function and u'' is Lebesgue integrable so that $\bar{u}'' = u''$ and $\bar{u}' = u'$. Suppose a function $g(x,y)$ exists, the Green's function, such that

$$u(x) = \int_0^1 f(y)g(x,y)\,dy \tag{3.62}$$

Because $u \in C^1$, then $\bar{\ell}u = \ell u$ by continuity of u and u'. We know we can take $\bar{\ell}$ into the integral in equation (3.62) but not ℓ because $g(x,y)$ may not belong to C^1. Therefore, using ℓ_x to indicate that derivatives in ℓ are with respect to x, we get

$$\ell u = \bar{\ell}u$$
$$= \bar{\ell}_x \int_0^1 f(y)g(x,y)\,dy$$
$$= \int_0^1 f(y)\bar{\ell}_x g(x,y)\,dy$$
$$= f(x) \tag{3.63}$$

from equation (3.61). We see that $\bar{\ell}_x g(x,y)$ must have the sifting property

$$\bar{\ell}_x g(x,y) = \delta(x-y) \tag{3.64}$$

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Because the BC's are linear, we also have

$$\text{BC}[u] = \int_0^1 f(y)\,\text{BC}_x[g(x, y)]\,dy \tag{3.65}$$

Therefore, other conditions on $g(x, y)$ are

$$\text{BC}_{1x}[g(x, y)] = 0 \tag{3.66a}$$

$$\text{BC}_{2x}[g(x, y)] = 0 \tag{3.66b}$$

where the x in the subscripts of the BC's indicates that $g(x, y)$ in the variable x satisfies the two boundary conditions.

From equation (3.64) we conclude that, because $\bar{\ell}_x$ is a second order ordinary differential equation, $g(x, y)$ must be continuous at $x = y$ and $\partial g/\partial x$ must have a jump discontinuity at $x = y$. The reason is that if $g(x, y)$ has a discontinuity at $x = y$, the first generalized derivative with respect to x will give a $\delta(x - y)$ by equation (3.39). A second generalized derivative would give $\delta'(x - y)$ in the result. But because $\delta'(x - y)$ is missing on the right of equation (3.64), $g(x, y)$ cannot be discontinuous at $x = y$. Assuming that $g(x, y)$ is defined by

$$g(x, y) = \begin{cases} g_1(x, y) & (0 \le x < y) \\ g_2(x, y) & (y < x \le 1) \end{cases} \tag{3.67}$$

equation (3.64) means that

$$\ell_x g_1(x, y) = \ell_x g_2(x, y) = 0 \tag{3.68a}$$

$$g_1(y, y) = g_2(y, y) \tag{3.68b}$$

$$\frac{\partial g_2}{\partial x}(y, y) - \frac{\partial g_1}{\partial x}(y, y) = \frac{1}{A(y)} \tag{3.68c}$$

Note that equation (3.68a) is the same as $\ell_x g = 0$ used above. This equation means that g_1 and g_2 in variable x are solutions of the homogeneous equation $\ell u = 0$. Equation (3.68b) expresses continuity of g at $x = y$ and equation (3.68c) gives the jump of $\partial g/\partial x$ at $x = y$. To get equation (3.68c), we note that

$$\bar{\ell}_x g = \ell_x g + A(y)\left[\frac{\partial g_2}{\partial x}(y, y) - \frac{\partial g_1}{\partial x}(y, y)\right]\delta(x - y)$$

$$= A(y)\left[\frac{\partial g_2}{\partial x}(y, y) - \frac{\partial g_1}{\partial x}(y, y)\right]\delta(x - y)$$

$$= \delta(x - y) \tag{3.69}$$

The last delta function follows from equation (3.64). Equation (3.68c) follows from the fact that the coefficient of $\delta(x - y)$ in the expression after the second equality sign must be equal to 1. The Green's function is now determined from equations (3.66) and (3.68a–c).

3.2.4. Leibniz rule of differentiation under the integral sign. We want to find the result of taking the derivative with respect to variable α in the following expression in which A, B, and

f are continuous functions and $B(\alpha) > A(\alpha)$ for $\alpha \in [a, b]$. Thus,

$$E(\alpha) = \frac{d}{d\alpha} \int_{A(\alpha)}^{B(\alpha)} f(x, \alpha)\, dx \tag{3.70}$$

Let us define the function $H(x, \alpha)$ as follows:

$$H(x, \alpha) = h[x - A(\alpha)]h[B(\alpha) - x] \tag{3.71}$$

where $h(x)$ is the Heaviside function. The function $H(x, \alpha) = 1$ when $A(\alpha) < x < B(\alpha)$ and $H(x, \alpha) = 0$ otherwise. Using $H(x, \alpha)$, we can write $E(\alpha)$ as

$$E(\alpha) = \frac{\bar{d}}{d\alpha} \int_{-\infty}^{\infty} H(x, \alpha) f(x, \alpha)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\overline{\partial H}}{\partial \alpha} f + H \frac{\overline{\partial f}}{\partial \alpha} \right) dx \tag{3.72}$$

We have

$$\frac{\overline{\partial H}}{\partial \alpha}(x, \alpha) = -A'(\alpha)h[B(\alpha) - x]\delta[x - A(\alpha)]$$

$$+ B'(\alpha)h[x - A(\alpha)]\delta[B(\alpha) - x]$$

$$= -A'(\alpha)\delta[x - A(\alpha)] + B'(\alpha)\delta[B(\alpha) - x] \tag{3.73}$$

Note that we have used

$$h[B(\alpha) - x]\delta[x - A(\alpha)] = h[B(\alpha) - A(\alpha)]\delta[x - A(\alpha)]$$

$$= \delta[x - A(\alpha)] \tag{3.74}$$

because $B(\alpha) - A(\alpha) > 0$; thus, the Heaviside function is 1. Similarly, we do the same as in equation (3.74) for the second product of the Heaviside and the delta functions in equation (3.73). Using equation (3.73) in equation (3.72) and integrating with respect to x, we get the Leibniz rule of differentiation under the integral sign,

$$E(\alpha) = \int_{A(\alpha)}^{B(\alpha)} \frac{\partial f}{\partial \alpha}(x, \alpha)\, dx + B'(\alpha)f[B(\alpha), \alpha] - A'(\alpha)f[A(\alpha), \alpha] \tag{3.75}$$

3.3. Multidimensional Delta Functions

In multidimensions, $\delta(\mathbf{x})$ has a simple interpretation given by

$$\int \phi(\mathbf{x})\delta(\mathbf{x})\, d\mathbf{x} = \phi(0) \tag{3.76}$$

Thus,

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\ldots\delta(x_n) \tag{3.77}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In this section, we confine ourselves to three-dimensional space. Of interest in applications are $\delta(f)$ and $\delta'(f)$ where $f = 0$ is a surface in three-dimensional space.

We can always assume that f is defined so that $|\nabla f| = 1$ at every point on $f = 0$. If f does not have this property, then $f_1 = f/|\nabla f|$ does. Thus, redefine the surface.

3.3.1. Interpretation of $\delta(f)$. Consider the integral

$$I = \int \phi(\mathbf{x})\delta(f)\,d\mathbf{x} \tag{3.78}$$

Assume that we define a curvilinear coordinate system (u^1, u^2) on the surface $f = 0$ and extend these variables locally to the space near this surface along local normals. Let $u^3 = f$ which, because $|\nabla f| = df/du^3 = 1$, u^3 is the local distance from the surface. Thus, $f = u^3 = \text{constant}$ $\neq 0$ is a surface parallel to $f = 0$. Of course, we assume u^3 is small. From differential geometry (refs. 8 and 9), we have

$$d\mathbf{x} = \sqrt{g_{(2)}\left(u^1, u^2, u^3\right)}\,du^1\,du^2\,du^3 \tag{3.79}$$

where $g_{(2)}\left(u^1, u^2, u^3\right)$ is the determinant of the first fundamental form of the surface $f = u^3 = $ Constant. Using equation (3.79) in equation (3.78) and integrating with respect to u^3 gives

$$I = \int \phi\left[\mathbf{x}\left(u^1, u^2, u^3\right)\right]\delta\left(u^3\right)\sqrt{g_{(2)}\left(u^1, u^2, u^3\right)}\,du^1\,du^2\,du^3$$

$$= \int \phi\left[\mathbf{x}\left(u^1, u^2, 0\right)\right]\sqrt{g_{(2)}\left(u^1, u^2, 0\right)}\,du^1\,du^2$$

$$= \int_{f=0} \phi(\mathbf{x})\,dS \tag{3.80}$$

That is, I is the surface integral of ϕ over the surface $f = 0$.

3.3.2. Interpretation of $\delta'(f)$. We want to interpret

$$I = \int \phi(\mathbf{x})\delta'(f)\,d\mathbf{x}$$

$$= \int \phi\left[\mathbf{x}\left(u^1, u^2, u^3\right)\right]\delta'\left(u^3\right)\sqrt{g_{(2)}\left(u^1, u^2, u^3\right)}\,du^1\,du^2\,du^3 \tag{3.81}$$

Here we have used the coordinate system (u^1, u^2, u^3) defined above. Integrating the above equation with respect to u^3 gives

$$I = -\int \frac{\partial}{\partial u^3}\left[\phi(\mathbf{x})\sqrt{g_{(2)}\left(u^1, u^2, u^3\right)}\right]_{u^3=0}\,du^1\,du^2 \tag{3.82}$$

Again, from differential geometry, we have

$$\frac{\partial}{\partial u^3}\sqrt{g_{(2)}\left(u^1, u^2, u^3\right)} = -2H_f\left(u^1, u^2, u^3\right)\sqrt{g_{(2)}\left(u^1, u^2, u^3\right)} \tag{3.83}$$

where H_f stands for the local mean curvature of the surface $f = u^3 =$ Constant. Taking the derivative of the integrand of equation (3.82) and using the result of (3.83), we obtain

$$
I = -\int \frac{\partial \phi}{\partial u^3} \left[\mathbf{x}\left(u^1, u^2, 0\right) \right] \sqrt{g_{(2)}\left(u^1, u^2, 0\right)}\, du^1\, du^2
$$

$$
+ \int 2H_f\left(u^1, u^2, 0\right) \phi \left[\mathbf{x}\left(u^1, u^2, 0\right) \right] \sqrt{g_{(2)}\left(u^1, u^2, 0\right)}\, du^1\, du^2
$$

$$
= \int_{f=0} \left[-\frac{\partial \phi}{\partial n} + 2H_f(\mathbf{x})\phi(\mathbf{x}) \right] dS \tag{3.84}
$$

where $\partial \phi / \partial n$ is the usual normal derivative of ϕ. Intuitively, the appearance of the term $2H_f\phi$ in the integrand is not at all obvious. This appearance is a clear indication of the importance of differential geometry in multidimensional generalized function theory.

3.3.3. A simple trick. We have already shown that $\phi(x)\delta(x) = \phi(0)\delta(x)$. By taking the derivative of both sides of this equation, we get

$$
\phi'(x)\delta(x) + \phi(x)\delta'(x) = \phi(0)\delta'(x) \tag{3.85}
$$

Obviously, the right side is simpler than the left side. Let us consider the expression

$$
E = \phi(\mathbf{x})\delta(f) = \phi\left[\mathbf{x}\left(u^1, u^2, u^3\right) \right] \delta\left(u^3\right) \tag{3.86}
$$

where again we have used the coordinate system $\left(u^1, u^2, u^3\right)$ defined in section 3.3.1 above. We know that

$$
\phi\left[\mathbf{x}\left(u^1, u^2, u^3\right) \right] \delta\left(u^3\right) = \phi\left[\mathbf{x}\left(u^1, u^2, 0\right) \right] \delta\left(u^3\right) \tag{3.87}
$$

We use the notation $\underset{\sim}{\phi}(\mathbf{x})$ for $\phi\left[\mathbf{x}\left(u^1, u^2, 0\right) \right]$; that is, $\underset{\sim}{\phi}(\mathbf{x})$ is the restriction of $\phi(\mathbf{x})$ to the support of the delta function that is the surface $f = 0$. We note that $\partial \underset{\sim}{\phi} / \partial n = (\partial / \partial u^3)\phi\left[\mathbf{x}\left(u^1, u^2, 0\right) \right] = 0$. Using $\underset{\sim}{\phi}(\mathbf{x})$, we can write E in two forms:

$$
\left. \begin{array}{ll} E = \phi(\mathbf{x})\delta(f) & \text{(First form)} \\[2mm] E = \underset{\sim}{\phi}(\mathbf{x})\delta(f) & \text{(Second form)} \end{array} \right\} \tag{3.88}
$$

Is there an advantage of using the second form compared with the first form? The answer is yes! Let us take the gradient of E for the two forms in equation (3.88). Thus,

$$
\left. \begin{array}{ll} \overline{\nabla}E = \nabla\phi\, \delta(f) + \phi(\mathbf{x})\, \nabla f\, \delta'(f) & \text{(First form)} \\[2mm] \overline{\nabla}E = \nabla_2\underset{\sim}{\phi}\, \delta(f) + \underset{\sim}{\phi}(\mathbf{x})\, \nabla f\, \delta'(f) & \text{(Second form)} \end{array} \right\} \tag{3.89}
$$

Here, $\nabla_2\underset{\sim}{\phi}$ is the surface gradient of $\underset{\sim}{\phi}(\mathbf{x})$ on $f = 0$. From equation (3.84), we note that in the integration of $\delta'(f)$ in the first form, the term $\partial \phi / \partial n$ cancels a similar term in the integration of $\nabla\phi\, \delta(f)$. In the second form, because $\partial \underset{\sim}{\phi} / \partial n = 0$, $\partial \underset{\sim}{\phi} / \partial n$ does not appear in the integration of $\delta'(f)$ and obviously is also absent in the integration of $\nabla_2\underset{\sim}{\phi}\, \delta(f)$. Therefore,

algebraic manipulations are reduced. It is, thus, expedient to restrict functions multiplying the Dirac delta function to the support of the delta function. Note carefully that functions multiplying $\delta'(x)$ cannot be restricted to the support of $\delta'(x)$; that is, $\phi(x)\delta'(x) \neq \phi(0)\delta'(x)$.

3.3.4. The divergence theorem revisited. Let Ω be a finite volume in space and let $\phi(\mathbf{x})$ be a C^1 vector field. Let us define the discontinuous vector field $\phi_1(\mathbf{x})$ as

$$\phi_1(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & (\mathbf{x} \in \Omega) \\ 0 & (\mathbf{x} \notin \Omega) \end{cases} \tag{3.90}$$

Let the surface $f = 0$ denote the boundary $\partial\Omega$ of region Ω in such a way that $\mathbf{n} = \nabla f$ points to the outside of $\partial\Omega$ and $|\nabla f| = 1$ on $f = 0$. We have

$$\overline{\nabla} \cdot \phi_1 = \nabla \cdot \phi_1 + \Delta\phi_1 \cdot \mathbf{n}\,\delta(f)$$
$$= \nabla \cdot \phi_1 - \phi(\mathbf{x}) \cdot \mathbf{n}\,\delta(f) \tag{3.91}$$

We note that $\Delta\phi_1 = \phi_1(f = 0+) - \phi_1(f = 0-) = -\phi(f = 0)$. Integrating over the unbounded three dimensional space, we get

$$\int \int \int \frac{\overline{\partial}\phi_{1,1}}{\partial x_1} dx_1\, dx_2\, dx_3 = \int \int \phi_1 \Big|_{-\infty}^{\infty} dx_2\, dx_3 = 0 \tag{3.92}$$

Similarly, we get zero for integrals of $\overline{\partial}\phi_{1,2}/\partial x_2$ and $\overline{\partial}\phi_{1,3}/\partial x_3$, where $\phi_{1,i}$ is the ith component of ϕ_1. Therefore,

$$\int \overline{\nabla} \cdot \phi_1\, d\mathbf{x} = 0 \tag{3.93}$$

Now, the integration of the right side of equation (3.91) using equation (3.80) gives

$$\int_\Omega \nabla \cdot \phi\, d\mathbf{x} - \int_{\partial\Omega} \phi_n\, dS = 0 \tag{3.94}$$

Here we have used the fact that, from equation (3.90),

$$\nabla \cdot \phi_1 = \begin{cases} \nabla \cdot \phi & (\mathbf{x} \in \Omega) \\ 0 & (\mathbf{x} \notin \Omega) \end{cases} \tag{3.95}$$

Also, we define $\phi_n = \phi \cdot \mathbf{n}$. Equation (3.94) is the divergence theorem.

We note that equation (3.93) is valid if ϕ_1 has a discontinuity across the surface $k = 0$ within Ω as shown in figure 4. Equation (3.94) is therefore valid if $\nabla \cdot \phi$ in the volume integral is replaced by $\overline{\nabla} \cdot \phi$, where the only jump of ϕ in the generalized divergence comes from the discontinuity on $k = 0$. That is, we write

$$\overline{\nabla} \cdot \phi = \nabla \cdot \phi + \Delta\phi \cdot \mathbf{n}'\,\delta(k) \tag{3.96}$$

where $\mathbf{n}' = \nabla k$ is the unit normal to $k = 0$. Equation (3.94) can now be written

$$\int_\Omega \overline{\nabla} \cdot \phi\, d\mathbf{x} = \int_{\partial\Omega} \phi_n\, dS \tag{3.97}$$
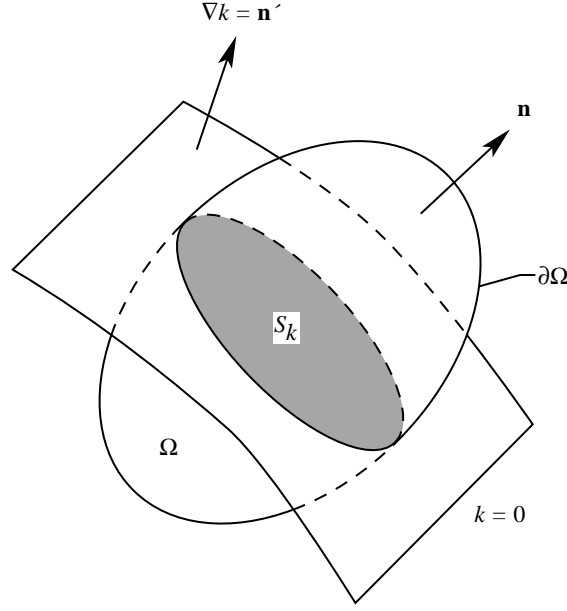
Figure 4. Control volume Ω intersecting surface of discontinuity of vector field ϕ used for deriving generalized divergence theorem.

which, by using equation (3.96), we can also write as

$$\int_\Omega \nabla \cdot \phi \, d\mathbf{x} = \int_{\partial\Omega} \phi_n \, dS - \int_{S_k} \Delta\phi_{n'} \, dS \tag{3.98}$$

where $\Delta\phi_{n'} = \Delta\phi \cdot \mathbf{n}'$ and S_k is the part of the surface $k = 0$ enclosed in region Ω. (See fig. 4.)

The divergence theorem is used in deriving conservation laws in fluid mechanics and physics in differential form. The fact that it remains valid for discontinuous vector fields, as shown in equation (3.97), implies that such conservation laws are valid when all the derivatives are interpreted as generalized derivatives. Thus, the jump conditions across the surface of discontinuities are inherent in these conservation laws as shown in section 3.4. This interpretation of conservation laws eliminates the need for the pillbox analysis of jump conditions.

3.3.5. Product of two delta functions. We have said earlier that the product of two arbitrary generalized functions generally may not be defined. Here we give the interpretation of the product of two multidimensional generalized functions for which multiplication is possible. Let $f = 0$ and $g = 0$ be two surfaces intersecting along a curve Γ as shown in figure 5. Assume $\nabla f = \mathbf{n}$ and $\nabla g = \mathbf{n}'$, where $|\mathbf{n}| = |\mathbf{n}'| = 1$. We want to interpret

$$I = \int \phi(\mathbf{x})\delta(f)\delta(g) \, d\mathbf{x} \tag{3.99}$$

On the local plane normal to the Γ-curve, define $u^1 = f$, $u^2 = g$, and $u^3 = \Gamma$, where Γ is the distance along the Γ-curve. Extend u^1 and u^2 to the space in the vicinity of the plane along a local normal to the plane. We have

$$d\mathbf{x} = \frac{du^1 \, du^2 \, du^3}{\sin\theta} \tag{3.100}$$
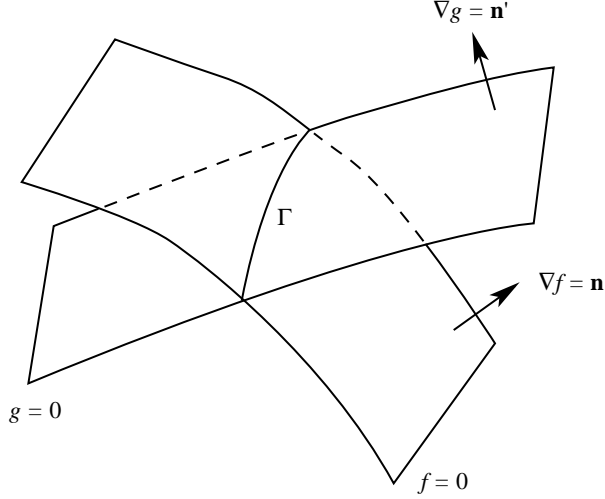
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Figure 5. Integration of $\delta(f)\,\delta(g)$ for two intersecting surfaces $f = 0$ and $g = 0$.

where $\sin\theta = |\mathbf{n} \times \mathbf{n}'|$. Using equation (3.100) in equation (3.99) and integrating the resulting integral with respect to u^1 and u^2, we get

$$I = \int \frac{\phi(\mathbf{x})}{\sin\theta}\,\delta\left(u^1\right)\,\delta\left(u^2\right)\,du^1\,du^2\,du^3 = \int_{\substack{f=0 \\ g=0}} \frac{\phi(\mathbf{x})}{\sin\theta}\,d\Gamma \tag{3.101}$$

This result is useful in applications. (See sec. 4.3.)

3.4. Finite Part of Divergent Integrals

The finite part of divergent integrals is important in aerodynamics. The classical procedure for finding the finite part of divergent integrals appears ad hoc and leads to questions about the validity of the procedure. First, could the appearance of divergent integrals in applications be the result of errors in modeling the physics of the problem? Second, will the method lead to a unique analytical expression or do different analytical expressions lead to equivalent numerical results? The generalized function theory clearly answers these questions.

Let us first examine the function $f(x) = \ln|x|$, which is locally integrable. The ordinary derivative of this function is

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \tag{3.102}$$

which is not locally integrable over any interval that includes $x = 0$. We know, however, that as a generalized function, $\ln|x|$ has generalized derivatives of all orders. What is the relation of the generalized derivative of $\ln|x|$ to the ordinary derivative $f'(x) = 1/x$?

Let us work with $F[\phi]$ representing $\ln|x|$ as follows:

$$F[\phi] = \int \ln|x|\phi(x)\,dx \qquad (\phi \in D) \tag{3.103}$$

We have, using the definition of generalized derivative,

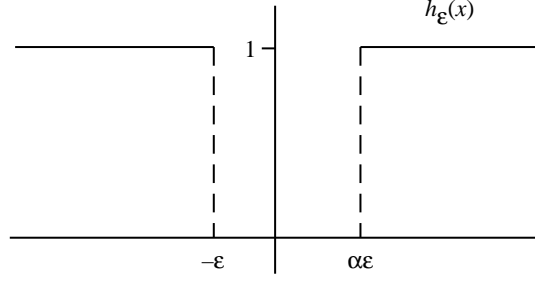$$F'[\phi] = -\int \ln|x|\phi'(x)\,dx \tag{3.104}$$

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Figure 6. Function $h_\varepsilon(x)$ used in defining finite part of divergent integral $\int [\phi(x)/x]\,dx$. $\varepsilon > 0$; $\alpha > 0$.

We need some integration by parts to get the term $1/x$ in the integrand of equation (3.104). However, this integration cannot be performed because $1/x$ is not locally integrable. We solve this problem by using a new functional depending on ε, the limit of which is $F'[\phi]$ as follows. Let $h_\varepsilon(x)$ be a function defined below for some constant $\alpha > 0$ and a parameter $\varepsilon > 0$. Thus,

$$h_\varepsilon(x) = \begin{cases} 0 & (-\varepsilon < x < \alpha\varepsilon) \\ 1 & (\text{Otherwise}) \end{cases} \tag{3.105}$$

This function is shown in figure 6. Then it is obvious that $\ln|x|$ can be written as the limit of an indexed generalized function as follows

$$\lim_{\varepsilon \to 0} h_\varepsilon(x)\ln|x| = \ln|x| \tag{3.106}$$

Note that if we define $F'_\varepsilon[\phi]$ as

$$F'_\varepsilon[\phi] = -\int h_\varepsilon(x)\ln|x|\phi'(x)\,dx \tag{3.107}$$

then we have from the completeness theorem of D' (sec. 3.1.5)

$$\lim_{\varepsilon \to 0} F'_\varepsilon[\phi] = F'[\phi] \tag{3.108}$$

The function $h_\varepsilon(x)\ln|x|$ has two jump discontinuities at $x = -\varepsilon$ and $x = \alpha\varepsilon$. We can either apply the classical integration by parts to equation (3.107) by breaking the real line into two intervals or by using the generalized derivative

$$F'_\varepsilon[\phi] = \int \frac{\bar{d}}{dx}[h_\varepsilon(x)\ln|x|]\phi(x)\,dx \tag{3.109}$$

Here we are integrating over $supp\,\phi$ and we do not worry about the terms coming from the limits of the integral in the integration by parts because $\phi = 0$ at the limit points.

We now take the derivative of the term in square brackets in equation (3.109):

$$\frac{\bar{d}}{dx}[h_\varepsilon(x)\ln|x|] = \frac{h_\varepsilon(x)}{x} - \ln\varepsilon\,\delta(x + \varepsilon) + \ln(\alpha\varepsilon)\delta(x - \alpha\varepsilon) \tag{3.110}$$

Here and below, we have used the result that $\phi(x)\delta(x - x_0) = \phi(x_0)\delta(x - x_0)$. Thus, after using equation (3.110) in equation (3.109) and integrating with respect to x, we have

$$F'_\varepsilon[\phi] = -\ln\varepsilon\,\phi(-\varepsilon) + \ln(\alpha\varepsilon)\phi(\alpha\varepsilon) + \int \frac{h_\varepsilon(x)}{x}\phi(x)\,dx$$

$$= \phi(0)\ln\alpha + \int \frac{h_\varepsilon(x)}{x}\phi(x)\,dx + o(\varepsilon) \tag{3.111}$$

where $o(\varepsilon)$ stands for terms of order ε and higher. Now from equation (3.108) we have

$$F'[\phi] = \lim_{\varepsilon\to 0} F'_\varepsilon[\phi]$$

$$= \phi(0)\ln\alpha + \lim_{\varepsilon\to 0}\int \frac{h_\varepsilon(x)}{x}\phi(x)\,dx$$

$$= \phi(0)\ln\alpha + \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}dx + \int_{\alpha\varepsilon}^{\infty}\frac{\phi(x)}{x}dx\right] \tag{3.112}$$

We can show that the limit of the integral on the right of equation (3.112) exists. If now $\alpha = 1$, then $\ln\alpha = 0$ and

$$F'[\phi] = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon}\frac{\phi(x)}{x}dx + \int_{\varepsilon}^{\infty}\frac{\phi(x)}{x}dx\right] \tag{3.113}$$

which is known as the Cauchy principal value (PV) of the integral. But $\alpha = 1$ need not be taken and equation (3.112) is numerically the same as equation (3.113). The above limit procedure is called taking the finite part of a divergent integral.

What have we achieved? Over any open interval that does not include $x = 0$ we have

$$\frac{\bar{d}}{dx}\ln|x| = \frac{1}{x} \tag{3.114}$$

but when $x = 0$ belongs to the open interval, then the classically divergent integral must be interpreted such that the functional $F'[\phi]$ corresponding to $(\bar{d}/dx)\ln|x|$ is recovered. As the above simple function demonstrates, more than one different analytical expression for the procedure can be used to find the finite part of a divergent integral. However, all the expressions are numerically equivalent. We define the principal value of $1/x$ as

$$\text{PV}\left(\frac{1}{x}\right) = \frac{\bar{d}}{dx}\ln|x| \tag{3.115}$$

Thus, when $x = 0$ is in the interval of integration of $1/x$, the finite part of the divergent integral must by taken to get the numerical value of $F'[\phi]$, where $F[\phi]$ is given by equation (3.103). Note that the term regularizing a divergent integral is also used in mathematics. The procedure given here corresponds to the canonical regularization of Gel'fand and Shilov. (See ref. 7.)

What is the use of this procedure in applications? Suppose we have reduced the solution of a problem to the evaluation of the expression

$$u(x) = \frac{d}{dx}\int_\Omega \phi(y)\ln|x - y|\,dy \tag{3.116}$$

where $x \in \Omega$. Let us assume that we know that the integral is continuous as a function of x so that d/dx can be replaced by $\bar{d}/dx$ and taken inside the integral. We get

$$u(x) = \int_\Omega \phi(y) \frac{\bar{d}}{dx} \ln|x - y| \, dy$$

$$= \int_\Omega \phi(y) \mathrm{PV}\left(\frac{1}{x - y}\right) dy \tag{3.117}$$

which is interpreted as the finite part of the divergent integral by the procedure defined earlier. We remind the readers that the procedure will result in exactly what equation (3.116) would give had we been able to perform the integration analytically. Also, assuming that $\phi = 0$ at the boundaries of Ω, an integration by parts of the first integral in equation (3.117) would give

$$u(x) = -\int_\Omega \phi'(y) \ln|x - y| \, dy \tag{3.118}$$

which is also a legitimate result if this integral exists. The problem is that often in applications, equation (3.116) is an integral equation for the unknown function $\phi(x)$, which has integrable singularities at the boundaries of the interval Ω. Thus, the above integration by parts is invalid and, in any case, the integral equation (3.118) is divergent. Therefore, the only choice left is the integral equation with the principal value of $1/(x - y)$, which is a well-known kernel in the theory of singular integral equations.

We now give an advanced example in three dimensions with a surprising implication in the numerical solution of an integral equation of transonic flow which we will discuss in section 4. Let us consider the integral

$$I(\mathbf{x}) = \frac{\partial^2}{\partial x_1^2} \int_\Omega \frac{\phi(\mathbf{y})}{r} \, d\mathbf{y} \tag{3.119}$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 \tag{3.120}$$

where Ω is a region in space and $\mathbf{x} \in \Omega$. In this problem $\phi(\mathbf{x})$ is a C^1 function and is the unknown of the aerodynamic problem. Assuming that the integral is a C^1 function in $\mathbf{x}$, we can replace $\partial^2/\partial x_1^2$ with $\overline{\partial}^2/\partial x_1^2$ and take the derivatives inside the integral

$$I(\mathbf{x}) = \int_\Omega \phi(\mathbf{y}) \frac{\overline{\partial}^2}{\partial x_1^2}\left(\frac{1}{r}\right) d\mathbf{y}$$

$$= \int_\Omega \phi(\mathbf{y}) \frac{\overline{\partial}^2}{\partial y_1^2}\left(\frac{1}{r}\right) d\mathbf{y} \tag{3.121}$$

We use generalized differentiation rather than ordinary differentiation because the latter will result in a divergent integral. Note that

$$\frac{\partial}{\partial y_1}\left(\frac{1}{r}\right) = \frac{r_1}{r^3} \tag{3.122a}$$

$$\frac{\partial^2}{\partial y_1^2}\left(\frac{1}{r}\right) = \frac{3r_1^2 - r^2}{r^5} \tag{3.122b}$$

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where $r_1 = x_1 - y_1$. Because r_1/r^3 is integrable, we write

$$\frac{\overline{\partial}^2}{\partial y_1^2}\left(\frac{1}{r}\right) = \frac{\overline{\partial}}{\partial y_1}\left(\frac{r_1}{r^3}\right) \tag{3.123}$$

and we proceed to find the finite part of the divergent integral in equation (3.121).

Let $f(\mathbf{y}, \mathbf{x}, \varepsilon) = g(r_1, r_2, r_3) - \varepsilon = 0$ be a piecewise smooth surface enclosing the point $\mathbf{y} = \mathbf{x}$ where $r_i = x_i - y_i$, $i = 1\text{--}3$ and g is a homogeneous function of order 1; that is, $g(\alpha r_1, \alpha r_2, \alpha r_3) = \alpha g(r_1, r_2, r_3)$. This condition assures that the surface $g(\alpha r_1, \alpha r_2, \alpha r_3) - \varepsilon = 0$ corresponds to $g(r_1, r_2, r_3) - \varepsilon/\alpha = 0$ for $\alpha \neq 0$. Thus, all the surfaces $g - \varepsilon = 0$ correspond to various values of α that are similar in shape. From the homogeneity of g, it follows that $f(\mathbf{y}, \mathbf{x}, 0) = g(r_1, r_2, r_3) = 0$ consists of a single-point $\mathbf{y} = \mathbf{x}$. For example, for a sphere with a center at $\mathbf{y} = \mathbf{x}$ and radius ε, we have

$$f(\mathbf{y}, \mathbf{x}, \varepsilon) = \sqrt{r_1^2 + r_2^2 + r_3^2} - \varepsilon = 0 \tag{3.124}$$

In addition, we assume $\nabla_y f = \mathbf{n}$, where $\mathbf{n}$ is the local unit outward normal to the surface. Let $f > 0$ outside and $f < 0$ inside this surface, respectively. We introduce the function $h_\varepsilon(\mathbf{y})$ by the relation

$$h_\varepsilon(\mathbf{y}) = \begin{cases} 1 & (f > 0) \\ 0 & (f < 0) \end{cases} \tag{3.125}$$

Now, we define the required generalized derivative in equation (3.123) by the relation

$$\frac{\overline{\partial}^2}{\partial y_1^2}\left(\frac{1}{r}\right) = \lim_{\varepsilon \to 0} \frac{\overline{\partial}}{\partial y_1}\left[\frac{h_\varepsilon(\mathbf{y}) r_1}{r^3}\right]$$

$$= \lim_{\varepsilon \to 0}\left[\frac{r_1 n_1}{r^3}\delta(f) + \frac{3r_1^2 - r^2}{r^5}h_\varepsilon(\mathbf{y})\right] \tag{3.126}$$

where n_1 is the component of $\mathbf{n}$ along the y_1-axis. Therefore, $I(x)$ can be written

$$I(\mathbf{x}) = \lim_{\varepsilon \to 0}\int_{f=0}\frac{r_1\, n_1}{r^3}\phi(\mathbf{y})\, dS$$

$$+ \lim_{\varepsilon \to 0}\int_\Omega \frac{3r_1^2 - r^2}{r^5}h_\varepsilon(\mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} \tag{3.127}$$

where we have used equation (3.80) to integrate $\delta(f)$ in equation (3.126).

Using a Taylor series expansion of $\phi(\mathbf{y})$ at $\mathbf{y} = \mathbf{x}$, we find that

$$\lim_{\varepsilon \to 0}\int_{f=0}\frac{r_1\, n_1}{r^3}\phi(\mathbf{y})\, dS = \alpha_f \phi(\mathbf{x}) \tag{3.128}$$

where α_f is a constant depending on the shape of the surface $f = 0$. For example, for the sphere given by equation (3.124), we have

$$\alpha_f = \frac{4\pi}{3} \tag{3.129}$$

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If we take the surface $f = 0$ to be a circular cylinder with its axis parallel to the y_1-axis such that the base radius is ε and its height is $\gamma, \frac{\varepsilon}{\gamma} \gg 1$, then

$$\alpha_f = 4\pi \tag{3.130}$$

Equation (3.127) is thus written

$$I(\mathbf{x}) = \alpha_f \phi(\mathbf{x}) + \lim_{\varepsilon \to 0} \int_\Omega \frac{3r_1^2 - r^2}{r^5} h_\varepsilon(\mathbf{y}) \phi(\mathbf{y}) \, d\mathbf{y} \tag{3.131}$$

Numerically, $I(\mathbf{x})$ is the same regardless of the shape of $f = 0$. Because $\left(3r_1^2 - r^2\right)/r^5$ near $\mathbf{y} = \mathbf{x}$ takes both positive and negative values, the shape of $f = 0$ as $\varepsilon \to 0$ affects the value of the integral in the summation process. This effect is similar to a well-known result for conditionally convergent series, which can be made to converge to any value by rearranging the terms of the series. The term $\alpha_f \phi(\mathbf{x})$ in equation (3.131) compensates for the change in the value of the volume integral when $f = 0$ is changed so that $I(\mathbf{x})$ is numerically the same.

What is the implication of the above result in applications? In practice, the volume integration is performed numerically. The volume integral has a hole enclosing $\mathbf{y} = \mathbf{x}$ whose boundary surface is given by $f = 0$. The value of α_f must, therefore, correspond to the grid system used in the volume integration. If the hole is rectangular, which is often the case, then neither of the above two α_f's in equations (3.129) and (3.130) is appropriate for the problem.

One question remains unanswered. When does the appearance of a divergent integral imply anything other than the breakdown of the physical modeling? The answer is when we have wrongly taken an ordinary derivative inside an improper integral. Such a step can make the integral divergent and is caused by the wrong mathematics (improper procedure) rather than the wrong physics. Thus, the analyst should always check the cause of the appearance of divergent integrals in applications. Because in classical aerodynamics, the inappropriate mathematics generally causes the appearance of divergent integrals, the finite part of divergent integrals must be used.

4. Applications

4.1. Introduction

In this section we give some applications in aerodynamics and aeroacoustics that show the power and the beauty of generalized function theory. We use the results of the previous sections here. Many areas of aerodynamics and aeroacoustics can use generalized function theory, especially because the approach is almost always more direct and simpler than other methods. In addition, for many problems involving partial differential equations, no alternate method is available for finding a solution. Below is a partial list of applications of generalized function theory in aerodynamics, fluid mechanics, and aeroacoustics:

Aerodynamics and fluid mechanics

    Derivation of transport theorems

    Derivation of governing conservation laws (such as two-phase flows)

    Derivation of jump conditions across flow discontinuities, velocity discontinuity as a vortex sheet

    Derivation of the governing equation for boundary element or field panel methods

    Subsonic, transonic, and supersonic aerodynamic theory

Sound from moving singularities

Derivation of the governing equation for the boundary element method

Derivation of the Kirchhoff formula for moving surfaces

Study of noise from moving surfaces using the acoustic analogy

Identification of new noise generation mechanisms and their source strength (such as shock noise)

In addition, in both aerodynamics and aeroacoustics, generalized function theory can help in the derivation of geometric identities involving curves, surfaces, and volumes, particularly under deformation and in motion.

4.2. Aerodynamic Applications

We give here four applications that have been previously derived by other classical methods. The method based on generalized function theory, as expected, is much shorter and more elegant. Other examples in aerodynamics are presented by De Jager. (See ref. 5.)

4.2.1. Two transport theorems. We give two results here that are used in the derivation of conservation laws. We want to take the time derivative inside the integral

$$I = \frac{d}{dt} \int_{\Omega(t)} Q(\mathbf{x}, t) \, d\mathbf{x} \tag{4.1}$$

where $\Omega(t)$ is a time-dependent region of space and $Q(\mathbf{x}, t)$ is a C^1 function. Let us assume the boundary $\partial\Omega(t)$ of Ω is piecewise smooth and is given by the surface $f = 0$ such that $f > 0$ in Ω. Assume also that $\nabla f = \mathbf{n}'$ where $\mathbf{n}'$ is the unit inward normal to the surface. Suppose we can ascertain that the integral in equation (4.1) is continuous in time. Then, we can replace d/dt with $\bar{d}/dt$ and bring the derivative inside the integral. We write

$$I = \frac{\bar{d}}{dt} \int h(f) Q(\mathbf{x}, t) \, d\mathbf{x}$$

$$= \int \left[\frac{\partial f}{\partial t} \delta(f) Q(\mathbf{x}, t) + h(f) \frac{\partial Q}{\partial t} \right] d\mathbf{x}$$

$$= \int_{\partial\Omega(t)} \frac{\partial f}{\partial t} Q(\mathbf{x}, t) \, dS + \int_{\Omega(t)} \frac{\partial Q}{\partial t} \, d\mathbf{x} \tag{4.2}$$

where $h(f)$ is the Heaviside function. Here we have used equation (3.80) to integrate $\delta(f)$ in the second step above. We can show that

$$\frac{\partial f}{\partial t} = -v_{n'} = v_n \tag{4.3}$$

where $v_{n'}$ and v_n are the local normal velocities in the direction of inward and outward normals, respectively. Thus,

$$I = \int_{\partial\Omega(t)} v_n Q(\mathbf{x}, t) \, dS + \int_{\Omega(t)} \frac{\partial Q}{\partial t} d\mathbf{x} \tag{4.4}$$

This equation is the generalization of the Leibniz rule of differentiation of integrals in one dimension.

For the second result, we want to take the time derivative inside the following integral by assuming again that the integral is continuous in time and that Q is a C^1 function. Thus,

$$I = \frac{d}{dt} \int_{\partial\Omega(t)} Q(\mathbf{x},t)\,dS \tag{4.5}$$

We first convert the surface integral into a volume integral

$$I = \frac{\bar{d}}{dt} \int \delta(f) \underset{\sim}{Q}(\mathbf{x},t)\,d\mathbf{x} \tag{4.6}$$

Here $f = 0$ describes $\partial\Omega(t)$ and $\nabla f = \mathbf{n}$, where $\mathbf{n}$ is the unit outward normal. Also, note that $\underset{\sim}{Q}$ is the restriction of Q to $f = 0$ as explained in section 3.3. Therefore,

$$I = \int \left[\frac{\partial f}{\partial t} \delta'(f) \underset{\sim}{Q}(\mathbf{x},t) + \delta(f) \frac{\partial \underset{\sim}{Q}}{\partial t} \right] d\mathbf{x} \tag{4.7}$$

We now must use the results of section 3.3 to integrate $\delta'(f)$ and $\delta(f)$. However, $\partial f/\partial t = -\underset{\sim}{v_n}$ and this function is restricted already to $f = 0$. Thus,

$$\frac{\partial}{\partial n} \left[\underset{\sim}{v_n} \underset{\sim}{Q}(\mathbf{x},t) \right] = 0 \tag{4.8}$$

Using equations (3.80) and (3.84), we get

$$I = \int_{\partial\Omega(t)} \left[\frac{\partial \underset{\sim}{Q}}{\partial t} - 2v_n H_f Q(\mathbf{x},t) \right] dS \tag{4.9}$$

where H_f is the local mean curvature of $\partial\Omega(t)$. What is $\partial \underset{\sim}{Q}/\partial t$? We have the following result, assuming that Q is nonimpulsive,

$$\frac{\partial \underset{\sim}{Q}}{\partial t} = \frac{\partial Q}{\partial t} + v_n \frac{\partial Q}{\partial n} \tag{4.10}$$

Derivation of equation (4.9) by other methods is not trivial.

4.2.2. Unsteady shock jump conditions. These conditions are usually obtained by the pillbox analysis. We present a method here based on generalized function theory. We have said that the conservation laws such as the mass continuity and the momentum equations are valid as they stand if we replace all ordinary derivatives with generalized derivatives. We derive here the jump conditions from these two conservation laws. Let $k(\mathbf{x},t) = 0$ describe an unsteady shock surface. Let $\nabla k = \mathbf{n}$, where $\mathbf{n}$ is the unit normal pointing in the downstream direction. We denote this downstream region as region 2 and the upstream region as region 1. We define the jump ΔQ in any parameter by

$$\Delta Q = [Q]_2 - [Q]_1 \tag{4.11}$$

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Figure 7. Thin wing in incompressible, irrotational flow with wake.

where the subscripts 1 and 2 refer to the upstream and downstream regions, respectively. Applying the rules of generalized differentiation to the mass continuity equation, we have

$$
\frac{\overline{\partial}\rho}{\partial t} + \overline{\nabla} \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})
$$
$$
+ \left[\Delta \rho \frac{\partial k}{\partial t} + \Delta(\rho \mathbf{u}) \cdot \mathbf{n} \right] \delta(k) = 0 \tag{4.12}
$$

where ρ is the density and $\mathbf{u}$ is the fluid velocity. The sum of the first two terms on the right of the first equality sign is the ordinary mass continuity equation and is 0. The coefficient of $\delta(k)$ must also be 0. Thus,

$$
-\Delta \rho v_n + \Delta(\rho u_n) = \Delta[\rho(u_n - v_n)] = 0 \tag{4.13}
$$

where $v_n = -\partial k / \partial t$ is the local shock normal velocity and $u_n = \mathbf{u} \cdot \mathbf{n}$ is the local fluid normal velocity. This expression is the first shock jump condition.

The momentum equation in tensor notation using the summation convention gives

$$
\frac{\overline{\partial}}{\partial t}(\rho u_i) + \frac{\overline{\partial}}{\partial x_j}(\rho u_i u_j) + \frac{\overline{\partial p}}{\partial x_i} = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i}
$$
$$
+ \left[\Delta(\rho u_i) \frac{\partial k}{\partial t} + \Delta(\rho u_i u_j) n_j + \Delta p n_i \right] \delta(k) = 0 \tag{4.14}
$$

where p is the pressure. The sum of the three terms after the first equality sign is the ordinary momentum equation. The coefficient of $\delta(k)$ must be zero; therefore,

$$
\Delta[\rho u_i (u_n - v_n)] + \Delta p n_i = 0 \tag{4.15}
$$

This expression is the second shock jump condition. We can derive a similar result from the energy equation.

Note that had we used the mass continuity and momentum equations in nonconservative form, we would have been faced with ambiguities of multiplication of generalized functions. This problem is discussed in detail by Colombeau. (See ref. 27.) In that reference, the remedy for the removal of these ambiguities is discussed from intuitive and mathematically rigorous aspects.

4.2.3. Velocity discontinuity as a vortex sheet. Let us consider a thin lifting wing in forward flight in an incompressible fluid as shown in figure 7. It can be shown that the velocity field can be idealized as irrotational (i.e., $\nabla \times \mathbf{u} = 0$) where $\mathbf{u}$ is the fluid velocity. However, a velocity
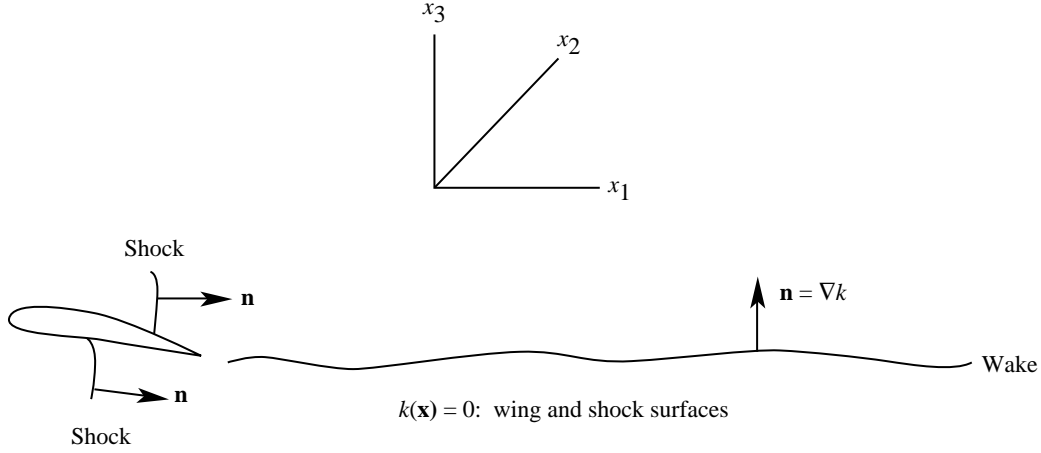
Figure 8. Diagram used in deriving Oswatitsch integral equation of transonic flow.

discontinuity can occur on the wing and on the wake. In this case $\overline{\nabla} \times \mathbf{u} \neq 0$ and the velocity discontinuity over the wing and the wake gives the vorticity distribution

$$\overline{\nabla} \times \mathbf{u} = \nabla \times \mathbf{u} + \mathbf{n} \times \Delta \mathbf{u} \delta(k)$$

$$= \mathbf{u} \times \Delta \mathbf{u} \delta(k)$$

$$= \mathbf{\Gamma} \delta(k) \tag{4.16}$$

where $k(\mathbf{x}, t) = 0$ describes the wing and wake surfaces and $\nabla k = \mathbf{n}$, the local unit normal to these surfaces. Here we define the vorticity distribution $\mathbf{\Gamma} = \mathbf{n} \times \Delta \mathbf{u}$. Note that again we define $\Delta \mathbf{u} = [\mathbf{u}]_2 - [\mathbf{u}]_1$, where $\mathbf{n}$ points into region 2. The Biot-Savart law gives the velocity field,

$$\mathbf{u}(\mathbf{x}) = \int \frac{\mathbf{\Gamma} \times \hat{\mathbf{r}}}{r^2} \delta(k) \, d\mathbf{y} = \int_{k=0} \frac{\mathbf{\Gamma} \times \hat{\mathbf{r}}}{r^2} \, dS \tag{4.17}$$

where

$$\hat{\mathbf{r}} = \frac{\mathbf{x} - \mathbf{y}}{r}$$

4.2.4. An integral equation of transonic flow. To derive the Oswatitsch integral equation of transonic flow (refs. 30 and 31), consider a thin wing with shock waves in transonic flow moving with uniform speed along the x_1-axis as shown in figure 8. Let u be the perturbation velocity along the x_1-axis. The governing equation for this flow parameter in nondimensional form is

$$\nabla^2 u - \frac{1}{2} \frac{\partial^2 u^2}{\partial x_1^2} = 0 \tag{4.18}$$

For simplicity, we assume that the airfoil, the shock surfaces, and the wake surface are all specified by $k(\mathbf{x}) = 0$. We set up this problem in generalized function space by converting the derivatives in equation (4.18) to generalized derivatives. We again define a jump in u or u^2 by $\Delta(\cdot) = [\cdot]_2 - [\cdot]_1$, where $\mathbf{n} = \nabla k$ points into region 2. For the airfoil itself, we define

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$[u]_1 = \left[u^2\right]_1 = 0$ because the airfoil is a closed surface. Thus,

$$\overline{\nabla}^2 u - \frac{1}{2}\frac{\overline{\partial}^2 u^2}{\partial x_1^2} = \Delta\left[\frac{\partial u}{\partial n} - \frac{\partial}{\partial n_1}\left(\frac{u^2}{2}\right)\right]\delta(k)$$

$$+ \nabla\cdot\left\{\Delta\left[u_n - \frac{u^2}{2}\mathbf{n}_1\right]\delta(k)\right\} \tag{4.19}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal to the surface $k = 0$ and $\mathbf{n}_1 = (n_1, 0, 0)$. Note that on the right side of equation (4.19) we dropped the sum of two terms, which by equation (4.18) is 0.

To get an integral equation, we use the Green's function of the Laplace equation, which is $-1/4\pi r$, and treat $-\partial^2 u^2/\partial x_1^2$ as a source term to obtain

$$4\pi u(\mathbf{x}) = -\frac{1}{2}\frac{\overline{\partial}^2}{\partial x_1^2}\int\frac{1}{r}u^2(\mathbf{y})\,d\mathbf{y}$$

$$-\int_{k=0}\frac{1}{r}\left[\frac{\partial u}{\partial n} - \frac{\partial}{\partial n_1}\left(\frac{u^2}{2}\right)\right]dS$$

$$-\nabla\cdot\int_{k=0}\frac{1}{r}\left(u\mathbf{n} - \frac{u^2}{2}\mathbf{n}_1\right)dS \tag{4.20}$$

Now if we bring the derivatives inside the first volume integral, which is over the unbounded space, we must use the finite part of the divergent integral introduced in section 3.4, equation (3.119). Taking $Q(\mathbf{y}) = u^2(\mathbf{y})$ in equation (3.119), from equation (3.131) we have

$$\frac{\overline{\partial}^2}{\partial x_1^2}\int\frac{1}{r}u^2(\mathbf{y})\,d\mathbf{y} = \alpha_f u^2(\mathbf{x}) + \lim_{\varepsilon\to 0}\int\frac{3r_1^2 - r^2}{r^5}h_\varepsilon(\mathbf{y})u^2(\mathbf{y})\,d\mathbf{y} \tag{4.21}$$

The last integral in equation (4.20) is

$$\nabla\cdot\int_{k=0}\frac{1}{r}\left(u\mathbf{n} - \frac{u^2}{2}\mathbf{n}_1\right)dS = -\int_{k=0}\frac{1}{r^2}\left(u\cos\theta - \frac{u^2}{2}n_1\cos\theta_1\right)dS \tag{4.22}$$

where $\cos\theta = \hat{\mathbf{r}}\cdot\mathbf{n}$, $\cos\theta_1 = 1/n_1\,\hat{\mathbf{r}}\cdot\mathbf{n}_1$, and $\hat{\mathbf{r}} = (\mathbf{x}_1 - \mathbf{y}_1)/r$. Our job is finished. The integral on the right side is convergent. Substitute equations (4.21) and (4.22) in equation (4.20). The result is the Oswatitsch integral equation of transonic flow. Further approximation is possible, but we stop at this point. This derivation is much shorter and more direct than the original one. (See refs. 30 and 31.)

4.3. Aeroacoustic Applications

In this section, we give four examples for the linear wave equation. Even for this equation, the use of generalized function theory leads to important and useful results. Before the examples, we give some standard forms of the inhomogeneous source terms appearing in aeroacoustic problems. These follow:

$$\Box^2\Phi = Q(\mathbf{x}, t) \tag{4.23a}$$

$$\Box^2\Phi = Q(\mathbf{x}, t)\delta(f) \tag{4.23b}$$

$$\Box^2 \Phi = \frac{\partial}{\partial t}[Q(\mathbf{x}, t)\delta(f)] \tag{4.23c}$$

$$\Box^2 \Phi = \nabla \cdot [\boldsymbol{Q}(\mathbf{x}, t)\delta(f)] \tag{4.23d}$$

$$\Box^2 \Phi = \underset{\sim}{Q}(\mathbf{x}, t)h\!\left(\tilde{f}\right)\delta'(f) \tag{4.23e}$$

$$\Box^2 \Phi = Q(\mathbf{x}, t)\delta(f)\delta\!\left(\tilde{f}\right) \tag{4.23f}$$

In these equations, $f(\mathbf{x}, t) = 0$ is a moving surface, usually assumed a closed surface. An open surface, such as a panel on a rotor blade, is described by $f = 0$ and $\tilde{f}(\mathbf{x}, t) > 0$, where $f(\mathbf{x}, t) = \tilde{f}(\mathbf{x}, t) = 0$ describes the edge of the open surface. (See fig. 9.) Also, we denote the Heaviside function as $h\!\left(\tilde{f}\right)$. In equation (4.23e), note that $\underset{\sim}{Q}$ is the restriction of Q to $f = 0$.

The solutions of the above equations have been given in many publications of the author and coworkers. (See refs. 33–36.) We give only a brief summary here.

The Green's function of the wave equation is

$$G(\mathbf{y}, \tau; \mathbf{x}, t) = \begin{cases} \frac{\delta(g)}{4\pi r} & (\tau \leq t) \\ 0 & (\tau > t) \end{cases} \tag{4.24}$$

where

$$g = \tau - t + \frac{r}{c} \tag{4.25}$$

In this equation, $(\mathbf{x}, t)$ and $(\mathbf{y}, \tau)$ are the observer and the field (source) space-time variables, respectively. The speed of sound is denoted by c and $r = |\mathbf{x} - \mathbf{y}|$. The two forms of the solution
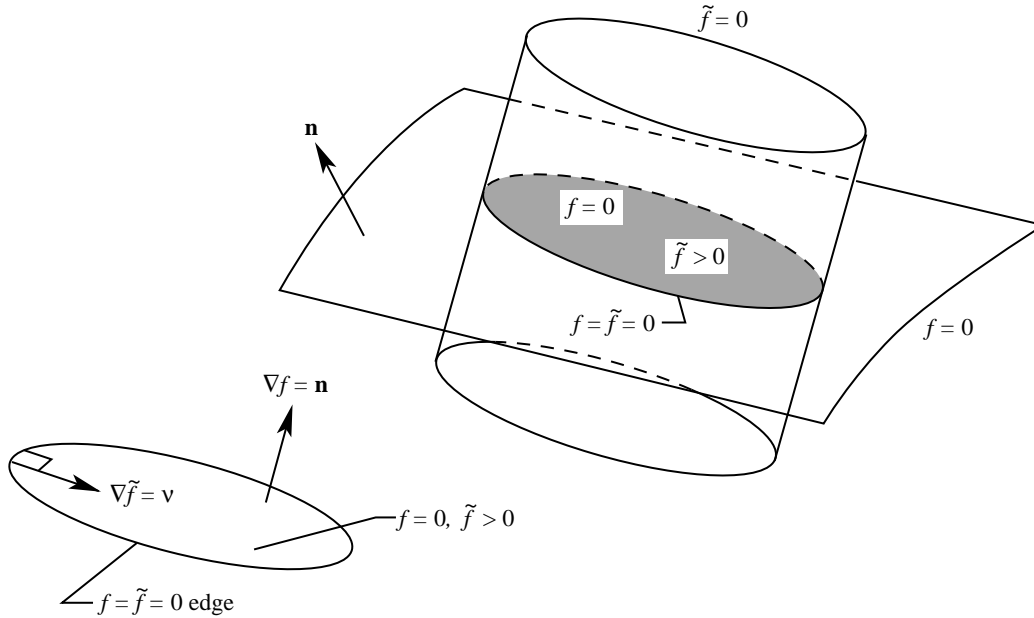

Figure 9. Definition of open surface by relations $f = 0$, $\tilde{f} > 0$. Edge is defined by $f = \tilde{f} = 0$ and $\boldsymbol{\nu}$ is the unit inward geodesic normal.

of equation (4.23a) are

$$4\pi\Phi(\mathbf{x},t) = \int \frac{1}{r}[Q]_{\text{ret}}\,d\mathbf{y} \tag{4.26}$$

and

$$4\pi\Phi(\mathbf{x},t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{\Omega(\tau)} Q(\mathbf{y},\tau)\,d\Omega \tag{4.27}$$

where the subscript ret stands for retarded time $t - r/c$. The surface $\Omega(\tau)$ is the sphere $r = c(t - \tau)$ (i.e., the sphere with center at the observer $\mathbf{x}$ and radius $c(t - \tau)$ with the element of the surface denoted by $d\Omega$). The two forms of the solution of equation (4.23a) are known as the retarded time and the collapsing sphere forms of the solution, respectively.

The solution of equation (4.23b) can also be written in several forms. (See refs. 33 and 34.) We give two forms here. For a rigid surface $f(\mathbf{x},t) = 0$, let $M_r = \mathbf{M}\cdot\hat{\mathbf{r}}$ be the local Mach number in the radiation direction. Then

$$4\pi\Phi(\mathbf{x},t) = \int_{f=0} \left[\frac{Q(\mathbf{y},\tau)}{r\,|1-M_r|}\right]_{\text{ret}} dS \tag{4.28}$$

Note that to get this equation, the formal Green's function solution, which is

$$4\pi\Phi(\mathbf{x},t) = \int \frac{1}{r} Q(\mathbf{y},\tau)\delta(f)\delta(g)\,d\mathbf{y}\,d\tau \tag{4.29}$$

is integrated as follows. First introduce a Lagrangian variable $\boldsymbol{\eta}$ on and near the surface $f = 0$ such that the Jacobian of the transformation is unity. Note that we have $\mathbf{y} = \mathbf{y}(\boldsymbol{\eta},\tau)$ and

$$r = |\mathbf{x} - y(\boldsymbol{\eta},\tau)| \tag{4.30}$$

Next let $\tau \to g$, which gives $\partial g/\partial\tau = 1 - M_r$. Integrate equation (4.29) next with respect to g and finally integrate $\delta(f)$ by the method of section 3.3 to get equation (4.28).

A more interesting method of integrating the delta functions in equation (4.29) is to let $\tau \to g$ and integrate with respect to g. The integration gives

$$4\pi\Phi(\mathbf{x},t) = \int \frac{1}{r}[Q(\mathbf{y},\tau)]_{\text{ret}}\delta(F)\,d\mathbf{y} \tag{4.31}$$

where $F(\mathbf{y};\mathbf{x},t) = [f(\mathbf{y},\tau)]_{\text{ret}} = f[\mathbf{y}, t - r/c]$. Note, however, that even if $|\nabla f| = 1$ by definition, we have $|\nabla F| \neq 1$ in equation (4.31). We will, therefore, give a slight modification of equation (3.80). In the following integral, assume $|\nabla f| \neq 1$, then

$$I = \int \phi(\mathbf{x})\delta(f)\,d\mathbf{x} = \int_{f=0} \frac{\phi(\mathbf{x})}{|\nabla f|}dS \tag{4.32}$$

This result applies to equation (4.31). It is easily shown by differentiation that

$$|\nabla F| = \left(1 + M_n^2 - 2M_n\cos\theta\right)^{\frac{1}{2}} \equiv \Lambda \tag{4.33}$$

where $M_n = v_n/c$, $v_n = -\partial f/\partial t$ is the local normal velocity on $f = 0$, and $\cos\theta = \mathbf{n}\cdot\hat{\mathbf{r}}$ is the cosine of the angle between the local normal to $f = 0$ and the radiation direction $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{y})/r$. Using equations (4.32) and (4.33) in equation (4.31), we get

$$4\pi\Phi(\mathbf{x},t) = \int_{F=0} \frac{1}{r}\left[\frac{Q(\mathbf{y},\tau)}{\Lambda}\right]_{\text{ret}} d\Sigma \tag{4.34}$$
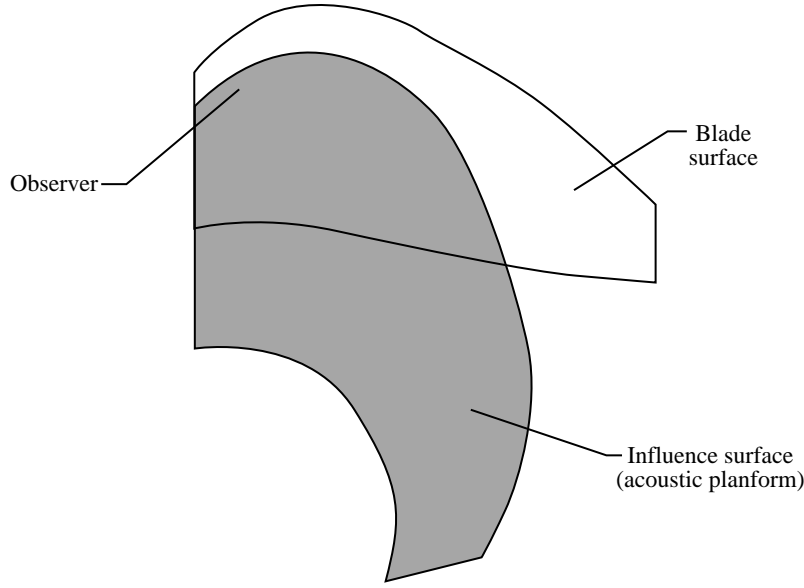
Figure 10. Influence surface for observer on propeller surface, forward Mach number = 0.334, helical Mach number = 0.880. (See ref. 44.)

where $d\Sigma$ is the element of the surface area at $F = 0$. Note that for supersonic surfaces, the condition $Mr = 1$ produces a singularity in equation (4.28). The use of equation (4.34) removes this singularity in most cases.

To visualize the surface Σ: $F = 0$, let the surface $f = 0$ move in space. Construct the intersection of the collapsing sphere $r = c(t - \tau)$ for a fixed $(\mathbf{x}, t)$ with the surface $f = 0$. The surface in space that is the locus of these curves of intersection is the Σ-surface or the influence surface for $(\mathbf{x}, t)$. Given $(\mathbf{x}, t)$, this surface is unique because the sphere $r = c(t - \tau)$ has center at $\mathbf{x}$ and $r = 0$ at $\tau = t$. Given $\tau \leq t$, because $(\mathbf{x}, t)$ is fixed, the sphere is specified and $f(\mathbf{y}, \tau) = 0$ is also specified. Therefore, the intersecting curve, if it exists, is specified. Figure 10 (ref. 32) shows the Σ-surface for a rotating propeller blade. This figure indicates that the Σ-surface is dependent on the motion and the geometry of the surface $f = 0$. A singularity in equation (4.34) may exist when $\Lambda = 0$; however, we will not address the singularity problem here. Such a problem can occur for supersonic propeller blades with blunt leading edges. That situation should be avoided because of excessive drag problems.

The solutions of equations (4.23c) and (4.23d) can be related to that of (4.23b). For example, the solution of equation (4.23c) is

$$4\pi \Phi(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_{f=0} \left[\frac{Q(\mathbf{y}, \tau)}{r \left| 1 - M_r \right|} \right]_{\text{ret}} dS \tag{4.35}$$

Now $\partial / \partial t$ can be brought inside the integral using the relation

$$\frac{\partial}{\partial t} = \frac{1}{1 - M_r} \frac{\partial}{\partial \tau} \tag{4.36}$$

Note, however, that $r = |\mathbf{x} - \mathbf{y}(\boldsymbol{\eta}, \tau)|$ so that $\partial r / \partial \tau \neq 0$. (See refs. 33 and 34.) Similar manipulations can be performed for the form of solution of equation (4.23c) based on equation (4.34), but it is better to work with the source terms of equation (4.23c) before using the Green's function approach. (See section 4.3.3.)

The solution of equation (4.23e) is by far the most difficult of the problems considered here. We first simplify the algebraic manipulations by defining $\tilde{f}$ such that $\nabla \tilde{f} = \boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is the unit outward geodesic normal to the edge. The geodesic normal is tangent to the surface $f = 0$, $\tilde{f} > 0$ and is orthogonal to edge $f = \tilde{f} = 0$. (See fig. 9.) The formal solution of equation (4.23e) is

$$4\pi \Phi(\mathbf{x}, t) = \int \frac{1}{r} \underset{\sim}{Q}(\mathbf{y}, \tau) h(\tilde{f}) \delta'(f) \delta(g) \, d\mathbf{y} \, d\tau$$

$$= \int \frac{1}{r} [\underset{\sim}{Q}(\mathbf{y}, \tau)]_{\mathrm{ret}} \, h(\widetilde{F}) \delta'(F) \, d\mathbf{y} \tag{4.37}$$

where, as before, $F = [f]_{\mathrm{ret}}$ and we define $\widetilde{F}(\mathbf{y}; \mathbf{x}, t) = [\tilde{f}(\mathbf{y}, \tau)]_{\mathrm{ret}}$. Let $\mathbf{N}$ be the unit normal to the surface $F = 0$. We can show that

$$\mathbf{N} = \frac{\mathbf{n} - M_n \hat{\mathbf{r}}}{\Lambda} \tag{4.38}$$

Equation (4.37) is now of the form of equation (3.81). Again, because $|\nabla F| = \Lambda \neq 1$, we must give the modification to equation (3.81) here. In this case, we have for $|\nabla f| \neq 1$

$$\int \phi(\mathbf{x}) \delta'(f) \, d\mathbf{y} = \int_{f=0} \left\{ -\frac{1}{|\nabla f|} \frac{\partial}{\partial n} \left[\frac{\phi}{|\nabla f|} \right] + \frac{2 H_f \phi}{|\nabla f|^2} \right\} dS \tag{4.39}$$

Next, using F in place of f here, we get from equation (4.39)

$$4\pi \Phi(\mathbf{x}, t) = \int_{F=0} \left\{ -\frac{1}{\Lambda} \frac{\partial}{\partial N} \left(\frac{[Q]_{\mathrm{ret}}}{r \Lambda} h(\widetilde{F}) \right) + \frac{2 H_F [Q]_{\mathrm{ret}} h(\widetilde{F})}{r \Lambda^2} \right\} d\Sigma \tag{4.40}$$

where H_F is the local mean curvature of the Σ-surface given by $F = 0$. Note that

$$\frac{\partial}{\partial N} h(\widetilde{F}) = \mathbf{N} \cdot \nabla \widetilde{F} \, \delta(\widetilde{F}) \tag{4.41}$$

so that we must integrate this delta function in equation (4.40). Using a curvilinear coordinate system on the Σ-surface, we can show that

$$\int_{F=0} \phi(\mathbf{x}) |\nabla \widetilde{F}| \delta(\widetilde{F}) \, d\Sigma = \int_{\substack{F=0 \\ \widetilde{F}=0}} \frac{\phi}{\sin \theta'} dL \tag{4.42}$$

where θ' is the angle between $\mathbf{N}$ and $\widetilde{\mathbf{N}} = \nabla \widetilde{F} / |\nabla \widetilde{F}|$. Also, dL is the element of length of the edge of the Σ-surface given by $F = \widetilde{F} = 0$. The final result of manipulations of equation (4.40) based on equation (4.42) is

$$4\pi \Phi(\mathbf{x}, t) = \int_{\substack{F=0 \\ \widetilde{F}>0}} \left\{ -\frac{1}{\Lambda} \frac{\partial}{\partial N} \left(\frac{[Q]_{\mathrm{ret}}}{r\Lambda} \right) + \frac{2 H_F [Q]_{\mathrm{ret}}}{r\Lambda^2} \right\} d\Sigma$$

$$- \int_{\substack{F=0 \\ \widetilde{F}=0}} \frac{[Q]_{\mathrm{ret}} \cot \theta'}{r\Lambda^2} dL \tag{4.43}$$

Note that we have defined $\widetilde{\mathbf{N}}$ as the unit normal to $\widetilde{F} = 0$ and we have

$$\widetilde{\mathbf{N}} = \frac{\boldsymbol{\nu} - M_\nu \hat{\mathbf{r}}}{\widetilde{\Lambda}} \tag{4.44a}$$

$$\widetilde{\Lambda} = |\nabla \widetilde{F}| \tag{4.44b}$$

$$\cos \theta' = \mathbf{N} \cdot \widetilde{\mathbf{N}} \tag{4.44c}$$

Because $\underset{\sim}{Q}$ is the restriction of Q to $f = 0$, we have $\partial \underset{\sim}{Q}/\partial n = 0$. In equation (4.43), we find the normal derivative of $[\underset{\sim}{Q}]_{\mathrm{ret}}$ first:

$$\frac{\partial}{\partial N}[\underset{\sim}{Q}]_{\mathrm{ret}} = \left[\frac{\partial \underset{\sim}{Q}}{\partial N}\right]_{\mathrm{ret}} + \frac{1}{c}\left[\frac{\partial \underset{\sim}{Q}}{\partial \tau}\mathbf{N} \cdot \hat{\mathbf{r}}\right]_{\mathrm{ret}} \tag{4.45}$$

Using equation (4.38), we get

$$\mathbf{N} = \frac{1}{\Lambda}[(1 - M_n \cos \theta)\,\mathbf{n} - M_n \sin \theta\, \mathbf{t}_1] \tag{4.46}$$

where $\mathbf{t}_1$ is the unit vector along the projection of $\hat{\mathbf{r}}$ on the local tangent plane to $f = 0$. Therefore, after using $\partial \underset{\sim}{Q}/\partial n = 0$ we get

$$\frac{\partial \underset{\sim}{Q}}{\partial N} = -\frac{M_n \sin \theta}{\Lambda}\frac{\partial \underset{\sim}{Q}}{\partial t_1} \tag{4.47}$$

where $\partial/\partial t_1$ is the directional derivative of Q along $\mathbf{t}_1$. In this case, we no longer need restriction of Q to $f = 0$ because $\partial \underset{\sim}{Q}/\partial t_1 = \partial Q/\partial t_1$. For $\partial Q/\partial \tau$ in equation (4.45), we must use a relation similar to that in equation (4.10). The curve $F = \widetilde{F} = 0$ in equation (4.43) is generated by the intersection of the collapsing sphere $g = 0$ and the edge curve $f = \tilde{f} = 0$ of the open surface $f = 0, \tilde{f} > 0$.

We next consider equation (4.23f). The formal solution is

$$4\pi\Phi(\mathbf{x}, t) = \int \frac{1}{r}Q(\mathbf{y}, \tau)\delta(f)\delta(\tilde{f})\delta(g)\, d\mathbf{y}\, d\tau$$

$$= \int \frac{1}{r}[Q]_{\mathrm{ret}}\delta(F)\delta(\widetilde{F})\, d\mathbf{y} \tag{4.48}$$

This equation is similar to equation (3.99) except that $|\nabla F| \neq 1$ and $|\nabla \widetilde{F}| \neq 1$. In this case, $\sin \theta$ in equation (3.100) is replaced by $|\nabla F \times \nabla \widetilde{F}|$, which by definition is Λ_0. Therefore, equation (4.48) gives

$$4\pi\Phi(\mathbf{x}, t) = \int_{\substack{F=0 \\ \widetilde{F}=0}} \frac{1}{r}\left[\frac{Q}{\Lambda_0}\right]_{\mathrm{ret}} dL \tag{4.49}$$

We now give four applications.

4.3.1. Lowson's formula for a dipole in motion. A dipole is an idealization of a point force. A point force in motion is described by the wave equation

$$\square^2 p' = -\frac{\partial}{\partial x_i}\{f_i(t)\delta[\mathbf{x} - \mathbf{s}(t)]\} \tag{4.50}$$

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where p' is the acoustic pressure, f_i is the component of the point force, and $\mathbf{s}(t)$ is the position of the force at time t. The formal solution of equation (4.50) is

$$4\pi p'(\mathbf{x}, t) = -\frac{\partial}{\partial x_i} \int \frac{f_i(\tau)}{r} \delta[\mathbf{y} - \mathbf{s}(\tau)] \delta(g) \, d\mathbf{y} \, d\tau \tag{4.51}$$

Let us integrate the above integral with respect to $\mathbf{y}$. We get

$$4\pi p'(\mathbf{x}, t) = -\frac{\partial}{\partial x_i} \int \frac{f_i(\tau)}{r^*} \delta(g^*) \, d\tau \tag{4.52}$$

where

$$r^* = |\mathbf{x} - \mathbf{s}(\tau)| \tag{4.53a}$$

and

$$g^* = \tau - t + \frac{r^*}{c} \tag{4.53b}$$

Now let $\tau \to g^*$ and note that

$$\frac{\partial g^*}{\partial \tau} = 1 - M_r \tag{4.54}$$

where $\mathbf{M}_r = \dot{\mathbf{s}} \cdot \hat{\mathbf{r}}/c$ is the Mach number of the point force in the radiation direction. Integrate the resulting equation with respect to g^* to get

$$4\pi p'(\mathbf{x}, t) = -\frac{\partial}{\partial x_i} \left[\frac{f_i(\tau)}{r \, |1 - M_r|} \right]_{\tau^*} \tag{4.55}$$

where τ^* is the emission time. The solution of $g^* = 0$ has only one root if the point force is in subsonic motion. The derivative in equation (4.55) can now be taken inside the square brackets. The resulting equation is a formula given by Lowson (ref. 37) that is useful in noise prediction of rotating blades where the dipole sources can by assumed compact.

4.3.2. Kirchhoff formula for moving surfaces. In the 1930's, Morgans published a paper in which he derived the Kirchhoff formula for moving surfaces. (See ref. 38.) The derivation of this formula was based on classical analysis and was lengthy. In 1988, Farassat and Myers gave a modern derivation of this result based on generalized function theory. (See refs. 39 and 40.) The derivation is short and avoids the use of four-dimensional Green's identity and the associated difficulties of dealing with surfaces and volumes in four dimensions. We present the basic idea behind this modern derivation here and refer the readers to reference 39.

Assume that the surface in motion on which conditions on $\phi(\mathbf{x}, t)$ are specified is given by $f(\mathbf{x}, t)$. This surface can be deformable. Assume that ϕ satisfies the wave equation in the exterior of the surface $f = 0$, which is the region defined by $f > 0$. Now extend ϕ to the entire unbounded space as follows:

$$\tilde{\phi}(\mathbf{x}, t) = \begin{cases} \phi(\mathbf{x}, t) & (f > 0) \\ 0 & (f < 0) \end{cases} \tag{4.56}$$

Clearly, $\tilde{\phi}$ satisfies the wave equation in the unbounded space. However, $\tilde{\phi}$ has discontinuities across $f = 0$ that appear as source terms of the wave equation. Note that the jumps in $\tilde{\phi}$ and its derivatives depend on corresponding values for ϕ because $\Delta\tilde{\phi} = \phi(f = 0+)$ and $\Delta\partial\tilde{\phi}/\partial t = \partial\phi/\partial t \, (f = 0+)$.

Applying the rules of generalized differentiation to $\tilde{\phi}$, we get

$$\overline{\square}^2\tilde{\phi} = -\left(\phi_n + \frac{1}{c}M_n\phi_t\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n\phi\delta(f)] - \nabla\cdot[\phi\mathbf{n}\delta(f)] \tag{4.57}$$

where $M_n = v_n/c$ and $v_n = -\partial f/\partial t$ is the local normal velocity of the surface $f = 0$. As before, we have assumed $\nabla f = \mathbf{n}$, the local outward unit normal to $f = 0$. The three types of source terms on the right of equation (4.57) are of the standard types given in equations (4.23a–f). The solution for a deformable surface is given by

$$4\pi\tilde{\phi}(\mathbf{x},t) = \int_{D(S)}\left[\frac{E_1\sqrt{g_{(2)}}}{r\,(1-M_r)}\right]_{\tau^*}du^1\,du^2 + \int_{D(S)}\left[\frac{\phi E_2\sqrt{g_{(2)}}}{r\,(1-M_r)}\right]_{\tau^*}du^1\,du^2 \tag{4.58}$$

where $D(S)$ is a time-independent region in u^1u^2-space onto which the surface $f = 0$ is mapped. The determinant of the coefficient of the first fundamental form is denoted $g_{(2)}$. In this equation τ^* is the emission time of the point (u^1, u^2) on the surface $f = 0$. The expression E_1 depends on ϕ, ϕ_n, $\nabla_2\phi$ (surface gradient of ϕ), and the kinematic and geometric parameters of the surface $f = 0$. The expression E_2 depends only on the kinematic and geometric parameters of the surface $f = 0$. (See ref. 39.)

4.3.3. Noise from moving surfaces. Let an impenetrable surface $f = 0$ be in motion such that $f > 0$ outside the body and $\nabla f = \mathbf{n}$, the unit outward normal. Let us assume that the fluid is extended inside this surface with conditions of undisturbed medium (i.e., density ρ_0 and speed of sound c). We know that the mass continuity and momentum equations are valid when the derivatives are written as generalized derivatives. Let us extract only the contribution of discontinuities across $f = 0$ and leave the effect of all other discontinuities (such as those across shock waves) in these equations. The mass continuity equation gives

$$\frac{\overline{\partial}\rho}{\partial t} + \overline{\nabla}\cdot(\rho\mathbf{u}) = -(\rho - \rho_0)v_n\delta(f) + \rho u_n\delta(f)$$

$$= \rho_0 v_n\delta(f) \tag{4.59}$$

where $\nu_n = -\partial f/\partial t$ is the local normal velocity of $f = 0$ and we have used the impenetrability condition on this surface, which is $u_n = \nu_n$. The momentum equation gives

$$\frac{\overline{\partial}}{\partial t}(\rho u_i) + \frac{\overline{\partial}}{\partial x_j}\left(\rho u_i u_j + \mathbf{P}_{ij}\right) = \mathbf{P}_{ij}n_j\delta(f) \tag{4.60}$$

where $\mathbf{P}_{ij} = \mathbf{E}_{ij} + (p - p_0)\delta_{ij}$ is the compressive stress tensor and $\mathbf{E}_{ij}$ is the viscous stress tensor. Now we take the generalized derivative of both sides of equation (4.59) and $\overline{\partial}/\partial x_i$ of both sides of equation (4.60), subtract the latter from the former, and finally subtract $c^2\overline{\partial}^2\rho/\partial x_i^2$ from both sides to get

$$\overline{\square}^2 p' = \frac{\overline{\partial}^2}{\partial x_i\,\partial x_j}\left[\mathbf{T}_{ij}h(f)\right] - \frac{\partial}{\partial x_i}\left[\mathbf{P}_{ij}n_j\delta(f)\right] + \frac{\partial}{\partial t}\left[\rho_0 v_n\delta(f)\right] \tag{4.61}$$

where $p' = c^2(\rho - \rho_0)$. Here $\mathbf{T}_{ij}$ is the Lighthill stress tensor. Now we have added $h(f)$, the Heaviside function, on the right side to indicate that $\mathbf{T}_{ij} \neq 0$ outside the surface $f = 0$. This is the Ffowcs Williams–Hawkings (FW-H) equation. (See ref. 41.) Note that in the far field, p' is the acoustic pressure.

The source terms of equation (4.61) are of the standard types in equations (4.23a–f). For a surface in subsonic motion, the solution for surface sources involving the Doppler factor is most appropriate for numerical work. (See refs. 33 and 34.) For supersonic surfaces such as an advanced propeller blade on which $|M_n| < 1$ everywhere, a different solution based on the Σ-surface must be used. We show here briefly how this can be done. In applications, we need to calculate the sound from an open surface such as a panel on a blade. We, therefore, define such an open surface by $f = 0, \tilde{f} > 0$ with the edge defined by $f = \tilde{f} = 0$ as before. The assumptions concerning the gradients of f and $\tilde{f}$ at the beginning of this section hold here. We are interested in the solution of equations of the types

$$\Box^2 p' = \frac{\partial}{\partial t} \left[\rho_0 v_n h(\tilde{f}) \delta(f) \right] \tag{4.62a}$$

$$\Box^2 p' = -\frac{\partial}{\partial x_i} \left[p n_i h(\tilde{f}) \delta(f) \right] \tag{4.62b}$$

where $h(\tilde{f})$ is the Heaviside function. Note that we have approximated $\mathbf{P}_{ij}$ in equation (4.61) by $p\boldsymbol{\delta}_{ij}$ where p is the surface (gauge) pressure. To derive solutions for equations (4.62a) and (4.62b) suitable for supersonic panel motion, we write

$$\frac{\partial}{\partial t} \left[\rho_0 \underset{\sim}{v}_n h(\tilde{f}) \delta(f) \right] = \rho_0 \frac{\partial \underset{\sim}{v}_n}{\partial t} h(\tilde{f}) \delta(f)$$

$$- \rho_0 \underset{\sim}{v}_n^2 h(\tilde{f}) \delta'(f)$$

$$- \rho_0 \, v_n \, v_\nu \delta(\tilde{f}) \delta(f) \tag{4.63}$$

where v_ν is the velocity of the edge in the direction of the geodesic normal. A similar operation can be performed on the right of equation (4.62b) such that

$$-\frac{\partial}{\partial x_i} \left[p n_{\underset{\sim}{i}} h(\tilde{f}) \delta(f) \right] = -\nabla \cdot \left[p \underset{\sim}{\mathbf{n}} h(\tilde{f}) \delta(f) \right]$$

$$= -p \underset{\sim}{h}(\tilde{f}) \delta'(f) + 2 p H_f h(\tilde{f}) \delta(f) \tag{4.64}$$

where H_f is the local mean curvature of the surface $f = 0$. The source terms of the right of equations (4.63) and (4.64) are of the types in equations (4.23a–f). (See refs. 35 and 36.)

4.3.4. Identification of shock noise source strength. The first term on the right of the FW-H equation (4.61) is known as the quadrupole source. As mentioned in the derivations, the discontinuities in the region $f > 0$ (i.e., outside the body) contribute source terms after generalized differentiation is performed. If a shock wave described by the equation $k(\mathbf{x}, t) = 0$ exists on a rotating blade, then the quadrupole term gives surface sources on the shock, the

strengths of which are determined as follows. Let us take the generalized second derivative of $\mathbf{T}_{ij}$ by

$$\frac{\overline{\partial}\mathbf{T}_{ij}}{\partial x_i} = \frac{\partial \mathbf{T}_{ij}}{\partial x_i} + \Delta\mathbf{T}_{ij}n'_i\delta(k) \tag{4.64a}$$

$$\frac{\overline{\partial}^2\mathbf{T}_{ij}}{\partial x_i \partial x_j} = \frac{\partial^2\mathbf{T}_{ij}}{\partial x_i \partial x_j} + \Delta\frac{\partial\mathbf{T}_{ij}}{\partial x_i}n'_j\delta(k) + \frac{\partial}{\partial x_j}\left[\Delta\mathbf{T}_{ij}n'_i\delta(k)\right] \tag{4.65b}$$

where $\mathbf{n}' = \nabla k$ is the unit normal to the shock surface pointing to the downstream region. The last two terms on the right of this equation are shock surface terms that are of monopole and dipole types, respectively. The first term on the right of equation (4.65b) is a volume term that is familiar in Lighthill's jet noise theory. In the rotating blade noise problem, this term primarily reflects nonlinearities other than turbulence. Farassat and Tadghighi (ref. 42) conjectured that the shock surface terms contributed relatively more than the volume term in equation (4.65b). Preliminary calculations have supported this conjecture. (See ref. 43.)

The interesting aspect in the above result is that the shock source strength is obtained purely by mathematics. Without the use of the operational properties of generalized functions, the identification of shocks as sources of sound and the determination of the source strength would be rather difficult. Other mechanisms of noise generation can also be identified by this method. (See ref. 44.)

5. Concluding Remarks

In this paper, we have given the rudiments of generalized function theory and some applications in aerodynamics and aeroacoustics. These applications depend on the concept of generalized differentiation and on the Green's function approach. We have briefly discussed the generalized Fourier transformation. Many more examples could be given. The power of this theory stems from its operational properties. In addition to the exchange of limit processes that leads to many useful results, discontinuous solutions of linear equations using the Green's function are easily obtained by posing the problem in generalized function space. As seen in the example of the Oswatitsch integral equation of transonic flow, a nonlinear partial differential equation with a discontinuous solution can be cast into an integral equation based on the fundamental solution of the linear part of the differential equation. The Schwartz generalized function theory has unified many ad hoc methods in mathematics and has answered some fundamental questions about linear partial differential equations. The nonlinear theory now being developed, in which multiplication of generalized functions is allowed, can be even more useful in applications. Generalized function theory is an extension of classical analysis and gives engineers and scientists added power in applications. This extension is much like the complex analysis that extends real analysis and is very important in applied mathematics. Finally, multidimensional generalized functions, particularly the delta function and its derivatives, are quite useful in many applications.

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13. ABSTRACT *(Maximum 200 words)*
Generalized functions have many applications in science and engineering. One useful aspect is that discontinuous functions can be handled as easily as continuous or differentiable functions and provide a powerful tool in formulating and solving many problems of aerodynamics and acoustics. Furthermore, generalized function theory elucidates and unifies many ad hoc mathematical approaches used by engineers and scientists. We define generalized functions as continuous linear functionals on the space of infinitely differentiable functions with compact support, then introduce the concept of generalized differentiation. Generalized differentiation is the most important concept in generalized function theory and the applications we present utilize mainly this concept. First, some results of classical analysis, are derived with the generalized function theory. Other applications of the generalized function theory in aerodynamics discussed here are the derivations of general transport theorems for deriving governing equations of fluid mechanics, the interpretation of the finite part of divergent integrals, the derivation of the Oswatitsch integral equation of transonic flow, and the analysis of velocity field discontinuities as sources of vorticity. Applications in aeroacoustics include the derivation of the Kirchhoff formula for moving surfaces, the noise from moving surfaces, and shock noise source strength based on the Ffowcs Williams–Hawkings equation.

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