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Progress in Computing Fuel-Optimal Orbit Transfers in Large Numbers of Burns

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ABSTRACT

This report describes the current state of development of methods for calculating optimal orbital transfers with large numbers of burns. Reported on first is the homotopy-motivated and so-called Direction Correction method. So far, this method has been partially tested with one solver, the final step has yet to be implemented. Second is the Patched Transfer method. This method is rooted in some simplifying approximations made on the original optimal control problem. The transfer is broken up into single-burn segments, each single-burn solved as a predictor step and the whole problem then solved with a corrector step.

I. INTRODUCTION

Electric propulsion, with its high specific impulse, promises very low fuel consumption but it produces less thrust than its counterparts. If one wants to use electric propulsion, one needs to be prepared to tolerate the long transfer times that will likely be incurred. The greater time spent thrusting must be spent wisely if fuel savings are to be realized. Furthermore, the effects of Earth's oblateness and atmospheric drag become more significant on the orbits of long transfer times.

To spend the thrusting time wisely, form an optimal control problem to maximize the mass at the end of the transfer. This, therefore, is the cost function

$$J = m(t_f) \tag{1}$$

subject to the boundary conditions

$$\psi\left(\mathbf{r}(0), \mathbf{v}(0), \mathbf{r}\left(t_f\right), \mathbf{v}\left(t_f\right)\right) = 0 \tag{2}$$

and the state dynamics

$$\dot{\mathbf{r}} = \mathbf{v} \tag{3}$$

$$\dot{\mathbf{v}} = \frac{T}{m}\mathbf{e}_T - \frac{\mu}{r^3}\mathbf{r} \tag{4}$$

$$\dot{m} = -\frac{T}{g_o I_{sp}} \tag{5}$$

where $\mathbf{e}_T$ is the thrust direction, a unit vector, and the thrust magnitude, T, is limited between zero and some maximum value T_{max}, μ is the gravitational constant, g_o is the gravitational acceleration at sea-level, and I_{sp} is the specific impulse of the motor. Sometimes $g_o I_{sp}$ is referred to as the exit velocity of the motor.

This results in the well-known bang-bang optimal control problem, discussed in detail by Lawden[1]. However, where the boundary conditions are often designed for the rendezvous problem, herein the boundary conditions are designed such that the initial and final points lie on the desired orbits without specifying the position, or true anomaly, on either orbit.

As found using the Euler-Lagrange necessary conditions, the optimal thrust direction for this problem is

$$\mathbf{e}_T = \frac{\boldsymbol{\lambda}_\mathbf{v}}{|\boldsymbol{\lambda}_\mathbf{v}|} \tag{6}$$

where $\boldsymbol{\lambda}_\mathbf{v}$ is found from the following differential equations

$$\dot{\boldsymbol{\lambda}}_\mathbf{r} = \mu\left[\frac{\boldsymbol{\lambda}_\mathbf{v}}{r^3} - 3\frac{(\boldsymbol{\lambda}_\mathbf{v}{}^{\mathrm{T}}\mathbf{r})\mathbf{r}}{r^5}\right] \tag{7}$$

$$\dot{\boldsymbol{\lambda}}_\mathbf{v} = -\boldsymbol{\lambda}_\mathbf{r} \tag{8}$$

$$\dot{\lambda}_m = \frac{T}{m^2}\boldsymbol{\lambda}_\mathbf{v}{}^{\mathrm{T}}\mathbf{e}_T = \frac{T}{m^2}|\boldsymbol{\lambda}_\mathbf{v}| \tag{9}$$

The optimal thrust magnitude for this problem is a bang-bang solution. Polarity for the on-off control is determined by applying the following switching law, Eqn. (10), to the switching function, Eqn. (11).

$$\begin{aligned} &H_S > 0, \quad T = T_{max} \\ &H_S < 0, \quad T = 0 \end{aligned} \tag{10}$$

$$H_S = \frac{|\lambda_v|}{m} - \frac{\lambda_m}{g_o I_{sp}} \tag{11}$$

Solutions of this problem with long transfer times and, therefore, large numbers of burns are desired. There are many methods that have been successfully used to compute n-burn transfers, where n is anywhere from 1 to about 6. However, fewer methods successfully compute transfers for larger values of n. Methods for the former attempt to solve the optimal control problem either directly, indirectly, or as a hybrid of the two. In this report, assume that a mostly indirect method, such as BOUNDSCO[2] or MBCM[3] or that of Brusch[4], et. al, or of Redding[5] is being used.

One idea to obtain interesting solutions is to first compute some n-burn transfer, where n may be less than the number of burns desired. Using this as a starting point, increase the allowed transfer time and compute the new transfer. It is expected that the new transfer is relatively similar to the starting transfer. New transfers are then successfully produced this way until the desired transfer is reached. This homotopy method seems to work well as long as the number of burns performed in the transfer does not need to increase so that optimality is satisfied. For example, in many cases BOUNDSCO is unable to find a three burn solution when the two burn solution to an almost identical problem is given as the initial guess. Introduced in this report, the Direction Correction Method is an attempt to alleviate this difficulty. Its purpose is to find an n+1 burn solution to an orbit transfer problem with allowed transfer time $t_f + dt_f$ using an n burn solution to the same problem but with allowed transfer time t_f.

Another idea is to patch together a set of n-burn transfers, where n is a small integer, perhaps unity, to produce an m-burn transfer, where m is the desired number of burns. This method requires that the sequence of transfer orbits be either guessed and iterated upon for optimality, or simply prespecified. From the theory of optimal control, this patched solution will be a suboptimal solution. This idea will be referred to herein as the Patched Transfer method.

More than likely, once an optimal transfer has been computed, interest will shift to developing a guidance law. Possible analytical solutions found from consideration of the patched transfer method for the one burn solution of two very close orbits may give a simple guidance law.

II. DIRECTION CORRECTION METHOD PROGRESS

The first idea, referred to herein as the Direction Correction method, is based on the common homotopy strategy. The Direction Correction is designed to aid a homotopy strategy in calculating successive optimal transfers. In particular, the difficulty arises when the desired transfer has one more burn arc than the current computed transfer.

The method is attractive because it only requires the solution of a relatively small set of nonlinear equations. These equations are of the following form

$$\left.\frac{\partial C}{\partial \mathbf{z}}\right|_{\mathbf{z}(0)} \Phi(0, t_a + dt_a)\delta\mathbf{z}(t_a + dt_a) = 0 \tag{12a}$$

$$\left.\frac{\partial D}{\partial \mathbf{z}}\right|_{\mathbf{z}(t_f)} \Phi(t_b + dt_b, t_f)\mathbf{f}(\delta\mathbf{z}(t_a + dt_a)) = -\left.\frac{\partial D}{\partial \mathbf{z}}\right|_{\mathbf{z}(t_f)} \dot{\mathbf{z}}(t_f)dt_f \tag{12b}$$

For reasons given in a previous report and a paper submitted to the 1994 AIAA Guidance, Navigation, and Control conference[6], both equations are evaluated at time t_a + dt_a. The first equation propagates a guess made for this instant in time back to the initial time, using it to check a condition on the boundary conditions at the initial time, denoted $C(\mathbf{z})$ where $\mathbf{z}$ is the state vector. The second equation is a similar situation, except that it is applied to the boundary conditions at the final time, denoted $D(\mathbf{z})$. The function $\mathbf{f}(\mathbf{z})$ takes into account the fact that the number of burns in the desired transfer, $\mathbf{z}(t)$ + $\delta\mathbf{z}(t)$, is one greater than in the computed transfer, $\mathbf{z}(t)$.

The solution information can easily be put into a form useful for a variety of numerical methods. For example, the change $\delta\mathbf{z}(0)$ can be propagated through the transition matrix to calculate the changes at each node point for a multiple point shooting method. This method is still under development but shows promise as relatively simple way of getting to the n+1 burn solution.

Using the IMSL routine DNEQBF to solve Eqns. (12a-b), the method has been used to predict the correct change, or 'direction,' for an example. The algorithm starts with information from a given transfer. Then, it iteratively improves upon an initial guess, using DNEQBF. The method has produced an approximate solution to Eqns. (12a-b). Comparing this solution to the correct answer, errors are only about 4%. The final step is to add the solution to the computed transfer and attempt to converge the desired transfer with a solver such as BOUNDSCO. Although this has not been implemented yet, success is expected.

The second method was inspired in part by the work of others. Zondervan, et. al made some simple guidance observations[7], specifically that in some regions the primer vector is relatively constant in a velocity-fixed reference frame. This implies that a simple control law is available in some cases. Marec presents a solution to the orbit correction problem[8]. This motivated a notion that solutions to linearized and/or approximated problems were available. In this spirit a solution was obtained for the optimal transfer between two close orbits. This solution has been presented in [6].

Most interesting about this transfer was the simplicity of the control. Over this short transfer between a circular orbit and a close target orbit, the optimal control of the thrust angle was almost linear in time. And, in addition, the control direction was almost coincident with the velocity direction.

To review, a modified optimal control problem is considered. The dynamics for this problem are again the equations of orbital motion; however, this time the state is defined relative to the initial orbit. Assume that the distance from a reference orbit is small compared to the radius of the reference orbit and ignore all terms to the order of $(\delta r/\rho)^2$. This assumption results in the following dynamics:

$$\delta\dot{\mathrm{r}} = \delta\mathrm{v} \tag{13a}$$

$$\delta\dot{\mathrm{v}} = \frac{T}{m}\mathrm{e}_T + 3\frac{\mu(\delta\mathrm{r}\bullet\rho)}{\rho^5}\rho - \frac{\mu}{\rho^3}\delta\mathrm{r} \tag{13b}$$

$$\dot{m} = -\frac{T}{g_o I_{sp}} \tag{13c}$$

Here, $\delta\mathbf{r}$ and $\delta\mathbf{v}$ represent displacement from an osculating orbit or the initial orbit, $\mathbf{e}_T$ is the thrust direction, T is the thrust, m is the mass, μ is the gravitational constant, and ρ represents the initial orbit which satisfies identical dynamics but without the thrust term.

Writing the Hamiltonian for this, the approximated system, gives

$$H = \lambda_{\mathrm{r}}{}^{\mathrm{T}}\delta\mathrm{v} + \lambda_{\mathrm{v}}{}^{\mathrm{T}}\left[\frac{T}{m}\mathrm{e}_T + 3\frac{\mu(\delta\mathrm{r}\bullet\rho)}{\rho^5}\rho - \frac{\mu}{\rho^3}\delta\mathrm{r}\right] - \lambda_m\frac{T}{g_o I_{sp}} \tag{14}$$

Evaluating the Euler-Lagrange equations results in the following differential equations involving the costates:

$$\dot{\lambda}_r = -3\frac{\mu(\lambda_v{}^T\rho)}{\rho^5}\rho + \frac{\mu}{\rho^3}\lambda_v \tag{15a}$$

$$\dot{\lambda}_v = -\lambda_r \tag{15b}$$

$$\dot{\lambda}_m = -\frac{T}{m^2|\lambda_v|} \tag{15c}$$

The control, $\mathbf{e}_T$ is

$$\mathbf{e}_T = \frac{\lambda_v}{|\lambda_v|} \tag{16}$$

and the control T is bang-bang, governed by the switching function, H_T, as

$$H_T = \frac{|\lambda_v|}{m} - \frac{\lambda_m}{g_o I_{sp}} \tag{17}$$

$$\begin{aligned} H_T > 0, \quad T = T_{max} \\ H_T < 0, \quad T = 0 \end{aligned} \tag{18}$$

Pleasantly, Eqns. (15) happen to be the differential equations for the costates on a coast arc coinciding with the initial orbit or the osculating orbit. The coast arc costates have been solved by Lawden and other authors[9,10]. However, it is Glandorf's formulation, actually based on work by Pines[11], that is currently being considered. His formulation is in the following form:

$$\begin{bmatrix} \lambda_v(t) \\ -\lambda_r(t) \end{bmatrix} = \mathrm{P}(t)\left[\mathrm{P}(t_0)\right]^{-1}\begin{bmatrix} \lambda_v(t_0) \\ -\lambda_r(t_0) \end{bmatrix} \tag{19}$$

Considering the form of the state dynamics, their solution can then be written as

$$\begin{bmatrix} \delta r(t) \\ \delta v(t) \end{bmatrix} = \mathrm{P}(t)\left[\mathrm{P}(t_0)\right]^{-1}\begin{bmatrix} \delta r(t_0) \\ \delta v(t_0) \end{bmatrix} + T\mathrm{P}(t)\int_{t_0}^{t}\left[\frac{\left[\mathrm{P}(\tau)\right]^{-1}\begin{bmatrix} 0 \\ \lambda_v(\tau) \end{bmatrix}}{m(\tau)|\lambda_v(\tau)|}\right]d\tau \tag{20}$$

An analytical expression for the integral has been rather elusive. Currently, work is focused on approximating the integral. For example, if the magnitude of the Lagrange multiplier is approximated as

$$|\lambda_v(\tau)| \approx g(\tau) \left\| \begin{bmatrix} \lambda_v(t_0) \\ -\lambda_r(t_0) \end{bmatrix} \right\| \tag{21}$$

where the function $g(t)$ represents a "curve fit" of sorts, then Eq.(20) becomes

$$\begin{bmatrix} \delta r(t) \\ \delta v(t) \end{bmatrix} \approx P(t)[P(t_0)]^{-1} \begin{bmatrix} \delta r(t_0) \\ \delta v(t_0) \end{bmatrix} + TP(t) \left[\int_{t_0}^{t} \left[\frac{[P(\tau)]^{-1} \begin{bmatrix} 0 \\ [I \quad 0]P(\tau) \end{bmatrix}}{m(\tau)g(\tau)} \right] d\tau \right] [P(t_0)]^{-1} \frac{\begin{bmatrix} \lambda_v(t_0) \\ -\lambda_r(t_0) \end{bmatrix}}{\left\| \begin{bmatrix} \lambda_v(t_0) \\ -\lambda_r(t_0) \end{bmatrix} \right\|} \tag{22}$$

Now, the integral only has to be evaluated once for each choice of burn, saving a considerable about of computation time. Finally, note that burns are not restricted in length, using osculating orbits (much as in Encke's method) the burn lengths are actually rather arbitrary. The only consideration for burn length, then, is the error accumulated by approximated functions during integration.

To formulate a method for computing the transfer, the above discussion hints to a burn-by-burn approach. Burns would be guessed by a user via a set of transfer orbits and burn times. Each burn would then be approached as a single-burn rendezvous problem. This produces a sub-optimal transfer and can be thought of as a predictor step. The corrector step would then consist of iterations to make it an optimal transfer; either a direct optimization of the transfer orbit elements and burn times or an indirect optimization by multiple-shooting.

IV. CONCLUSIONS

The development of the Direction Correction method is proceeding rather well. At the time of this report we are not prepared to say whether the method will be successful. The ideas that it is based upon have been validated individually. It has also produced a fair approximation to the solution of a known problem. Further testing of the method is required in order to determine just how robust it is; but at this point it seems pretty clear that method will work.

The Patched Transfer Method is very promising. Glandorf's formulation for the Lagrange multipliers been checked numerically and a suitable approximation for the

Lagrange multiplier magnitude is forthcoming. The next steps are to refine the predictor-corrector idea, code the method, and test it.

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