Modelling and Control of a Rotor Supported by Magnetic Bearings

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Abstract: *In this paper we develop a dynamical model of a rotor and the active magnetic bearings used to support the rotor. We use this model to develop a stable state feedback control of the magnetic bearing system.*

We present the development of a rigid body model of the rotor, utilizing both Rotation Matrices (Euler Angles) *and* Euler Parameters (Quaternions). *In the latter half of the paper we develop a stable state feedback control of the actively controlled magnetic bearing to control the rotor position under imbalances. The control law developed takes into account the variation of the model with rotational speed. We show stability over the whole operating range of speeds for the magnetic bearing system. Simulation results are presented to demonstrate the closed loop system performance. We develop the model of the magnetic bearing, and present two schemes for the excitation of the poles of the actively controlled magnetic bearing. We also present a scheme for averaging multiple sensor measurements and splitting the actuation forces amongst redundant actuators.*

1 Introduction

Several representations of rigid body rotations, including *Rotation Matrices*, *Cayley -Kline parameters*, *Euler Parameters* & *Spinors* ([K.W86] [PP80] [OB79]) have been developed. Conventional methods of deriving rigid body dynamics utilize the Euler angle parametrization of the space of orientations of the rigid body. Such a parametrization of $SO(3)$ suffers from coordinate singularities. The singularities are entirely a result of the choice of parametrization. A parametrization that is globally nonsingular is the parametrization utilizing Euler parameters (unit quaternions). In this paper we develop the dynamical equations describing the rigid body model of a rotor supported by actively controlled magnetic bearings, using rotation matrices (parametrized by euler angles) and using euler parameters.

We begin this paper by giving a brief description of the various mathematical terms and ideas that will be used in defining *rotation matrices* and *euler parameters* [YCBDB82]. We present some of the properties of *rotation matrices* and *quaternions*. We derive the dynamical equations of the rotor supported by active magnetic bearings using both *rotation matrices* and *quaternions*. We present the development of a stable state feedback control law and simulation results of the system when controlled by this state feedback control law. Most magnetic bearing systems are comprised of redundant sensors and actuators. We present a linear algebraic technique utilizing the method of least squares to average multiple measurements and split the actuation forces amongst redundant actuators.

2 Preliminaries

The magnetic bearing system utilizes many frames of reference, in which various quantities such as positions, velocities and angles are described. In this section we will set out the notation by which we will refer to the various quantities. Also we will present some of the definitions and facts necessary for the derivation of the dynamical equations.

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2.1 Notation

In this section we will set out the notation by which we will refer to the various quantities.

- Vectors will be referred to by lower case letters, with an arrow on top. Position vectors will be represented in either Cartesian or spherical coordinate systems. Representation of vectors in Cartesian coordinate systems will be the $\vec{X}, \vec{Y}, \vec{Z}$ components of the vector. Vectors would also be represented as column matrices. The components of the column matrix would not contain an arrow.

  Example 2.1 *The vector $\vec{a}$ would be represented in either of the two ways.*

  $$\vec{a} = a_x\vec{X} + a_y\vec{Y} + a_z\vec{Z}$$

  $$\vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

  The components of a vector would be written without the arrow on top.

- Matrices and tensors will be referred to by upper case letters. Dimensions and components of the matrix will always accompany the notation. Example: $A \in \Re^{3\times 3}$
- Variables and constants will be denoted by lower case letters, and will be accompanied by a statement concerning their dimension. Example: $\theta \in S$.
- When referring to variables, which are described in a frame of reference, the subscript of the variable will refer to the frame of reference. Example: $\vec{\omega}^{rotor}_{inertial}$
- We will use three frames of reference primarily.

  1. The first frame of reference is the fixed inertial frame of reference. This frame will be referred to as the **inertial frame**, and the subscript *inertial* will accompany variables described in this frame of reference. The orthonormal coordinate vectors of this reference frame will be denoted as $\vec{X}_{inertial}, \vec{Y}_{inertial}, \vec{Z}_{inertial}$.
  2. The second frame of reference is rigidly attached to the center of mass of the rotor, and moves with the rotor. This frame will be referred to as the **rotor based reference frame**, and the subscript *rotor* will accompany variables described in this frame of reference. The orthonormal coordinate vectors of this reference frame will be denoted as $\vec{X}_{rotor}, \vec{Y}_{rotor}, \vec{Z}_{rotor}$. The orientation of the rotor frame is along the principal axes of inertia of the rotor. The origin of the rotor frame is at the center of mass of the rotor.
  3. The third frame of reference is attached to the center of the magnetic bearing. We will say more about this frame later. The origin of this frame is coincident with the center of the bearing. This frame will be referred to as the **bearing based reference frame**, and the subscript *bearing* will accompany variables described in this reference frame. The orthonormal coordinate vectors of this reference frame will be denoted as $\vec{X}_{bearing}, \vec{Y}_{bearing}, \vec{Z}_{bearing}$.

- Rotation matrices relate the orientation of vectors in one frame relative to another. The convention we employ through this report would be as follows.

$$\begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-1} = R^{frame-2}_{frame-1} \begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-2} \tag{1}$$

  where $R^{frame-2}_{frame-1}$ is a rotation matrix that expresses the basis vectors of $frame-2$ in terms of the basis vectors of $frame-1$. The rotation matrix can be expressed as a combination of basis vectors of both the frames in the following manner. Given that the basis vectors of frames 1 and 2 are represented with the appropriate subscript,

$$R^{frame-2}_{frame-1} = \begin{bmatrix} \vec{X}_{frame-2}\cdot\vec{X}_{frame-1} & \vec{Y}_{frame-2}\cdot\vec{X}_{frame-1} & \vec{Z}_{frame-2}\cdot\vec{X}_{frame-1} \\ \vec{X}_{frame-2}\cdot\vec{Y}_{frame-1} & \vec{Y}_{frame-2}\cdot\vec{Y}_{frame-1} & \vec{Z}_{frame-2}\cdot\vec{Y}_{frame-1} \\ \vec{X}_{frame-2}\cdot\vec{Z}_{frame-1} & \vec{Y}_{frame-2}\cdot\vec{Z}_{frame-1} & \vec{Z}_{frame-2}\cdot\vec{Z}_{frame-1} \end{bmatrix} \tag{2}$$

- Position vectors of objects will be referred to in the following manner.

$$\vec{r}^{\,object}_{frame} = \begin{bmatrix} r^{object}_{x-frame} \\ r^{object}_{y-frame} \\ r^{object}_{z-frame} \end{bmatrix} \tag{3}$$

- The cross product form of a vector is referred to as $S(\cdot)$. That is

$$\begin{aligned}
\vec{a} \times \vec{b} &= S(a)b \\
S(a) &= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix} \\
\vec{a} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \\
\vec{b} &= \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}
\end{aligned}$$

- A unit vector in the direction of an object will be represented as

$$\vec{u}^{object}_{frame} = \begin{bmatrix} u^{object}_{x-frame} \\ u^{object}_{y-frame} \\ u^{object}_{z-frame} \end{bmatrix} \tag{4}$$

$$= \frac{\vec{r}^{object}_{frame}}{||\vec{r}^{object}_{frame}||_2} \tag{5}$$

$$||\vec{r}^{object}_{frame}||_2 = \sqrt{[r^{object}_{x-frame}]^2 + [r^{object}_{y-frame}]^2 + [r^{object}_{z-frame}]^2} \tag{6}$$

- By norm, we refer to the Euclidean norm of the vector throughout this report.
- If the object being referred to is the origin of a coordinate frame, it will be referred to as $\vec{r}^{frame-2-origin}_{frame-1}$. To set the ideas clear, consider the following examples.

Example 2.2 *The origin of the rotor coordinate frame, as observed in an inertial reference frame would be represented as*

$$\vec{r}^{rotor-origin}_{inertial} = \begin{bmatrix} r^{rotor-origin}_{x-inertial} \\ r^{rotor-origin}_{y-inertial} \\ r^{rotor-origin}_{z-inertial} \end{bmatrix} \tag{7}$$

We tag the word origin when we explicitly refer to the origin of a coordinate frame.

- Homogenous Transforms

Mappings between points in the Euclidean group $SE(3)$ to points in $SE(3)$ are represented as 4×4 matrix transformations that map a position and orientation of a frame to another position and orientation. Such mappings are termed *homogenous transforms*, and in coordinates are specified as follows.

$$\begin{aligned}
T^{frame-2}_{frame-1} &: SE(3) \rightarrow SE(3) \\
\begin{bmatrix} \vec{X}_{frame-1} \\ \vec{Y}_{frame-1} \\ \vec{Z}_{frame-1} \\ 1 \end{bmatrix} &= T^{frame-2}_{frame-1} \begin{bmatrix} \vec{X}_{frame-2} \\ \vec{Y}_{frame-2} \\ \vec{Z}_{frame-2} \\ 1 \end{bmatrix} \\
T^{frame-2}_{frame-1} &= \begin{bmatrix} R^{frame-2}_{frame-1} & \begin{bmatrix} r^{frame-2-origin}_{x-frame-1} \\ r^{frame-2-origin}_{y-frame-1} \\ r^{frame-2-origin}_{z-frame-1} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}
\end{aligned}$$

The inverse of a homogenous transform $T^{frame-2}_{frame-1}$ is represented and given as follows.

$$\begin{aligned}
[T^{frame-2}_{frame-1}]^{-1} &= T^{frame-1}_{frame-2} \\
&= \begin{bmatrix} [R^{frame-2}_{frame-1}]^T & -[R^{frame-2}_{frame-1}]^T \begin{bmatrix} r^{frame-2-origin}_{x-frame-1} \\ r^{frame-2-origin}_{y-frame-1} \\ r^{frame-2-origin}_{z-frame-1} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}
\end{aligned}$$

Example 2.3 *The homogenous tranformation tranforming coordinates of the origin of the bearing coordinate frame as observed in the inertial coordinate frame to the same coordinates as observed from the rotor reference frame would be represented as follows.*

$$\begin{bmatrix} r^{bearing-origin}_{x-rotor} \\ r^{bearing-origin}_{y-rotor} \\ r^{bearing-origin}_{z-rotor} \\ 1 \end{bmatrix} = T^{inertial}_{rotor} \begin{bmatrix} r^{bearing-origin}_{x-inertial} \\ r^{bearing-origin}_{y-inertial} \\ r^{bearing-origin}_{z-inertial} \\ 1 \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} R^{inertial}_{rotor} & \begin{bmatrix} r^{inertial-origin}_{x-rotor} \\ r^{inertial-origin}_{y-rotor} \\ r^{inertial-origin}_{z-rotor} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} r^{bearing-origin}_{x-inertial} \\ r^{bearing-origin}_{y-inertial} \\ r^{bearing-origin}_{z-inertial} \\ 1 \end{bmatrix} \tag{9}$$

2.2 Definitions

We will use the properties of linear vector spaces, quaternions and quaternion algebra. In this section we begin by defining vector spaces and algebras. We then proceed to state/derive the properties of rotation matrices and euler parameters (quaternions and quaternion algebra) [L.A79] [AR88] [AR89] [RA87] [K.W86].

Groups: A *group* is a set X with an internal operation $X \times X \rightarrow X$, such that

- the operation is associative

  $$(xy)z = x(yz) \quad \forall\, x, y, z \in X,$$

- there is an element $e \in X$ called the identity such that

  $$xe = ex = x \quad \forall\ x \in X,$$

- for each $x \in X$ there is an element of X called the inverse of x (written x^{-1}), such that

  $$x^{-1}x = xx^{-1} = e.$$

Usually this group operation is referred to as multiplication. If the operation is commutative then it is referred to as addition and the group is called an *Abelian Group.*

Ring: A *ring* is a set X with two internal operations called multiplication and addition, such that

- X is an abelian group under addition,
- multiplication is associative and distributive with respect to addition.

If the group has an element $e \in X$ such that $ex = xe = x \quad \forall\, x \in X$ it is called a *ring with identity.* Also, if $x \in X$ has an inverse, then it is said to be *regular.*

Field: A *field* is a ring with identity, all the elements of which (except zero, the additive identity) are regular.

Module: A *module* X over a ring R is an abelian group X with an external operation, called scalar multiplication, such that

$$\begin{aligned} \alpha(x+y) &= \alpha x + \alpha y \\ (\alpha+\beta)x &= \alpha x + \beta x \\ (\alpha\beta)x &= \alpha(\beta x) \end{aligned}$$

for all $\alpha, \beta \in \Re$ and $x, y \in X$.

Algebra: An *algebra* A is a module over a ring R with identity with an internal operation called multiplication such that

- A is a ring,
- the external operation $(\alpha, x) \rightarrow \alpha x$ is such that

  $$\alpha(xy) = (\alpha x)y = x(\alpha y).$$

Vector Spaces: A *vector or linear space* is a module for which the ring of operators is a field. Its elements are called vectors.

Quaternion: Quaternions can be viewed in many ways. A *quaternion* is defined as an operator with a scalar q_0 and a vector part $\vec{q}$, expressed either as a sum of its parts,

$$\mathbf{q} = \{q_0 + \vec{q}\}$$

or as a four dimensional vector,

$$\mathbf{q} = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}.$$

If $q_0 = 0$ then the quaternion is called a *vector quaternion*, and if $\vec{q} = \vec{0}$ then it is said to be a *scalar quaternion*. In this paper we notate the *quaternions* by boldface letters.

2.3 Properties of Rotation Matrix

The rotation matrix that relates the orientation of one frame relative to another requires the specification of three angles, and can be parametrized in a number of ways. We now indicate two commonly utilized parametrizations.

- **Fixed Axis Rotations:** Let $frame-1$ and $frame-2$ be coincident to begin with.
  - Rotate $frame-2$ through an angle θ_x about the vector $\vec{X}_{frame-1}$.
  - Rotate $frame-2$ through an angle θ_y about the vector $\vec{Y}_{frame-1}$.
  - Rotate $frame-2$ through an angle θ_z about the vector $\vec{Z}_{frame-1}$.

The resulting rotation matrix, relating the coordinate vectors of $frame-2$ to the coordinate vectors of $frame-1$ can be given as

$$R_{frame-1}^{frame-2} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_x & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

Comment 2.1 Note that as the successive rotations are performed about the fixed axes, the rotation matrices are premultiplied in the order in which the rotations are performed.

$$R_{frame-1}^{frame-2} = \begin{bmatrix} \cos\theta_z\cos\theta_y & \cos\theta_z\sin\theta_y\sin\theta_x - \sin\theta_z\cos\theta_x & \cos\theta_z\sin\theta_y\cos\theta_x + \sin\theta_z\sin\theta_x \\ \sin\theta_z\cos\theta_y & \sin\theta_z\sin\theta_y\sin\theta_x + \cos\theta_z\cos\theta_x & \sin\theta_z\sin\theta_y\cos\theta_x - \cos\theta_z\sin\theta_x \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_y\cos\theta_x \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-1} = R_{frame-1}^{frame-2} \begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-2} \tag{11}$$

- **Moving Axis Rotations:** Let $frame-1$ and $frame-2$ be coincident to begin with.
  - Rotate $frame-2$ through an angle θ_z about the vector $\vec{Z}_{frame-2}$.
  - Rotate $frame-2$ through an angle θ_y about the vector $\vec{Y}_{frame-2}$.
  - Rotate $frame-2$ through an angle θ_x about the vector $\vec{X}_{frame-2}$.

  The resulting rotation matrix, relating the coordinate vectors of $frame-2$ to the coordinate vectors of $frame-1$ can be given as

$$R_{frame-1}^{frame-2} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_x & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

  Comment 2.2 Note that as the rotations are performed about the moving axes, the rotation matrices are post-multiplied in the order in which the rotations are performed.

$$R_{frame-1}^{frame-2} = \begin{bmatrix} \cos\theta_z\cos\theta_y & \cos\theta_z\sin\theta_y\sin\theta_x - \sin\theta_z\cos\theta_x & \cos\theta_z\sin\theta_y\cos\theta_x + \sin\theta_z\sin\theta_x \\ \sin\theta_z\cos\theta_y & \sin\theta_z\sin\theta_y\sin\theta_x + \cos\theta_z\cos\theta_x & \sin\theta_z\sin\theta_y\cos\theta_x - \cos\theta_z\sin\theta_x \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_y\cos\theta_x \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-1} = R_{frame-1}^{frame-2} \begin{bmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{bmatrix}_{frame-2} \tag{13}$$

We now note three important properties of rotation matrices.

$$\dot{R}^{frame-2}_{frame-1} = S(\vec{\omega}^{frame-2}_{frame-1})R^{frame-2}_{frame-1} \tag{14}$$

$$\frac{d}{dt}[R^{frame-2}_{frame-1}]^T = -[R^{frame-2}_{frame-1}]^T S(\vec{\omega}^{frame-2}_{frame-1}) \tag{15}$$

$$S(R^{frame-2}_{frame-1}a) = R^{frame-2}_{frame-1}S(a)R^{frame-1}_{frame-2} \tag{16}$$

We will now derive expressions for angular velocity of the object as a function of the derivatives of the parametrizations of the orientation angles. We will now derive the derivatives of the elementary rotation matrices.

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \tag{17}$$

$$\frac{d}{dt}R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\theta_x & -\cos\theta_x \\ 0 & \cos\theta_x & -\sin\theta_x \end{bmatrix}\dot{\theta}_x \tag{18}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \tag{19}$$

$$= S(\dot{\theta}_x)R(\theta_x) \tag{20}$$

$$\frac{d}{dt}R(\theta_y) = S(\dot{\theta}_y)R(\theta_y) \tag{21}$$

$$\frac{d}{dt}R(\theta_z) = S(\dot{\theta}_z)R(\theta_z) \tag{22}$$

For the case of fixed axis rotations, we note that

$$R^{frame-2}_{frame-1} = R(\theta_z)R(\theta_y)R(\theta_x) \tag{23}$$

$$\frac{d}{dt}R^{frame-2}_{frame-1} = [\frac{d}{dt}R(\theta_z)]R(\theta_y)R(\theta_x) + R(\theta_z)[\frac{d}{dt}R(\theta_y)]R(\theta_x) \tag{24}$$

$$+R(\theta_z)R(\theta_y)[\frac{d}{dt}R(\theta_x)] \tag{25}$$

$$\frac{d}{dt}R^{frame-2}_{frame-1} = S(\dot{\theta}_z)R(\theta_z)R(\theta_y)R(\theta_x) + R(\theta_z)S(\dot{\theta}_y)R(\theta_y)R(\theta_x) \tag{26}$$

$$+R(\theta_z)R(\theta_y)S(\dot{\theta}_x)R(\theta_x) \tag{27}$$

$$\frac{d}{dt}R^{frame-2}_{frame-1} = S(\dot{\theta}_z)R(\theta_z)R(\theta_y)R(\theta_x) \tag{28}$$

$$+R(\theta_z)S(\dot{\theta}_y)R^T(\theta_z)R(\theta_z)R(\theta_y)R(\theta_x) \tag{29}$$

$$+R(\theta_z)R(\theta_y)S(\dot{\theta}_x)[R(\theta_z)R(\theta_y)]^T R(\theta_z)R(\theta_y)R(\theta_x) \tag{30}$$

$$\frac{d}{dt}R^{frame-2}_{frame-1} = [S(\dot{\theta}_z) + S(R(\theta_z)\dot{\theta}_y) + S(R(\theta_z)R(\theta_y)\dot{\theta}_x)]R^{frame-2}_{frame-1} \tag{31}$$

$$S(\vec{\omega}^{frame-2}_{frame-1})R^{frame-2}_{frame-1} = [S(\dot{\theta}_z) + S(R(\theta_z)\dot{\theta}_y) + S(R(\theta_z)R(\theta_y)\dot{\theta}_x)]R^{frame-2}_{frame-1} \tag{32}$$

$$S(\vec{\omega}^{frame-2}_{frame-1}) = S(\dot{\theta}_z) + S(R(\theta_z)\dot{\theta}_y) + S(R(\theta_z)R(\theta_y)\dot{\theta}_x) \tag{33}$$

We simplify the above expression to get,

$$S(\vec{\omega}^{frame-2}_{frame-1}) = \begin{bmatrix} 0 & -\dot{\theta}_z & 0 \\ \dot{\theta}_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$+\begin{bmatrix} 0 & 0 & \cos\theta_z\dot{\theta}_y \\ 0 & 0 & \sin\theta_z\dot{\theta}_y \\ -\cos\theta_z\dot{\theta}_y & -\sin\theta_z\dot{\theta}_y & 0 \end{bmatrix} \tag{35}$$

$$+\begin{bmatrix} 0 & \sin\theta_y\dot{\theta}_x & \cos\theta_y\sin\theta_z\dot{\theta}_x \\ -\sin\theta_y\dot{\theta}_x & 0 & -\cos\theta_y\cos\theta_z\dot{\theta}_x \\ -\cos\theta_y\sin\theta_z\dot{\theta}_x & \cos\theta_y\cos\theta_z\dot{\theta}_x & 0 \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} 0 & \omega^{frame-2}_{z-frame-1} & \omega^{frame-2}_{y-frame-1} \\ \omega^{frame-2}_{z-frame-1} & 0 & -\omega^{frame-2}_{x-frame-1} \\ -\omega^{frame-2}_{y-frame-1} & \omega^{frame-2}_{x-frame-1} & 0 \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \omega^{frame-2}_{x-frame-1} \\ \omega^{frame-2}_{y-frame-1} \\ \omega^{frame-2}_{z-frame-1} \end{bmatrix} = \begin{bmatrix} \cos\theta_y \cos\theta_z \dot{\theta}_x - \sin\theta_z \dot{\theta}_y \\ \cos\theta_y \sin\theta_z \dot{\theta}_x + \cos\theta_z \dot{\theta}_y \\ \dot{\theta}_z - \sin\theta_y \dot{\theta}_x \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} \cos\theta_y \cos\theta_z & -\sin\theta_z & 0 \\ \cos\theta_y \sin\theta_z & \cos\theta_z & 0 \\ -\sin\theta_y & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \tag{39}$$

Note from equation (39) that determinant of the matrix relating the angular velocities of $frame-2$ and the derivatives of the parametrization $\cos\theta_y$. Therefore, the matrix is invertible for small values of the angle θ_y.

2.4 Properties of Quaternions

We will present here the properties of quaternion algebra that we use in this paper [1] [K.W86] [OB79]. We will also derive the derivatives of quaternions.

Quaternion addition: The sum of two quaternions $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_0 + y_0 \\ \vec{x} + \vec{y} \end{bmatrix}.$$

Quaternion product: The product of two quaternions $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{xy} = \begin{bmatrix} -\vec{x}^T & x_0 \\ x_0\vec{1} + \vec{x}\times & \vec{x} \end{bmatrix} \mathbf{y} = \begin{bmatrix} x_0 y_0 - \vec{x}\cdot\vec{y} \\ x_0\vec{y} + y_0\vec{x} + \vec{x}\times\vec{y} \end{bmatrix}.$$

Quaternion conjugate: The conjugate of a quaternion $\mathbf{q}$ is given by

$$\mathbf{q}^* = \begin{bmatrix} q_0 \\ -\vec{q} \end{bmatrix}.$$

Quaternion norm: The norm of a quaternion $\mathbf{q}$ is defined to be

$$||\mathbf{q}||^2 = \mathbf{q}^*\mathbf{q} = {q_0}^2 + \vec{q}\cdot\vec{q}.$$

This is analogous to the euclidean vector norm of a four dimensional vector.

Quaternion Inverse: The inverse of a quaternion $\mathbf{q}$ is defined as

$$\mathbf{q}^{-1} = \frac{1}{||\mathbf{q}||^2}\mathbf{q}^*.$$

It can be verified that this inverse has the property that

$$\mathbf{q}^{-1}\mathbf{q} = \mathbf{q}\mathbf{q}^{-1} = 1.$$

Rotation operation: A rotation of a vector $\vec{x}$ by θ about an axis $\vec{n}$ is given by

$$\mathbf{q}\vec{x}\mathbf{q}^*$$

where $\mathbf{q}$ is the quaternion given by

$$\mathbf{q} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})\vec{n} \end{bmatrix}$$

The derivatives of the *Quaternion* representing a rotation operation are given by

$$\dot{\mathbf{q}} = \mathbf{q}\{\frac{1}{2}\vec{w}\}$$

$$\ddot{\mathbf{q}} = \mathbf{q}\{\frac{1}{2}\vec{\dot{w}}\} + \mathbf{q}\{\frac{1}{2}\vec{w}\}\{\frac{1}{2}\vec{w}\}$$

where $\vec{w}$ is the angular velocity and $\vec{\dot{w}}$ is the angular acceleration.
The angular velocity and angular accelerations are given in terms of the quaternions through the following relations:

$$\vec{w} = 2\mathbf{q}^*\dot{\mathbf{q}}$$

$$\vec{\dot{w}} = 2\mathbf{q}^*\ddot{\mathbf{q}} + 2\dot{\mathbf{q}}^*\dot{\mathbf{q}}$$

[1] Boldface letters represent *quaternions*

Let us now calculate the derivative of the conjugate of the quaternion. As the quaternions representing a rotation operation are *unit quaternions* (unity norm), the inverse is the conjugate. Hence

$$\begin{aligned} 1 &= \mathbf{q}\mathbf{q}^* \\ 0 &= \dot{\mathbf{q}}\mathbf{q}^* + \mathbf{q}\dot{\mathbf{q}}^* \\ \dot{\mathbf{q}}^* &= -\mathbf{q}^{-1}\dot{\mathbf{q}}\mathbf{q}^* \\ &= -\mathbf{q}^*\dot{\mathbf{q}}\mathbf{q}^* \\ &= -\mathbf{q}^*\mathbf{q}\{\frac{1}{2}\vec{w}\}\mathbf{q}^* \end{aligned}$$

2.5 Relation between Rotation Matrices & Euler Parameters

The operation of rotation by an angle θ about an axis $\vec{n}$ can be represented by both a rotation matrix (R) and euler parameters ($\mathbf{q}$) as

$$R = \cos(\theta)\vec{1} + (1 - \cos(\theta))\vec{n}\vec{n}^T + \sin(\theta)\vec{n}\times$$

$$\mathbf{q} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})\vec{n} \end{bmatrix}$$

These two representations can be related as follows:

$$R = ({q_0}^2 - \vec{q}^T\vec{q})I_{3\times 3} + 2\vec{q}\vec{q}^T + 2q_0\vec{q}\times$$

3 Dynamical Equations of the Rotor

In this section we derive the equations of motion of the rigid body rotor supported by active magnetic bearings. We begin this section, with a derivation using rotation matrices, and then proceed to do the same using euler parameters (quaternions).

3.1 Dynamical Equations using *Rotation Matrices*

To eliminate ambiguity regarding the specification of reference frames, we will primarily work in the rotor reference frame, and finally transform the coordinates to the inertial reference frame. We derive the dynamic equations of the magnetic bearing in a systematic manner. For an excellent exposition on kinematics, refer to [RS94].

Step 1.

We compute the angular momentum of the rotor about the origin of the rotor reference frame, denoted as $\vec{H}^{rotor}_{rotor}$ as

$$\vec{H}^{rotor}_{rotor} = I^{rotor}_{rotor}\vec{\omega}^{rotor}_{rotor} \tag{40}$$

Step 2.

We utilize the principle of torque balance to relate the rate of change of angular momentum to the net torque. We note here that by net torques (T^{rotor}) we refer to the summation of the applied torques (τ^{rotor}, and the moments of the applied forces ($\vec{F}^i$) about the origin of the rotor reference frame. It is to be understood of course that the quantities on either side of the equality will be referenced in one coordinate frame. Indeed to avoid ambiguity, we will henceforth refer to each of the aforementioned quantities in a single coordinate frame. Expressing all quantities in the inertial reference frame, we get

$$\vec{H}^{rotor}_{rotor} = I^{rotor}_{rotor}R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}$$

Step 3.

We utilize Newton's torque balance equations to derive the following.

$$T^{rotor}_{rotor} = \frac{d}{dt}[\vec{H}^{rotor}_{rotor}]$$

$$\vec{\tau}^{rotor}_{rotor} + \sum_{i=1}^{n}\vec{r}^{\,i}_{rotor}\times\vec{F}^{i}_{rotor} = I^{rotor}_{rotor}\dot{\vec{\omega}}^{rotor}_{rotor} + \vec{\omega}^{rotor}_{rotor}\times\vec{H}^{rotor}_{rotor}$$

$$R^{inertial}_{rotor}\vec{\tau}^{rotor}_{inertial} + \sum_{i=1}^{n}\vec{r}^{\,i}_{rotor}\times R^{inertial}_{rotor}\vec{F}^{i}_{inertial} = I^{rotor}_{rotor}\dot{\vec{\omega}}^{rotor}_{rotor}$$

$$+R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial} \times R^{inertial}_{rotor} R^{rotor}_{inertial} \vec{H}^{rotor}_{rotor}$$

$$R^{inertial}_{rotor}\dot{\vec{\tau}}^{rotor}_{inertial} + \sum_{i=1}^{n} \vec{r}_{rotor} \times R^{inertial}_{rotor}\vec{F}^{i}_{inertial} = I^{rotor}_{rotor}\dot{\vec{\omega}}^{rotor}_{rotor}$$

$$+R^{inertial}_{rotor}(\vec{\omega}^{rotor}_{inertial} \times R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial})$$

$$R^{inertial}_{rotor}\dot{\vec{\tau}}^{rotor}_{inertial} + \sum_{i=1}^{n} S(\vec{r}_{rotor}) R^{inertial}_{rotor}\vec{F}^{i}_{inertial} = I^{rotor}_{rotor}\dot{\vec{\omega}}^{rotor}_{rotor}$$

$$+R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial}) R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}$$

where $\vec{r}_{rotor}$ is the point of application of the force $\vec{F}^{i}_{inertial}$, and is given as

$$\begin{bmatrix} r^{i}_{x-rotor} \\ r^{i}_{y-rotor} \\ r^{i}_{z-rotor} \\ 1 \end{bmatrix} = T^{inertial}_{rotor} \begin{bmatrix} r^{i}_{x-inertial} \\ r^{i}_{y-inertial} \\ r^{i}_{z-inertial} \\ 1 \end{bmatrix} \tag{41}$$

$$= \begin{bmatrix} R^{inertial}_{rotor} & -R^{inertial}_{rotor} \begin{bmatrix} r^{rotor-origin}_{x-inertial} \\ r^{rotor-origin}_{y-inertial} \\ r^{rotor-origin}_{z-inertial} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} r^{i}_{x-inertial} \\ r^{i}_{y-inertial} \\ r^{i}_{z-inertial} \\ 1 \end{bmatrix} \tag{42}$$

We now compute $\dot{\vec{\omega}}^{rotor}_{rotor}$ in the following manner utilizing (14) - (16)

$$\dot{\vec{\omega}}^{rotor}_{rotor} = \frac{d}{dt}[R^{rotor}_{inertial}]^{T}\vec{\omega}^{rotor}_{inertial} \tag{43}$$

$$= [R^{rotor}_{inertial}]^{T}\dot{\vec{\omega}}^{rotor}_{inertial} - [R^{rotor}_{inertial}]^{T} S(\vec{\omega}^{rotor}_{inertial})\vec{\omega}^{rotor}_{inertial} \tag{44}$$

$$= [R^{rotor}_{inertial}]^{T}\dot{\vec{\omega}}^{rotor}_{inertial} \tag{45}$$

$$= R^{inertial}_{rotor}\dot{\vec{\omega}}^{rotor}_{inertial} \tag{46}$$

Substituting the expression for $\dot{\vec{\omega}}^{rotor}_{rotor}$ from equation(46) in the torque balance equation, we get

$$R^{inertial}_{rotor}\dot{\vec{\tau}}^{rotor}_{inertial} + \sum_{i=1}^{n} S(\vec{r}_{rotor}) R^{inertial}_{rotor}\vec{F}^{i}_{inertial} = I^{rotor}_{rotor} R^{inertial}_{rotor}\dot{\vec{\omega}}^{rotor}_{inertial}$$

$$+R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial}) R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}$$

We now recast the above equation in the following form

$$\dot{\vec{\omega}}^{rotor}_{inertial} = [I^{rotor}_{rotor} R^{inertial}_{rotor}]^{-1}[R^{inertial}_{rotor}\dot{\vec{\tau}}^{rotor}_{inertial} + \sum_{i=1}^{n} S(\vec{r}_{rotor}) R^{inertial}_{rotor}\vec{F}^{i}_{inertial}] \tag{47}$$

$$-[I^{rotor}_{rotor} R^{inertial}_{rotor}]^{-1}[R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial}) R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}] \tag{48}$$

$$\dot{\vec{\omega}}^{rotor}_{inertial} = R^{rotor}_{inertial}[I^{rotor}_{rotor}]^{-1}[R^{inertial}_{rotor}\dot{\vec{\tau}}^{rotor}_{inertial} + \sum_{i=1}^{n} S(\vec{r}_{rotor}) R^{inertial}_{rotor}\vec{F}^{i}_{inertial}] \tag{49}$$

$$-R^{rotor}_{inertial}[I^{rotor}_{rotor}]^{-1} R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial}) R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial} \tag{50}$$

Step 4.

We derive the force balance equations by first calculating the expression for the linear momentum of the rotor in the following manner.

$$\vec{L}^{rotor}_{rotor} = m^{rotor}\vec{v}^{rotor}_{rotor}$$

$$= m^{rotor} R^{inertial}_{rotor}\vec{v}^{rotor}_{inertial}$$

Step 5

Newton's law asserts that the rate of change of linear momentum equals the applied force. That is,

$$R^{inertial}_{rotor}\sum_{i=1}^{n}\vec{F}^{i}_{inertial} = \frac{d}{dt}[\vec{L}^{rotor}_{rotor}] \tag{51}$$

$$= m^{rotor}\dot{\vec{v}}^{rotor}_{rotor} + \vec{\omega}^{rotor}_{rotor} \times \vec{L}^{rotor}_{rotor} \tag{52}$$

$$= m^{rotor}\dot{\vec{v}}^{rotor}_{rotor} + S(R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}) m^{rotor} R^{inertial}_{rotor}\vec{v}^{rotor}_{inertial} \tag{53}$$

We now compute $\dot{\vec{v}}^{rotor}_{rotor}$ in (53) in the following manner utilizing (14) - (16).

$$\dot{\vec{v}}^{rotor}_{rotor} = \frac{d}{dt}[R^{rotor}_{inertial}]^T \vec{v}^{rotor}_{inertial} \tag{54}$$

$$= [R^{rotor}_{inertial}]^T \dot{\vec{v}}^{rotor}_{inertial} - [R^{rotor}_{inertial}]^T S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{55}$$

$$= R^{inertial}_{rotor} \dot{\vec{v}}^{rotor}_{inertial} - R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{56}$$

Substituting the expression for $\dot{\vec{v}}^{rotor}_{rotor}$ from equation (56) in the force balance equation (53), we recast the force balance equation as,

$$R^{inertial}_{rotor} \sum_{i=1}^{n} \vec{F}^i_{inertial} = m^{rotor} R^{inertial}_{rotor} \dot{\vec{v}}^{rotor}_{inertial} - m^{rotor} R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{57}$$

$$+ m^{rotor} S(R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}) R^{inertial}_{rotor} \vec{v}^{rotor}_{inertial} \tag{58}$$

$$\tag{59}$$

We now rewrite the force equation (58) in the following form

$$\dot{\vec{v}}^{rotor}_{inertial} = \frac{1}{m^{rotor}} \sum_{i=1}^{n} \vec{F}^i_{inertial} + S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{60}$$

$$- R^{rotor}_{inertial} S(R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}) R^{inertial}_{rotor} \vec{v}^{rotor}_{inertial} \tag{61}$$

Note that the last term in equation (61) may be simplified in the following manner.

$$R^{rotor}_{inertial} S(R^{inertial}_{rotor}\vec{\omega}^{rotor}_{inertial}) R^{inertial}_{rotor} \vec{v}^{rotor}_{inertial} = R^{rotor}_{inertial} S(\vec{\omega}^{rotor}_{rotor})[R^{rotor}_{inertial}]^T \vec{v}^{rotor}_{inertial} \tag{62}$$

$$= S(R^{rotor}_{inertial}\vec{\omega}^{rotor}_{rotor})\vec{v}^{rotor}_{inertial} \tag{63}$$

$$= S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{64}$$

Substituting the expression in equation (64) in equation (61), we arrive at the force balance equations,

$$\dot{\vec{v}}^{rotor}_{inertial} = \frac{1}{m^{rotor}} \sum_{i=1}^{n} \vec{F}^i_{inertial} + S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} - S(\vec{\omega}^{rotor}_{inertial})\vec{v}^{rotor}_{inertial} \tag{65}$$

$$= \frac{1}{m^{rotor}} \sum_{i=1}^{n} \vec{F}^i_{inertial} \tag{66}$$

Collecting the force and torque balance equations, we write the dynamic equations of the magnetic bearing as follows.

$$\dot{\vec{\omega}}^{rotor}_{inertial} = R^{rotor}_{inertial}[I^{rotor}_{rotor}]^{-1}[R^{inertial}_{rotor}\dot{\vec{r}}^{rotor}_{inertial} + \sum_{i=1}^{n} S(\vec{r}^i_{rotor}) R^{inertial}_{rotor} \vec{F}^i_{inertial}] \tag{67}$$

$$- R^{rotor}_{inertial}[I^{rotor}_{rotor}]^{-1} R^{inertial}_{rotor} S(\vec{\omega}^{rotor}_{inertial}) R^{rotor}_{inertial} I^{rotor}_{rotor} R^{inertial}_{rotor} \vec{\omega}^{rotor}_{inertial} \tag{68}$$

$$\dot{\vec{v}}^{rotor}_{inertial} = \frac{1}{m^{rotor}} \sum_{i=1}^{n} \vec{F}^i_{inertial} \tag{69}$$

3.2 Dynamical Equations using *Quaternions*

As we saw in the previous section in equation 69 the force balance equations essentially are a restatement of $F = ma$ in the inertial coordinates. So we will only consider the angular momentum equations.

$$T^{rotor}_{rotor} = \frac{d}{dt}[\vec{H}^{rotor}_{rotor}] \tag{70}$$

$$\dot{\vec{r}}^{rotor}_{rotor} + \sum_{i=1}^{n} \vec{r}^i_{rotor} \times \vec{F}^i_{rotor} = I^{rotor}_{rotor} \dot{\vec{\omega}}^{rotor}_{rotor} + \vec{\omega}^{rotor}_{rotor} \times \vec{H}^{rotor}_{rotor}$$

Let us look at the first term on the right hand side of 71 initially.

$$I^{rotor}_{rotor} \dot{\vec{\omega}}^{rotor}_{rotor} = I^{rotor}_{rotor} \frac{d}{dt}(\mathbf{q}^{inertial}_{rotor} \vec{\omega}^{rotor}_{inertial} \mathbf{q}^{inertial^*}_{rotor})$$

$$= I^{rotor}_{rotor}(\dot{\mathbf{q}}^{inertial}_{rotor} \vec{\omega}^{rotor}_{inertial} \mathbf{q}^{inertial^*}_{rotor} + \mathbf{q}^{inertial}_{rotor} \dot{\vec{\omega}}^{rotor}_{inertial} \mathbf{q}^{inertial^*}_{rotor}$$

$$
\begin{aligned}
&+\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*}) \\
&= I_{rotor}^{rotor}(\mathbf{q}_{rotor}^{inertial}\{\frac{1}{2}\vec{\omega}_{inertial}^{rotor}\}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} + \mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} \\
&\quad -\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*}\mathbf{q}_{rotor}^{inertial}\{\frac{1}{2}\vec{\omega}_{inertial}^{rotor}\}\mathbf{q}_{rotor}^{inertial*}) \\
&= I_{rotor}^{rotor}(\mathbf{q}_{rotor}^{inertial}\{\frac{1}{2}\vec{\omega}_{inertial}^{rotor}\}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} + \mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} \\
&\quad -\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\{\frac{1}{2}\vec{\omega}_{inertial}^{rotor}\}\mathbf{q}_{rotor}^{inertial*}) \\
&= I_{rotor}^{rotor}(\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*})
\end{aligned}
$$

Substituting this back into 71 we get

$$
\vec{\tau}_{rotor}^{rotor} + \sum_{i=1}^{n}\vec{r}_{rotor}^{i} \times \vec{F}_{rotor}^{i} = I_{rotor}^{rotor}(\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*}) + \vec{\omega}_{rotor}^{rotor} \times \vec{H}_{rotor}^{rotor} \tag{71}
$$

$$
\begin{aligned}
&\mathbf{q}_{rotor}^{inertial}\vec{\tau}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} + \sum_{i=1}^{n}(\mathbf{q}_{rotor}^{inertial}\vec{r}_{inertial}^{i}\mathbf{q}_{rotor}^{inertial*} \times \mathbf{q}_{rotor}^{inertial}\vec{F}_{inertial}^{i}\mathbf{q}_{rotor}^{inertial*}) \\
&= I_{rotor}^{rotor}(\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*}) + \mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} \times \mathbf{q}_{rotor}^{inertial}\vec{H}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
\vec{\omega}_{inertial}^{rotor} &= \mathbf{q}^{inertial}rotor^{*}[I_{rotor}^{rotor}]^{-1}[\mathbf{q}_{rotor}^{inertial}\vec{\tau}_{inertial}^{rotor} + \sum_{i=1}^{n}(\mathbf{q}_{rotor}^{inertial}\vec{r}_{inertial}^{i}\mathbf{q}_{rotor}^{inertial*} \times \mathbf{q}_{rotor}^{inertial}\vec{F}_{inertial}^{i}) \\
&\quad -\mathbf{q}_{rotor}^{inertial}\vec{\omega}_{inertial}^{rotor}\mathbf{q}_{rotor}^{inertial*} \times \mathbf{q}_{rotor}^{inertial}\vec{H}_{inertial}^{rotor}]
\end{aligned} \tag{73}
$$

Comparing equations 68 and 73 we see the equivalence of both these derivations.

We have briefly derived the dynamics equations of the magnetic bearing system using euler parameters, as they have many advantages. Euler Parameters are defined everywhere and they have a nonsingular mapping with the rotational velocity $\vec{w}$. Using Quaternion algebra the above expressions can be further simplified. Simple expressions for all composite rotations and rotating reference frames can be developed [K.W86]. Euler parameters have also been shown to be as efficient computationally as rotation matrices and more compact in storage [JR90].

4 Small Angle Assumption

We have derived a detailed nonlinear model of the rotor supported by active magnetic bearings. We will now present the standard assumptions made in deriving the dynamical equations of the magnetic bearing and the simplification achieved on the nonlinear model [FK90].

In the magnetic bearing system, let the spin axis be x and the pitch and yaw axes be y and z axes. Let the spin angle, pitch and yaw angles be $\theta_x, \theta_y, \theta_z$. Usually we assume that the angles θ_y, θ_z are very small so that $\cos(\theta_y), \cos(\theta_z) \approx 1$, $\sin(\theta_y), \sin(\theta_z) \approx 0$. Also we can reasonably assume that the product of velocities and angular velocities are small and can be ignored. The external forces acting on the system are the forces at the two radial bearing systems F_y^1, F_z^1 and F_y^2, F_z^2; the force at the axial bearing F_z and the external torque τ_{motor} applied along the spin axis. With these assumptions the equations of motion of a rotor supported by magnetic bearings reduce to

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta x} \\ \dot{\theta y} \\ \dot{\theta z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\theta x} \\ \ddot{\theta y} \\ \ddot{\theta z} \end{bmatrix} = F \begin{bmatrix} x \\ y \\ z \\ \theta x \\ \theta y \\ \theta z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta x} \\ \dot{\theta y} \\ \dot{\theta z} \end{bmatrix} + G \begin{bmatrix} F_x \\ F_y^1 \\ F_y^2 \\ F_z^1 \\ F_z^2 \\ \tau_{motor} \end{bmatrix}
$$

where $F \in \Re^{12\times 12}$ and $G \in \Re^{12\times 6}$.

$$F = \begin{bmatrix} 0_{6\times 6} & & & & I_{6\times 6} & & \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0_{6\times 6} & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & -\omega a \\ & 0 & 0 & 0 & 0 & \omega a & 0 \end{bmatrix}$$

where $a \in \Re^{+}$ is a constant dependent on the inertias J_x, J_y.

$$G = \begin{bmatrix} & & & 0_{6\times 6} & & \\ \frac{1}{M} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{M} & \frac{1}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{M} & \frac{1}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{J_x} \\ 0 & 0 & 0 & -\frac{l_1}{J_y} & \frac{l_2}{J_y} & 0 \\ 0 & \frac{l_1}{J_y} & -\frac{l_2}{J_y} & 0 & 0 & 0 \end{bmatrix}$$

5 Feedback control of the rotor motion

The rotor system motion is decoupled between the spin axes and its pitch and yaw axes.Hence for the design of a linear state feedback controller we shall consider the dynamical equations of only the pitch and yaw axes motions of the rotor system, given by

$$\dot{\vec{x}} = \begin{bmatrix} 0_{4\times 4} & I_{4\times 4} \\ 0_{4\times 4} & A[\omega] \end{bmatrix} \vec{x} + \begin{bmatrix} 0_{4\times 4} \\ B \end{bmatrix} u \tag{74}$$

where $\vec{x} = [x_1^1 x_1^2 x_1^3 x_1^4 x_2^1 x_2^2 x_2^3 x_2^4]^T \in \Re^8$, $u = [u^1 u^2 u^3 u^4]^T \in \Re^4$, $A : \Re^{+} \rightarrow \Re^{4\times 4}$, $B \in \Re^{4\times 4}$.

$$A[\omega] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega a \\ 0 & 0 & \omega a & 0 \end{bmatrix} \& B = \begin{bmatrix} \frac{1}{M} & \frac{1}{M} & 0 & 0 \\ 0 & 0 & \frac{1}{M} & \frac{1}{M} \\ 0 & 0 & -\frac{l_1}{J_y} & \frac{l_2}{J_y} \\ \frac{l_1}{J_y} & -\frac{l_2}{J_y} & 0 & 0 \end{bmatrix} \tag{75}$$

Note that the x_1^1, x_2^1 subsystem and x_1^2, x_2^2 subsystem form two decoupled 2×2 systems, while $x_1^3, x_1^4, x_2^3, x_2^4$ form a coupled 4×4 system. Let us choose the control

$$u = B^{-1} \begin{bmatrix} -k_1^1 x_1^1 - k_2^1 x_2^1 \\ -k_1^2 x_1^2 - k_2^2 x_2^2 \\ -k_1^3 x_1^3 - k_2^3 x_2^3 \\ -k_1^4 x_1^4 - k_2^4 x_2^4 \end{bmatrix} \tag{76}$$

such that $k_i^j > 0 for i = 1, 2 : j = 1 \cdots 4$. We can now show that this control stabilizes the $x_3^1, x_3^2, x_4^1, x_4^2$ system at all speeds ω.

Theorem 5.1

Given *(G1) The system given by equation 74*

(G2) Feedbacks u given by equation 76

Then *(T1)* u^1, u^2 *stabilize the system at all speeds* ω.

Proof: ♠▷ Let us choose a Lyapunov function candidate V as follows:

$$V = \frac{\sum_{j=1}^{j=4} k_1^j {x_1^j}^2 + {x_2^j}^2}{2}$$

Taking the derivative of the Lyapunov function we get

$$\begin{aligned} \dot{V} &= \sum_{j=1}^{j=4} k_1^j x_1^j x_2^j - k_1^j x_1^j x_2^j - k_2^j {x_2^j}^2 \\ &\quad -\omega a x_2^3 x_2^4 + \omega a x_2^3 x_2^4 \\ &= \sum_{j=1}^{j=4} -k_2^j {x_2^j}^2 \\ &< 0 \quad if x_2^j \not= 0 for j = 1 \cdots 4 \end{aligned}$$

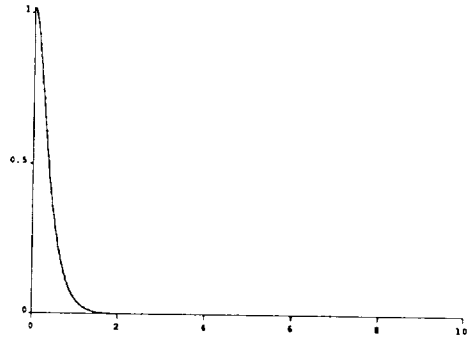

Figure 1: y vs time : w = 0

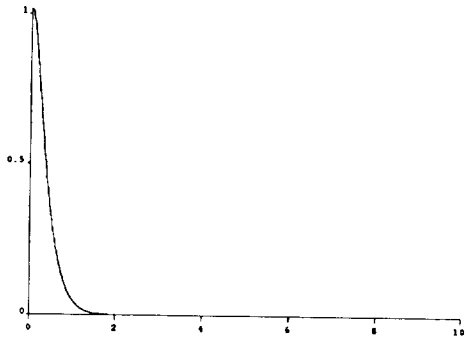

Figure 2: θ_y vs time : w = 0

Hence by Lyapunov theorem x_2^j converge to 0 for $j = 1 \cdots 4$. As the maximum invariant set containing the set $x_2^j = 0$ is $x_1^j = 0$, by LaSalle theorem [M.V78] x_1^j also converges to 0. Hence the system is stable at all ω. ◁♠

5.1 Simulation Results

We simulate this system using the control system simulation package *SIMNON*. We present the simulation results of applying the state feedback control given by equation 76 to the magnetic bearing system. We simulate the system response y and θ_y under the feedback when there is no spin (Figures 1,2) and with a spin of $100 rad/sec$ (Figures 3, 4).

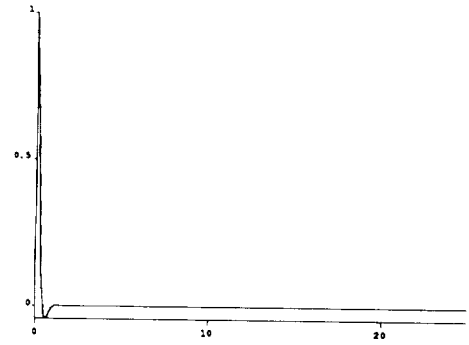

Figure 3: y vs time : w = 100

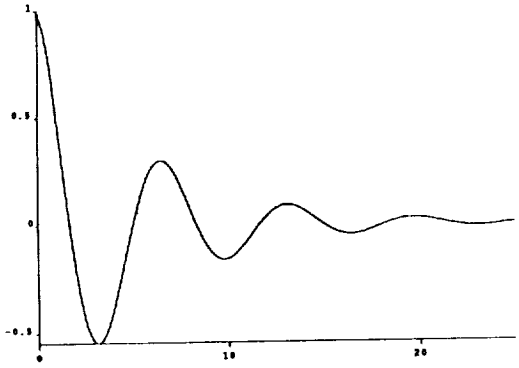

Figure 4: θ_y vs time : w = 100

5.2 Magnetic Bearing Model

In this section we present the basic equations for a single axis magnetic bearing, and two associated pole excitation schemes. The dynamical equations of the magnetic bearing may be written as follows.

$$\dot{\phi}_0 = u_0 \tag{77}$$

$$\dot{\phi}_\pi = u_\pi \tag{78}$$

where

$$\phi_0 \in \Re \quad \text{flux at pole location 0} \tag{79}$$

$$\phi_\pi \in \Re \quad \text{flux at pole location } \pi \tag{80}$$

$$u_0 \in \Re \quad \text{Control at pole location 0} \tag{81}$$

$$u_\pi \in \Re \quad \text{Control at pole location } \pi \tag{82}$$

Let the control force F generated by the magnetic bearing be the net force produced by the bearing elements at the angles 0 and π (the positive and negative poles in a pair). Indeed,

$$F = F_0 - F_\pi \tag{83}$$

We shall design the feedback control of the rotor using the net force as the control actuation. Treating this requisite force as the commanded output of the magnetic bearing subsystem described by equations 77,78, we design the flux feedback as a deadbeat controller. Inherent in this approach is the assumption that the flux feedback loop would be run at a much faster rate than the bandwidth of the force feedback system.

Discretizing the flux equations in the following manner.

$$\begin{bmatrix} \phi_0(k+1) \\ \phi_\pi(k+1) \end{bmatrix} = \begin{bmatrix} \phi_0(k) \\ \phi_\pi(k) \end{bmatrix} + \begin{bmatrix} Tu_0(k) \\ Tu_\pi(k) \end{bmatrix} \tag{84}$$

where

$$u_0(k) = \text{is the net control voltage at pole 0} \tag{85}$$

$$u_\pi(k) = \text{is the net control voltage at pole } \pi \tag{86}$$

We now note the relation between force and flux is given the following form

$$F(k+1) = K_{force-flux}[\phi_0^2(k+1) - \phi_\pi^2(k+1)] \tag{87}$$

where the magnetic constant $K_{force-flux} \in \Re$ relating the forces produced due to fluxes applied is assumed to be known. Choose the control inputs in equation (84) to be of the following form.

$$u_0(k) = \frac{-\phi_0(k) + v_0(k)}{T} \tag{88}$$

$$u_\pi(k) = \frac{-\phi_\pi(k) + v_\pi(k)}{T} \tag{89}$$

where

$$v_0(k) \in \Re \text{ is an exogenous control input, specified later} \tag{90}$$

$$v_\pi(k) \in \Re \text{ is an exogenous control input, specified later} \tag{91}$$

Substituting equations (88) and (89) in (84), we get,

$$\begin{bmatrix} \phi_0(k+1) \\ \phi_\pi(k+1) \end{bmatrix} = \begin{bmatrix} v_0(k) \\ v_\pi(k) \end{bmatrix} \tag{92}$$

Substituting equation (92) in (87), we get,

$$F(k+1) = K_{force-flux}[v_0^2(k) - v_\pi^2(k)] \tag{93}$$

We consider the following choices for choosing the control inputs $v_0(k)$ and $v_\pi(k)$.

5.2.1 Mutually Exclusive Scheme

Choose the control to be

$$w(k) = \frac{-F(k+1)}{K_{force-flux}} \tag{94}$$

Now choose the controls $v_0(k)$ and $v_\pi(k)$ in the following manner.

$$w(k) > 0 \rightarrow \begin{cases} v_0(k) = \sqrt{w(k)} \\ v_\pi(k) = 0 \end{cases} \tag{95}$$

$$w(k) < 0 \rightarrow \begin{cases} v_0(k) = 0 \\ v_\pi(k) = \sqrt{|w(k)|} \end{cases} \tag{96}$$

5.2.2 Biasing Scheme

Choose the following structure for the controls $v_0(k)$ and $v_\pi(k)$.

$$v_0(k) = v_{bias} + v_{variable}(k) \tag{97}$$

$$v_\pi(k) = v_{bias} - v_{variable}(k) \tag{98}$$

where

$$v_{bias} \in \Re \text{ is a constant biasing input} \tag{99}$$

$$v_{variable} \in \Re \text{ is a varying control input} \tag{100}$$

Note that such a structure for $v_0(k)$ and $v_\pi(k)$ implies that

$$\begin{aligned} K_{force-flux}[v_0^2(k) - v_\pi^2(k)] &= K_{force-flux}[(v_{bias} + v_{variable}(k))^2 - [(v_{bias} - v_{variable}(k))^2] \\ &= K_{force-flux}[4v_{bias}v_{variable}k] \end{aligned} \tag{101}$$

We now choose the control $v_{variable}(k)$ to be

$$v_{variable}(k) = \frac{F(k+1)}{K_{force-flux}v_{bias}} \tag{102}$$

where the control F is chosen to stabilize the rotor motion.
Both these excitation schemes have their advantages and disadvantages. In the *constant biasing* scheme, we note that the force to flux equations become linear. Also by choosing as the control is scaled by v_{bias}, change in force (or equivalently currents) required for a certain net force can be reduced. But maintaining a constant biasing voltage may increase the losses. An alternative might be to use permanent magnets to provide the bias voltage. In the mutually exclusive scheme we provide a force (or current) in only one pole, from a pair, at any time. On the other hand, the force to flux relations are nonlinear.

6 Multiple Sensors & Redundant Actuators

In many situations, we measure the same output with multiple sensors and the measurements have to be averaged in some manner. Similarly, in the case when we have redundant actuators (more than the necessary three orthogonal pairs), we need to apportion the actuation forces in an optimal sense, between all the actuators. Linear least squares theory provides us with a method for doing these [RH88] [Aub79] [J.L55]. In this section we will look at using the least squares estimation schemes for averaging measurements from multiple sensors and splitting the forces among redundant actuators.

6.1 Linear Least Squares

Definition 6.1 *A complete inner product space X is called a Hilbert space.*

Definition 6.2 *Given a Hilbert space X, and a subset $U \in X$, the orthogonal complement of U, denoted by $U^{\perp}$ is defined as follows.*

$$U^{\perp} = \{x \in X : < x, u >= 0 \ \forall \ u \in U\} \tag{103}$$

That is, the orthogonal complement of a set $U \in X$ is the set of all vectors in X that are orthogonal to every vector in U.

Theorem 6.1 *Projection Theorem*

Given *(G1) A Hilbert space X.*

If *(I1) $U \in X$ is a closed subspace of X.*

Then *(T1) The Hilbert space X can be decomposed into the direct sum,*

$$X = U \oplus U^{\perp} \tag{104}$$

Definition 6.3 *Let $U \in X$ be a closed subspace of a Hilbert space X. Decompose a vector $\vec{x} \in X$ into the direct sum $\vec{x} = \vec{x}_0 + \vec{x}_1$ where $\vec{x}_0 \in U$ and $\vec{x}_1 \in U^{\perp}$. Then $\vec{x}_0$ is called the orthogonal projection of the $\vec{x} \in X$ onto the subspace $U \in X$.*

Theorem 6.2 *Projection Theorem*

Given *(G1) A Hilbert space X.*

G(2) A direct sum decomposition of $X = U \oplus U^{\perp}$.

G(3) A vector $\vec{x} \in X$.

If *(I1) $\vec{x}_0$ is the orthogonal projection of $\vec{x}$ onto the closed subspace $U \in X$.*

Then *(T1) $\vec{x} - \vec{x}_0$ is the orthogonal projection of $\vec{x}$ onto the close subspace $U^{\perp}$*

Theorem 6.3 *Minimum Norm*

Given *(G1) A Hilbert space X, and a vector $\vec{x} \in X$.*

(G2) A closed subspace $U \in X$.

If *(I1) $\vec{x}_0$ is the orthogonal projection of $\vec{x}$ onto the subspace U.*

Then *(T1) For each $\vec{u} \in U$,*

$$||\vec{x} - \vec{x}_0|| \leq ||\vec{x} - \vec{u}|| \tag{105}$$

Given two Hilbert spaces X, Y, let the operator A be such that $A : X \rightarrow Y$. We now make the following definitions.

Definition 6.4 *The range of $A : X \rightarrow Y$ denoted as $\mathcal{R}(A) = \{A[x] \in Y \ \forall \ x \in X\}$.*

Note that the range of A is the set of all vectors in Y that are obtained by the action of the operator A on every element in X. That is, $\mathcal{R}(A) \subset Y$.

Definition 6.5 *The null space of $A : X \rightarrow Y$ denoted $\mathcal{N}(A) = \{x \in X : \ A[x] = 0_Y\}$*

Note that the null space of A is the set of all vectors in X that are mapped by A into the zero element of Y. It is clear that $\mathcal{N}(A) \subset X$.

Definition 6.6 *The adjoint of a linear operator $A : X \rightarrow Y$, denoted as A^*, is defined as follows.*

- $A^* : Y \rightarrow X$
- $< A[x], y >_Y = < x, A^*[y] >_X \quad \forall \ x \in X, \ y \in Y.$

where $< \cdot >_X$ is the inner product defined in space X, and $< \cdot >_Y$ is the inner product defined in space Y.

The usefulness of the adjoint operator will become evident in the solution of linear equations. The following properties of the adjoint operator are vital to its use.

- Given an operator on a Hilbert space $A : X \rightarrow Y$, and its adjoint $A^* : Y \rightarrow X$, it can be shown that
  1. $\mathcal{N}(A) = \mathcal{N}(A^*A)$
  2. $\mathcal{R}(A) = \mathcal{R}(AA^*)$
- Given an operator on a Hilbert space $A : X \rightarrow Y$, and its adjoint $A^* : Y \rightarrow X$, it can be shown that there exist orthogonal direct sum decompositions of Hilbert spaces X and Y of the following form.
  1. $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$
  2. $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$

6.2 Least Squares Solution of $y = A[x]$

Given a linear operator on the Hilbert space $A : X \to Y$, and a specific $y_1 \in Y$, we define the solution of the linear equation $y_1 = A[x]$ as $\{x \in X : y_1 = A[x]\}$.

There are three cases that merit consideration.

- If the operator $A : X \to Y$ is such that the $\mathcal{R}(A) = Y$ and the $\mathcal{N}(A) = \{0_X\}$, then the solution of $y_1 = A[x]$ exists and is unique. The solution is given as $x = A^{-1}[y_1]$. Note that the inverse $A^{-1} : Y \to X$ exists. Such a solution corresponds to a system of linear equations with as many equations as there are unknowns.
- If the operator $A : X \to Y$ is such that the $\mathcal{R}(A) \subset Y$ and the $\mathcal{N}(A) = \{0_X\}$, then we note the following,

$$\begin{aligned} y_1 &= A[x] \\ A^*[y_1] &= A^*[A[x]] \\ A^*[y_1] &= A^*A[x] \end{aligned}$$

  where the operator $A^*A : X \to X$. Note that $\mathcal{N}(A^*A) = \mathcal{N}(A) = \{0_X\}$. This implies that the inverse $(A^*A)^{-1} : X \to X$ exists. The solution therefore can be written as

$$x = (A^*A)^{-1}A^*y_1 \tag{106}$$

  There is a simple geometric interpretation of the above result. Given $y_1 \in Y$, there is a unique direct sum decomposition of y_1 as, $y_1 = (y_{11} \in \mathcal{R}(A)) \oplus (y_{12} \in \mathcal{N}(A^*))$. That is, the vector in $\mathcal{R}(A)$ closest to y is the vector $y - y_{12}$. Indeed, the best one could do is to find a solution $x \in X$ such that $A[x] = y - y_{12} = y_{11}$. So we attempt the following solution,

$$\begin{aligned} y_1 - y_{12} &= A[x] \\ A^*[y_1 - y_{12}] &= A^*A[x] \\ A^*[y_1] - A^*[y_{12}] &= A^*A[x] \\ A^*[y_1] - 0_Y &= A^*A[x] \\ A^*[y_1] &= A^*A[x] \\ x &= (A^*A)^{-1}A^*[y_1] \end{aligned}$$

  The solution (106) is called the least-squares solution of the linear equation $y_1 = A[x]$. Such a solution corresponds to an overdetermined set of linear equations.
- Given a linear operator $A : X \to Y$ is such that the $\mathcal{R}(A) = Y$ and the $\{0_X\} \subset \mathcal{N}(A)$, we follow the geometric intuition as follows.
  - Solutions exist as $\mathcal{R}(A) = Y$.
  - Consider any solution $x_i \in X : y_1 = A[x_i]$. This solution has a unique direct sum decomposition of the form $x_i = (x_{i1} \in \mathcal{R}(A^*)) \oplus (x_{i2} \in \mathcal{N}(A))$. Indeed, there is no contribution of $x_{i2} \in \mathcal{N}(A)$ to the solution of $y_1 = A[x]$. Furthermore, as $x_{i1} \in \mathcal{R}(A^*)$, it is true that there exists $w \in Y$ such that $x_{i1} = A^*[w]$. Note that

$$\begin{aligned} y_1 &= A[x_i] && (107) \\ &= A[x_{i1} + x_{i2}] && (108) \\ &= A[x_{i1}] && (109) \\ &= A[A^*[w]] && (110) \\ &= AA^*[w] && (111) \end{aligned}$$

    Now note that $AA^* : Y \to Y$. Also $\mathcal{R}(AA^*) = \mathcal{R}(A) = Y$. This implies that $\mathcal{N}(AA^*) = 0_Y$. This guarantees that $(AA^*)^{-1}$ exists. We therefore solve for w in equation (111) as

$$w = (AA^*)^{-1}y_1 \tag{112}$$

    Note that the minimum norm solution is certainly one that does not include elements from $\mathcal{N}(A)$. Therefore, the minimum norm solution of $y_1 = A[x]$ is $x_{i1} = A^*[w] = A^*(AA^*)^{-1}y_1$. This solution corresponds to an underdetermined set of linear equations.

6.3 Least squares solution to multiple sensors and redundant actuators

Let us consider the case when there exists a multiplicity of sensors for the same measurement. Let the actual measurement we are looking for be x and the multiple sensor measurements be $y = Ax$. Then, to get a mean measurement, with minimum error to the actual measurement, corresponds to exactly the overdetermined case in the least squares estimation. The measurement is then given by

$$x = (A^*A)^{-1}A^*[y_1]$$

This $(A^*A)^{-1}A^*$ is indeed the pseudoinverse of the A matrix.

Now consider the case when we have redundant sensors and we are looking for a force split that minimizes the norm of the total force applied. Given forces x produced by the redundant actuators, the net force applied is given by $y = Bx$. Now, given a force to be applied y, splitting it among the redundant actuators with minimum norm, is exactly the underdetermined case derived in the least squares estimation. The solution is given by

$$x = B^*[w] = B^*(BB^*)^{-1}y$$

Note that $B^*(BB^*)^{-1}$ is the pseudoinverse of B.

7 Summary

In this paper we have developed the detailed dynamical equations of a rigid body rotor supported by actively controlled magnetic bearings. We have done this using both *Rotation Matrices* and *Quaternions* to see the equivalence. Quaternions are more convenient to use, as they provide a nonsingular (invertible) transformation to the angular velocity $\vec{w}$. Also euler parameters are computationally as efficient and more compact in storage than rotation matrices. In addition, in developing the model of the magnetic bearing system, we have considered two schemes for pole excitation.

We notice that the model of the bearing system depends on the angular velocity in the spin direction. We have developed a state feedback controller that stabilizes the system for all speeds of rotation. We also note that this controller essentially decouples the system into 2×2 subsystems. We have presented simulation results showing the performance of the controller.

Finally we also present a least squares scheme for minimizing the residual in measurements of output with multiple sensors, and for minimizing the norm of the actuation forces when there are redundant actuators.

References

[AR88] R.Mukundan A.K.Pradeep, P.J.Yoder and R.J.Schilling. Crippled motion in robots. *IEEE Transactions on Aerospace and Electronic Systems*, 24(1):2–14, 1988.

[AR89] P.J.Yoder A.K.Pradeep and R.Mukundan. Dual matrix exponentials in robotic kinematics. *International Journal of Robotics Research*, 8(5):57–66, 1989.

[Aub79] Jean-Pierre Aubin. *Applied Functional Analysis*. Wiley, 1979.

[FK90] M.Fajita F.Matsumura and K.Okawa. Modelling and control of magnetic bearing systems achieving a rotation around axis of inertia. *Proceedings of 2nd international symposium on magnetic bearing*, 1990.

[J.L55] J.L.Kelley. *General Topology*. Springer-Verlag, 1955.

[JR90] R.H.Taylor J.Funda and R.P.Paul. On homogeneous transforms, quaternions and computational efficiency. *IEEE Transactions on Robotics and Automation*, 6(3), 1990.

[K.W86] K.W.Spring. Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: A review. *Mechanism and Machine Theory*, 21(5):365–373, 1986.

[L.A79] L.A.Pars. *A Treatise on Analytical Dynamics*. Ox Bow Press, 1979.

[M.V78] M.Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall Electrical Engineering Series. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1978.

[OB79] O.Bottema and B.Roth. *Theoretical Kinematics*. North-Holland, 1979.

[PP80] P.Kelland and P.Q.Tait. *Introduction to Quaternions*. Macmillan, 1880.

[RA87] R.Mukundan and A.K.Pradeep. Comments on the computational aspects of matrix exponentials and their use in robotic kinematics. *IEEE Transactions on automatic control*, AC32(11), 1987.

[RH88] R.J.Schilling and H.Lee. *Engineering Analysis*. Wiley Interscience, 1988.

[RS94] Z.Li R.M.Murray and S.S.Sastry. *A mathematical introduction to robotic manipulation*. CRC Press, To appear 1994.

[YCBDB82] C. D. Morette Y. C. Bruhat and M. Dillard-Bleick. *Analysis, Manifolds And Physics*. Elsevier Science Publishers, 1982.