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VIBRATION SUPPRESSION in  
FLEXIBLE STRUCTURES  
via the SLIDING-MODE CONTROL  
APPROACH

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## Sliding-Mode Control of Differential-Difference Systems

Sliding modes in ordinary differential equations.

$$\dot{x}(t) = Ax(t) + Bu(t). \quad (1)$$

$$u(t, x) = \begin{cases} u^+(t, x), & \text{if } s(x) > 0 \\ u^-(t, x), & \text{if } s(x) < 0 \end{cases} \quad (2)$$

“Sliding modes” in continuous-time difference systems.

$$x(t + \tau) = Ax(t) + Bu(t). \quad (3)$$

$$S(x(t + \tau)) = 0 \Rightarrow u(t). \quad (4)$$

We consider two configurations:

**Configuration A**

$$\begin{aligned}\dot{x}(t) &= A_{11}x(t) + A_{12}z(t) \\ z(t + \tau) &= A_{21}x(t) + A_{22}z(t) + B_0u(t).\end{aligned}$$

**Configuration B**

$$\begin{aligned}z(t + \tau) &= A_{11}z(t) + A_{12}x(t) \\ \dot{x}(t) &= A_{21}x(t) + A_{22}z(t) + B_0u(t).\end{aligned}$$

It is assumed that  $x \in R^{n_1}$ ,  $z \in R^{n_2}$  and  $u \in R^m$ .  $A_{11}$ ,  $A_{12}$ ,  $A_{21}$ ,  $A_{22}$ ,  $B_0$  are constant matrices of appropriate dimensions.

### Sliding-Mode Control Design

$$A_{12} = B_1 C_2 \quad (5)$$

Quasicontrol:

$$v = C_2 z \quad (6)$$

#### Configuration A

1. Sliding mode in differential subsystem.

$$v = v^*(x) \Rightarrow S_0(x) = 0 \quad (7)$$

2. Sliding mode in difference subsystem.

$$u \Rightarrow S(x, z) = v^*(x) - C_2 z = 0 \quad (8)$$

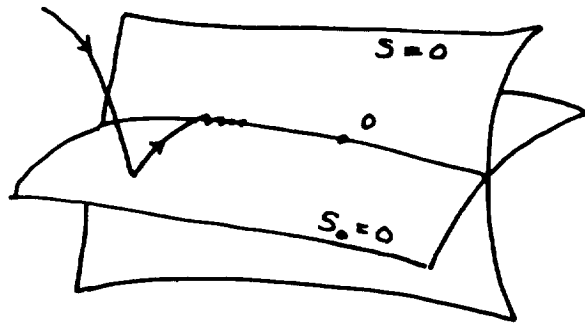
### Configuration B

1. "Sliding mode" in difference subsystem.

$$v = Dz(t) \Rightarrow S_0(z) = 0 \quad (9)$$

2. Sliding mode in differential subsystem.

$$u \Rightarrow S = Dz(t) - C_2x(t) = 0 \quad (10)$$



## Sliding Mode Control of Nondispersive Flexible Structures

Flexible rod in compression.

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = \frac{\partial^2 Q(t, x)}{\partial x^2} \quad (11)$$

$$\frac{\partial Q(t, 0)}{\partial t} = -u(t)$$

$$\frac{\partial Q(t, 1)}{\partial t} = 0.$$

Laplace transform approach.

$$p^2 \hat{Q}(p, x) = \hat{Q}''(p, x) \quad (12)$$

$$\hat{Q}'(p, 0) = -\hat{u}(p) \quad (13)$$

$$\hat{Q}'(p, 1) = 0, \quad (14)$$

where  $\hat{Q}(p, x) = \mathcal{L}Q(t, x)$ ,  $\hat{u}(p) = \mathcal{L}u(t)$ .

The solution of the boundary value problem

$$\hat{Q}(p, x) = \frac{e^{p(x-1)} + e^{-p(x-1)}}{e^p - e^{-p}} \cdot \frac{1}{p} \cdot \hat{u}(p). \quad (15)$$

If an output variable is

$$y(t) = Q(t, 1) \quad (16)$$

then

$$\hat{y}(p) = \hat{Q}(p, 1) = \frac{2}{e^p - e^{-p}} \frac{1}{p} \hat{u}(p). \quad (17)$$

In the time domain:

$$\dot{y}(t+1) - \dot{y}(t-1) = 2u(t) \quad (18)$$

or

$$\dot{y}(t) - \dot{y}(t-2) = 2u(t-1). \quad (19)$$

The equation can be written in the form of the difference-differential system as

**Configuration A:**

$$\begin{aligned}\dot{y}(t) &= z_1(t) \\ z_1(t+1) &= z_2(t) + 2u(t) \\ z_2(t+1) &= z_1(t)\end{aligned}$$

or

**Configuration B:**

$$\begin{aligned}y_1(t+1) &= y_2(t) + 2v(t) \\ y_2(t+1) &= y_1(t) \\ \dot{v}(t) &= u(t),\end{aligned}$$

where  $y_1(t) = y(t)$ .



**Control Design  
Configuration A**

$$z_1(t) = -\lambda \operatorname{sgn}(y(t)). \quad (20)$$

The equality is valid if

$$s(t) = z_1(t) + \lambda \operatorname{sgn}(y(t)) = 0. \quad (21)$$

To achieve this the control should be

$$u(t) = -\frac{1}{2}z_2(t) - \frac{1}{2}\lambda \operatorname{sgn}(y(t+1)). \quad (22)$$

$$y(t+1) = y(t) + \int_t^{t+1} z_1(\tau) d\tau. \quad (23)$$

As a result control is

$$u(t) = -\frac{1}{2}\dot{y}(t-1) - \frac{1}{2}\lambda \operatorname{sgn}(y(t) + y(t-1) - y(t-2)) + 2 \int_{t-1}^t u(\tau) d\tau.$$

With this control the system is stabilized in finite time.

### **Control Design Configuration B**

$$s(t) = (1 - \lambda)y_2(t) + 2v(t), \quad (24)$$

where  $|\lambda| < 1$ . If the control is

$$u(t) = -\mu \operatorname{sgn}(2v(t) + (1 - \lambda)y(t-1)) - (1 - \lambda)\dot{y}(t-1)$$

then

$$\dot{s} = -2\mu \operatorname{sgn}(s). \quad (25)$$

### Rod with attached masses

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = \frac{\partial^2 Q(t, x)}{\partial x^2} \quad (26)$$

$$\frac{\partial Q(t, 0)}{\partial t} = -u(t) \quad (27)$$

$$\frac{\partial Q(t, 1)}{\partial t} = -\frac{\partial^2 Q(t, 1)}{\partial t^2}. \quad (28)$$

### Configuration A:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -x_2(t) + z_1(t)$$

$$z_1(t) = z_2(t-1) + 2u(t-1)$$

$$z_2(t) = -z_1(t-1) + 2x_2(t-1),$$

where  $y(t) = Q(t, 1) = x_1(t)$ .

$$u(t) = -\frac{1}{2}z_2(t) - \frac{1}{2}\mu \operatorname{sgn}(\lambda x_1(t+1) + x_2(t+1)).$$

## State Estimation and Prediction

Extrapolator

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \Phi(1) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \int_t^{t+1} \Phi(t+1-\tau)h(\tau)d\tau,$$

where

$$h(\tau) = \begin{bmatrix} 0 \\ z_1(\tau-2) + 2u(\tau-1) - 2x_2(\tau-2) \end{bmatrix}.$$

Observer

$$\hat{x}_1(t) = \hat{x}_2(t) + L_1(\hat{x}_1(t) - y(t))$$

$$\hat{x}_2(t) = -\hat{x}_2(t) + \hat{z}_1(t) + L_2(\hat{x}_1(t) - y(t))$$

$$\hat{z}_1(t) = \hat{z}_2(t-1) + 2u(t-1) + L_3(\hat{x}_1(t) - y(t))$$

$$\hat{z}_2(t) = -\hat{z}_1(t-1) + 2\hat{x}_2(t-1) + L_4(\hat{x}_1(t) - y(t)).$$

The same approach can be used for systems of connected rods with many attached masses, multiple controls and also for the case of distributed actuators

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = \frac{\partial^2 Q(t, x)}{\partial x^2} + \varphi(x)u(t) \quad (29)$$

where  $\varphi(x)$  is quasipolynomial. In all cases the solution of the boundary value problem for Laplace transformed variables leads to **Configuration A** or **Configuration B**.

### Dispersive Structures

Euler-Bernoulli beam.

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = -\frac{\partial^4 Q(t, x)}{\partial x^4} \quad (30)$$

$$Q(t, 0) = 0 \quad (31)$$

$$Q'_x(t, 0) = 0 \quad (32)$$

$$Q''_{xx}(t, 1) = 0 \quad (33)$$

$$Q'''_{xxx}(t, 1) = u(t). \quad (34)$$

Second order dispersive structure.

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = a(x)\frac{\partial^2 Q(t, x)}{\partial x^2} + b(x)\frac{\partial Q(t, x)}{\partial x}. \quad (35)$$

General fourth order equation.

$$\frac{\partial^2 Q(t, x)}{\partial t^2} = -a(x) \frac{\partial^4 Q(t, x)}{\partial x^4} + b(x) \frac{\partial^2 Q(t, x)}{\partial x^2} + c(x) \frac{\partial Q(t, x)}{\partial x}.$$

The boundary conditions

$$Q(t, 0) = 0 \quad (36)$$

$$\frac{\partial Q(t, 0)}{\partial x} = 0. \quad (37)$$

$$\frac{\partial^2 Q(t, 1)}{\partial x^2} = u_1(t) \quad (38)$$

$$\frac{\partial^3 Q(t, 1)}{\partial x^3} = u_2(t). \quad (39)$$

### Integral Transform

$$P(t, \xi) = \int_0^1 \mathcal{D}(\xi, x) Q(t, x) dx \quad (40)$$

If  $\mathcal{D}$  satisfies an adjoint homogeneous boundary value problem then  $P(\xi, x)$  satisfies equation

$$\frac{\partial^2 P(t, \xi)}{\partial t^2} = \frac{\partial^2 P(t, \xi)}{\partial \xi^2} + \varphi(\xi) u(t) \quad (41)$$

$$\varphi(\xi) = -a(0) \mathcal{D}(\xi, 0). \quad (42)$$

### Euler-Bernoulli beam

If  $\mathcal{D}$  is a solution of the boundary value problem:

$$\frac{\partial^2 \mathcal{D}(\xi, x)}{\partial \xi^2} = -\frac{\partial^4 \mathcal{D}(\xi, x)}{\partial x^4}. \quad (43)$$

$$\mathcal{D}(\xi, 0) = 0 \quad (44)$$

$$\mathcal{D}'(\xi, 0) = 0 \quad (45)$$

$$\mathcal{D}''(\xi, 1) = 0 \quad (46)$$

$$\mathcal{D}'''(\xi, 1) = 0. \quad (47)$$

then  $P(t, \xi)$  satisfies an equation:

$$\frac{\partial^2 P(t, \xi)}{\partial t^2} = \frac{\partial^2 P(t, \xi)}{\partial \xi^2} + \varphi(\xi)u(t), \quad (48)$$

where

$$\varphi(\xi) = -\mathcal{D}(\xi, 1). \quad (49)$$

Initial values:  $\mathcal{D}(0, x)$  and  $\mathcal{D}'_{\xi}(0, x)$ .

If  $\mathcal{D}'_{\xi}(0, x) = 0$  then

$$P'_{\xi}(t, 0) = 0. \quad (50)$$

The possibility to choose  $\mathcal{D}(0, x)$  is an additional degree of freedom that can be used to assign the desired value of  $\varphi(\xi)$ .

Nonsingularity condition

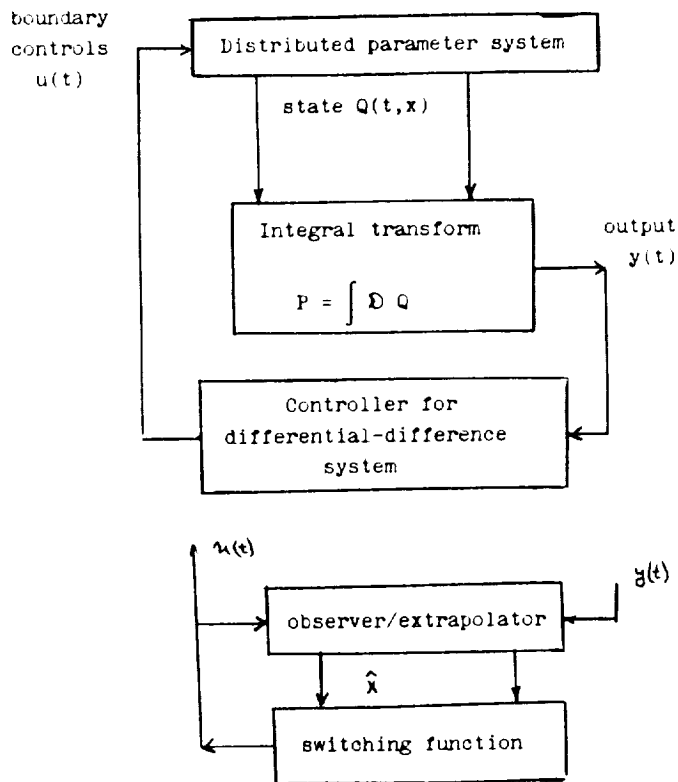
$$P \equiv 0 \Rightarrow Q \equiv 0. \quad (51)$$

Output

$$y(t) = P(t, 0) = \int_0^1 \mathcal{D}(0, x)Q(t, x)dx. \quad (52)$$



### Controller structure



### **Conclusions**

Sliding mode control became very popular recently because it makes the closed loop system highly insensitive to external disturbances and parameter variations. Sliding algorithms for flexible structures have been used previously, but these were based on finite-dimensional models. An extension of this approach for differential-difference systems is obtained. That makes it possible to apply sliding-mode control algorithms to the variety of nondispersive flexible structures which can be described as differential-difference systems.

The main idea of using this technique for dispersive structures is to reduce the order of the controlled part of the system by applying an integral transformation. We can say that transformation "absorbs" the dispersive properties of the flexible structure as the controlled part becomes dispersive.

## References

1. Bellman, R., Cooke K.L., *Differential-difference equations*, N.Y., Academic Press, 1963.
2. Drakunov, S.V., Izosimov D.B., Luk'yanov A.G., Utkin V.A. and Utkin V.I., *The Block Control Principle, Part II*, Automation and Remote Control, V. 51, no. 6,1990,pp. 737-746.
3. J. Martin, Ü. Özgüner, S. Yurkovich, *An Active Vibration Damper for Flexible Structures*, Proc. of 17th Pitts. Conf. on Modeling and Sim, April,1986.
4. Meirovitch L., *Elements of vibrational analysis*, McGraw-Hill Book Company, 1986.
5. Utkin V.I., *Sliding Modes and Their Application In Variable Structure Systems*. Moscow: MIR, 1978.
6. Drakunov S.V. and Utkin V.I., Sliding Mode Control in Dynamic Systems, *International Journal of Control*, v. 55, N. 4, pp.1029-1037, 1992.