DYNAMICAL OBSERVER FOR A FLEXIBLE BEAM VIA FINITE ELEMENT APPROXIMATIONS

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Motivation

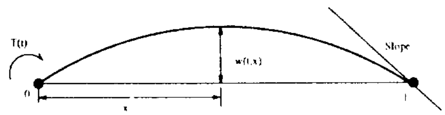
1. The purpose of this work is a computational investigation of the closed-loop output feedback control of an Euler-Bernoulli beam based on finite element approximation.

2. The observer is part of the classical observer + state feedback control, but it is finite-dimensional.

3. In the theoretical work on the subject it is assumed (and sometimes proved) that increasing the number of finite elements will improve accuracy of the control. In applications,this may be difficult to achieve because of numerical problems.

4. The main difficulty in computing the observer and simulating its work is the presence of high frequency eigenvalues in the finite-element model and poor numerical conditioning of some of the system matrices (e.g. poor observability properties) when the dimension of the approximating system increases. This work dealt with some of these difficulties.

Euler-Bernoulli Beam



$$\rho w_{tt} + EI w_{xxxx} = 0 \qquad (1)$$

on $[0\ l]$

where

$w(\bullet,\ 0) = w_0$

$w_t(\bullet,\ 0) = w_1$

$w_{xx}(t,\ 0) = u(t)$ **(Torque)**

$w_{xx}(t,\ l) = 0$

$z(t) = w_x(t,\ l)$ **(Measurement)**

Cubic FE Model

$$\phi_1 = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3 \qquad (2)$$

$$\phi_2 = -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2 \qquad (3)$$

$$\phi_3 = 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3 \qquad (4)$$

$$\phi_4 = -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right] \qquad (5)$$

Where

$\phi_1(x_e) = 1 \qquad \phi_i(x_e) = 0 \qquad i \neq 1$

$\phi_3(x_e + h_e) = 1$

$-\frac{d\phi_2}{dx}|_{x=x_e} = 1$

$-\frac{d\phi_4}{dx}|_{x=x_e+h_e} = 1$

Numerical Aspects

• System (8) is poorly conditioned numerically for large N.

• To improve numerical accuracy it is important to

a). solve directly (6) rather than (8).

b). use a Cholesky decomposition of M to avoid direct inversion of M in (6).

c). use a numerical integration method that is energy preserving on principal modes.

F-E Approximation to E-B beam

$$M\ddot{y} + Ly = Qu(t) \tag{6}$$

$$z = C\begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{7}$$

First-order system

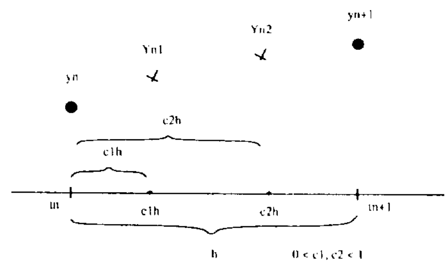
$$\dot{X}^N = \begin{bmatrix} 0 & I \\ -M^{-1}L & 0 \end{bmatrix} X^N + \begin{bmatrix} 0 \\ Q \end{bmatrix} u(t) \tag{8}$$

$$z = CX^N \tag{9}$$

Dimension of $\begin{bmatrix} 0 & I \\ -M^{-1}L & 0 \end{bmatrix}$:

$N\times$position
+ $N\times$slope
+ $N\times$velocity
+ $N\times$derivative of the slope
= $4N$ + 1 or 2, depending on Boundary Conditions

DIRK – Diagonally Implicit Runge-Kutta Method for Oscillatory Problems



$$\ddot{y} = f(t, y) \qquad (10)$$

$$\begin{aligned} Y_{nj} &= y_n + c_j h \dot{y}_n + h^2 \sum_{i=1}^{l} a_{ji} f(Y_{ni},\ t_n + c_i h) \\ y_{n+1} &= y_n + h \dot{y}_n + h^2 \sum_{i=1}^{l} b_i f(Y_{ni},\ t_n + c_i h) \\ \dot{y}_{n+1} &= \dot{y}_n + h \sum_{i=1}^{l} f(Y_{ni},\ t_n + c_i h) \end{aligned}$$

Butcher Array for $l = 2$

$$\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ \hline b_1 & b_2 & \\ b_1' & b_2' & \end{array}$$

Explicit RK:

$$\begin{array}{cc|} 0 & 0 \\ a_{21} & 0 \end{array}$$

Diagonally Implicit RK:

$$\begin{array}{cc|} a_{11} & 0 \\ a_{21} & a_{22} \end{array}$$

Example of Butcher Array for DIRK:

$$\begin{array}{cc|c} 1/2 & 0 & 1/2 \\ -5/12 & 1/2 & 1/2 \\ \hline 0 & 1/2 & \\ 0 & 1 & \end{array}$$

Using DIRK on E-B beam (1)

E-B Beam via FE Method

$$M\ddot{y} + Ly = Qu(t) \qquad (11)$$

$u(t) = 0$ **in the observer problem**

$$\ddot{y} = -M^{-1}Ly = Fy \qquad (12)$$

(Actually $M^{-1}L$ is replaced by a Cholesky decomposition).

$$\begin{aligned} Y_{n1} &= y_n + c_1 h\dot{y}_n + h^2[a_{11}FY_{n1} + a_{12}FY_{n2}] \\ Y_{n2} &= y_n + c_2 h\dot{y}_n + h^2[a_{21}FY_{n1} + a_{22}FY_{n2}] \end{aligned}$$

$$\begin{aligned} y_{n+1} &= y_n + h\dot{y}_n + h^2[b_1 FY_{n1} + b_2 FY_{n2}] \\ \dot{y}_{n+1} &= \dot{y}_n + h[b_1' FY_{n1} + b_2' FY_{n2}] \end{aligned}$$

Where $a_{12} = 0$

Using DIRK on E-B beam (2)

Solve to get Y_{n1} **and** Y_{n2}

$$\begin{aligned} Y_{n1} &= (I - h^2 a_{11}F)^{-1}(y_n + c_1 h\dot{y}_n) \\ Y_{n2} &= (I - h^2 a_{22}F)^{-1}(y_n + c_2 h\dot{y}_n + h^2 a_{21}FY_{n1}) \end{aligned}$$

Discrete-Time Model:

$$\begin{bmatrix} y_{n+1} \\ \dot{y}_{n+1} \end{bmatrix} = AV \begin{bmatrix} y_n \\ \dot{y}_n \end{bmatrix}$$

Where matrix AV **is**

$$\begin{bmatrix} I + h^2F(b_1A_1 + b_2A_3) & hI + h^2F(b_1A_2 + b_2A_4) \\ hF(b_1'A_1 + b_2'A_3) & I + hF(b_1'A_2 + b_2'A_4) \end{bmatrix}$$

And

$$A_1 = (I - h^2a_{11}F)^{-1} \qquad A_2 = c_1hA_1$$

$$A_3 = (I - h^2a_{22}F)^{-1}(I + h^2a_{21}FA_1)$$

$$A_4 = (I - h^2a_{22}F)^{-1}(c_2hI + h^2Fa_{21}A_2)$$

Digital Observer Design

Define

$$X_n = \begin{bmatrix} y_n \\ \dot{y}_n \end{bmatrix}$$

Time-discretized FE System:

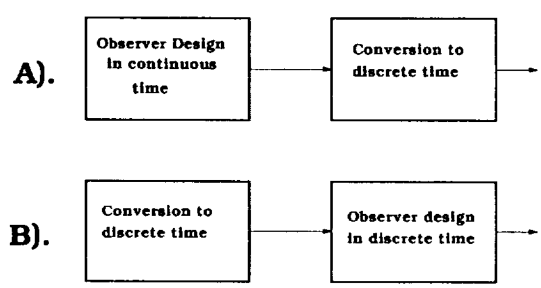
$$X_{n+1} = AV_{N_1} X_n \qquad (13)$$
$$Z_n = C_{N_1} X_n \qquad (14)$$

Observer:

$$\hat{X}_{N+1} = AV_{N_2} \hat{X}_n + G(Z_n - \hat{Z}_n) \qquad (15)$$
$$\hat{Z}_n = C_{N_2} \hat{X}_n \qquad (16)$$

Where N_1 and N_2 can be different, for example $N_1 = 64$ and $N_2 = 4$, 8 or 16.

Two Ways for Observer Design

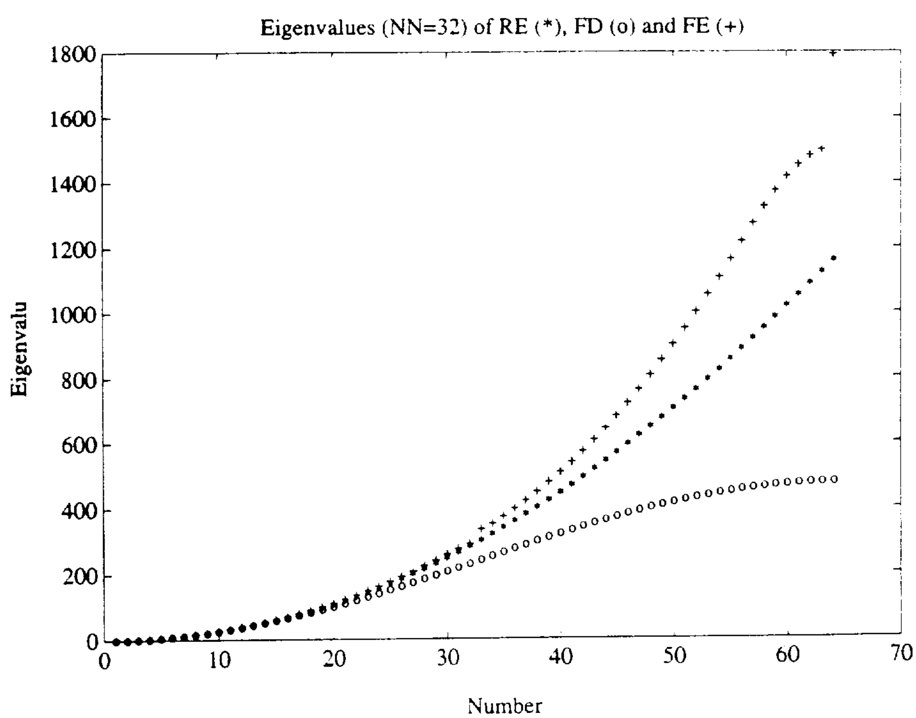


The above operations are not equivalent ("do not commute"). The discrete-time observer designed by variant B provides a more accurate tracking of beam's motion.

Eigenvalues of E-B Beam Using Finite Element Model

(Imaginary Part)

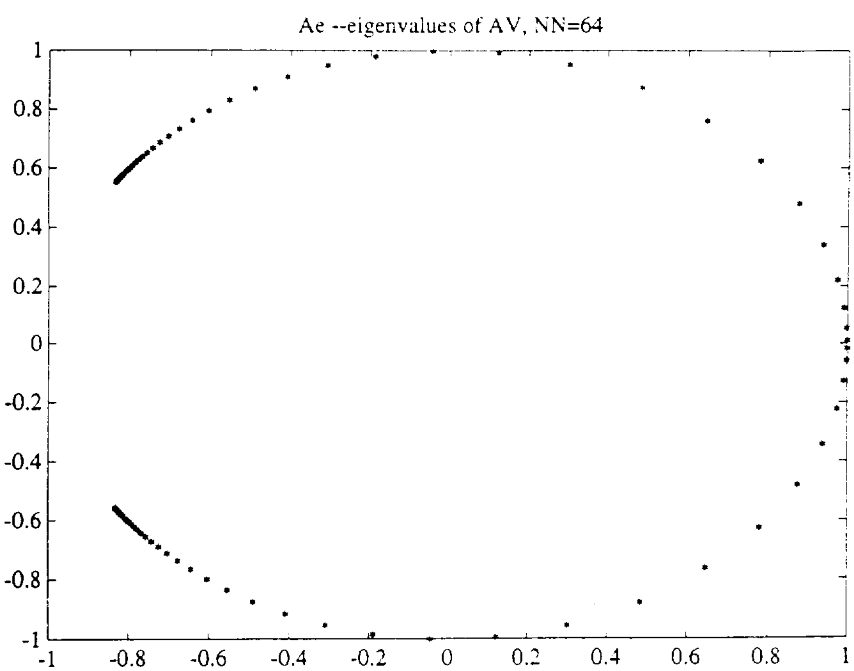
N	Max	Min	Ratio
4	135.9341	2.1817	6.2306e+01
8	541.9612	2.1805	2.4854e+02
16	2.1678e+03	2.1805	9.9419e+02
32	8.6712e+03	2.1805	3.9768e+03
64	3.4685e+04	2.1805	1.5907e+04
128	1.3874e+05	2.1805	6.3628e+04

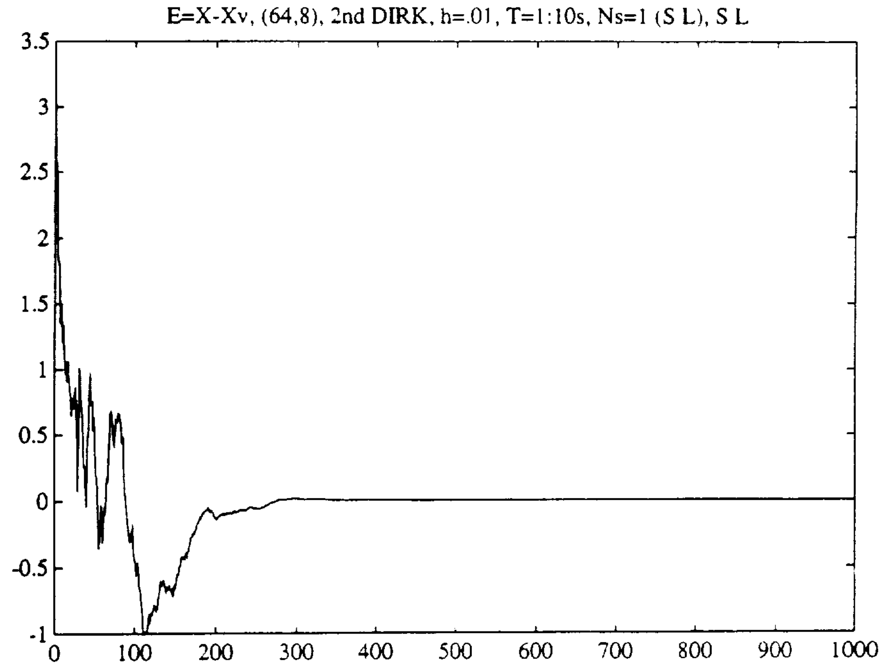


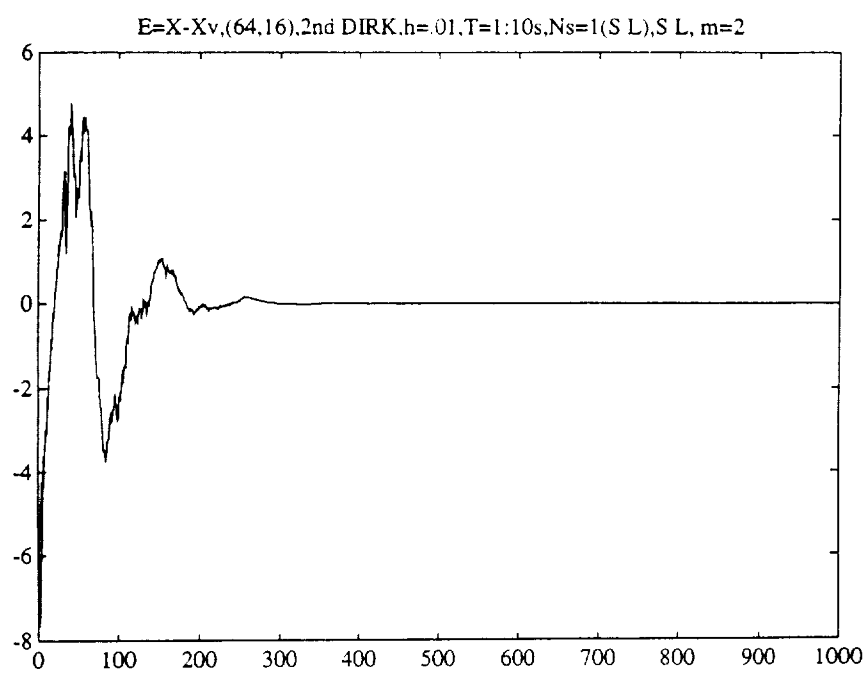
ideal beam: NN=64, Wst=4.3371, Wmax=1.8885
L=1, EI=0.02, K=0, m=1, A=1
Initial condition: sin(pi*x)

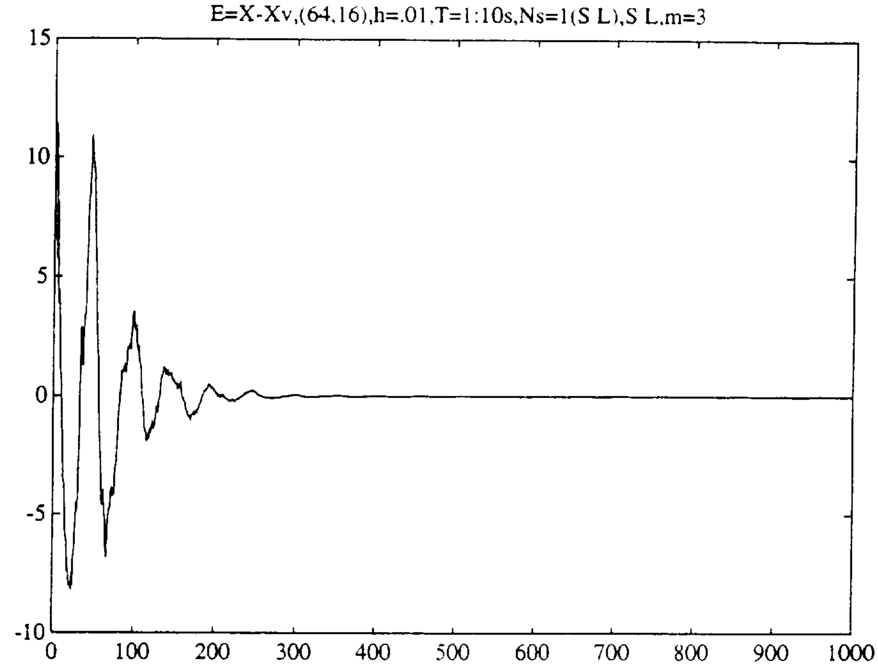
FE model: T= 10 s h= 0.01 hx=0.05

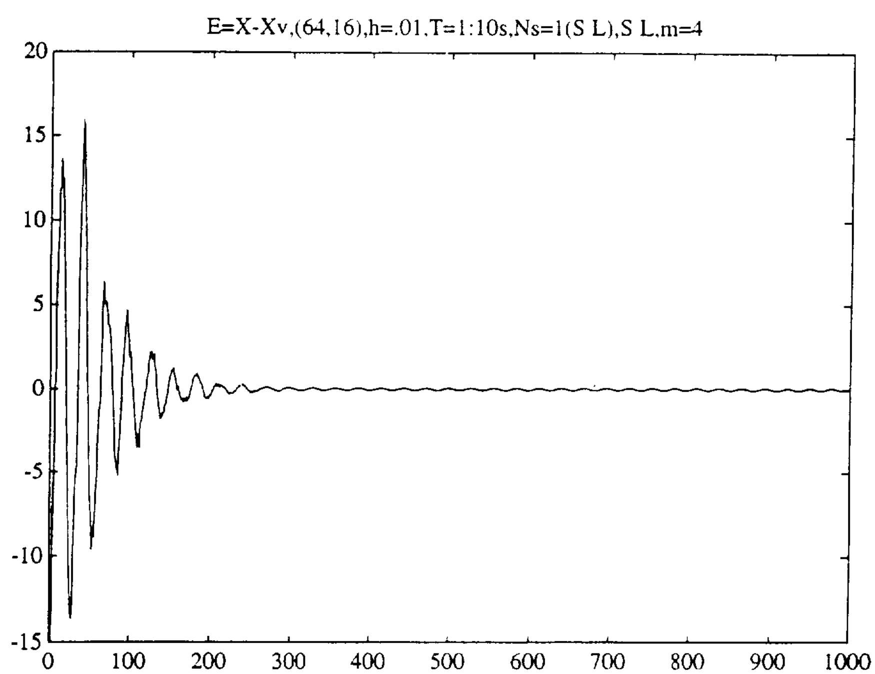
NN	E1 (FE Model)	E2(Max) (FE Model)
2	3.015 e-03	8.662 e-02
4	1.298 e-04	2.231 e-02
8	6.107 e-06	5.787 e-03
16	2.414 e-07	1.337 e-03
32	4.591 e-09	2.138 e-04

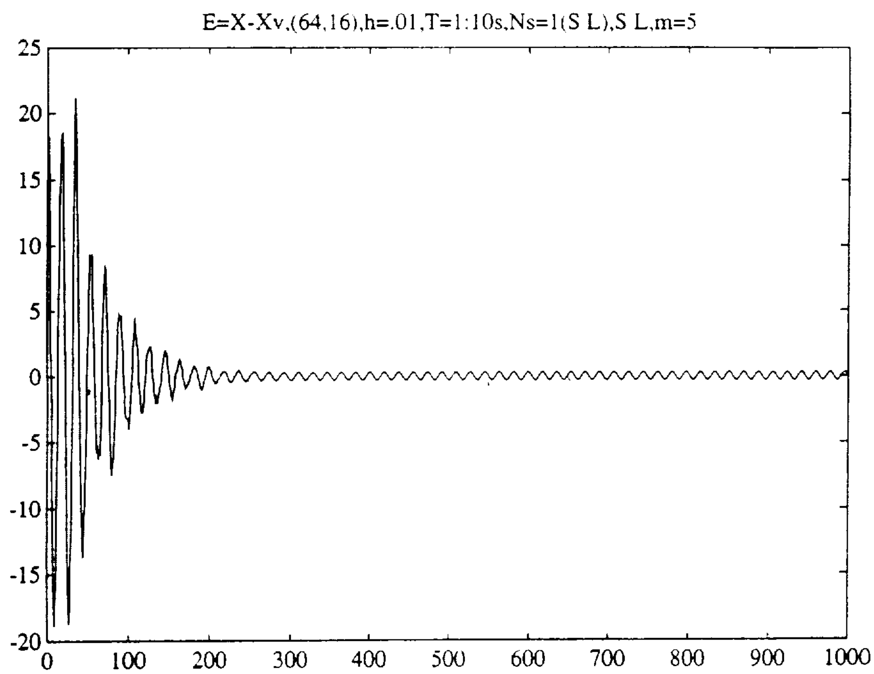


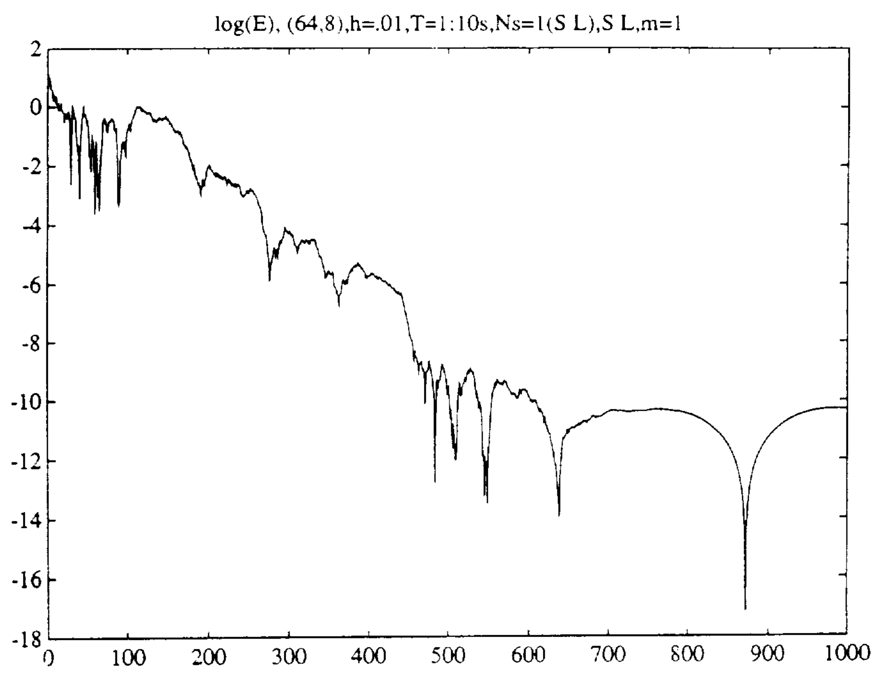

E=X-Xv, (64,8), 2nd DIRK, h=.01, T=1:10s, Ns=1 (S L), S L
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0
100
200
300
400
500
600
700
800
900
1000

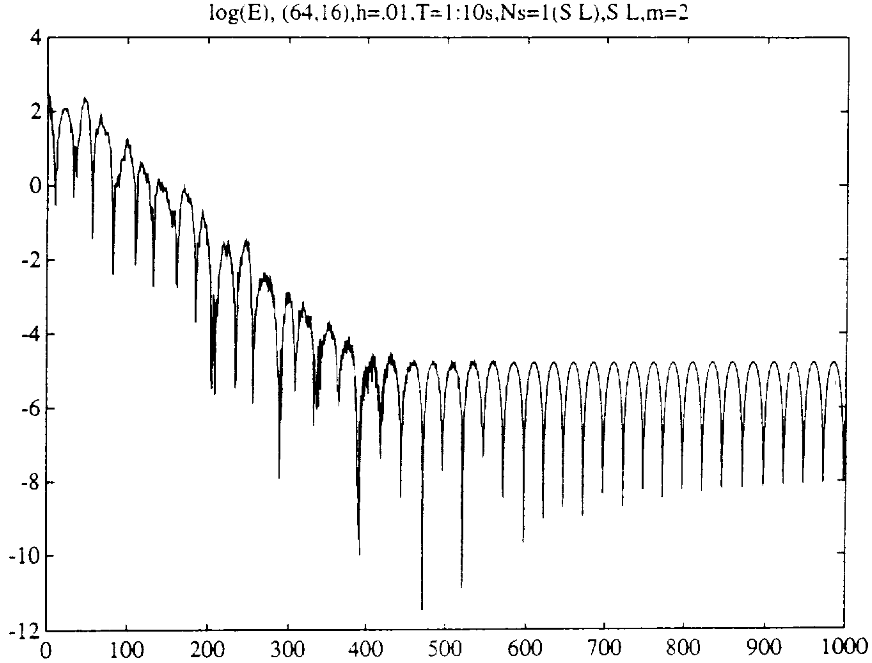

E=X-Xv,(64,16),2nd DIRK,h=.01,T=1:10s,Ns=1(S L),S L, m=2
6
4
2
0
-2
-4
-6
-8
0
100
200
300
400
500
600
700
800
900
1000

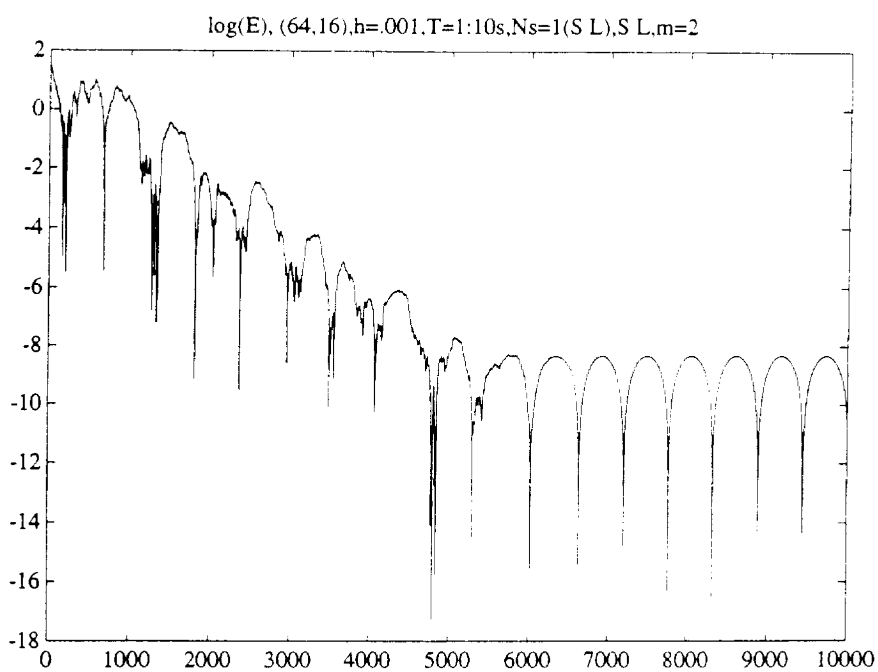

E=X-Xv,(64,16),h=.01,T=1:10s,Ns=1(S L),S L,m=3
15
10
5
0
-5
-10
0
100
200
300
400
500
600
700
800
900
1000

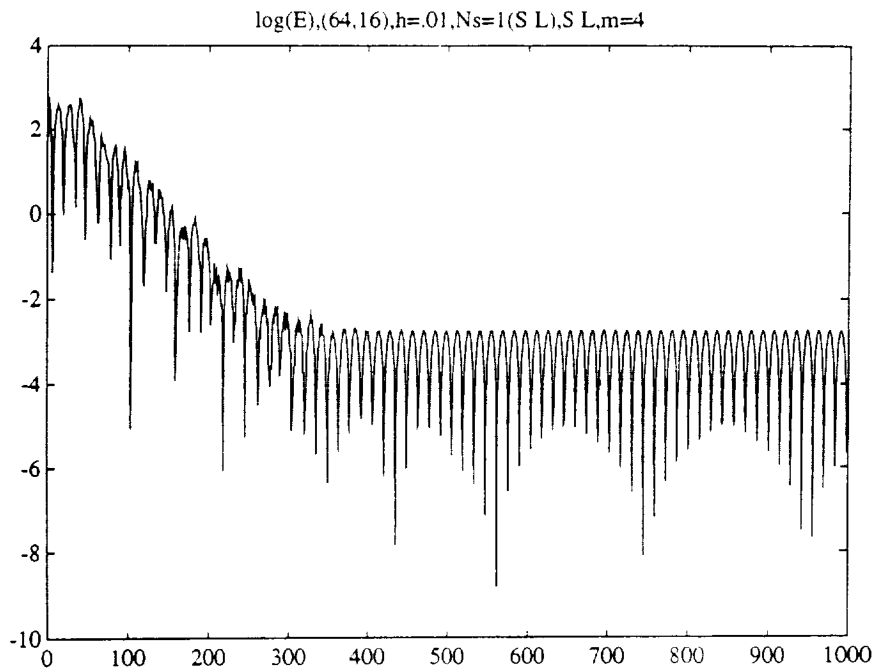

E=X-Xv,(64,16),h=.01,T=1:10s,Ns=1(S L),S L,m=4
20
15
10
5
0
-5
-10
-15
0
100
200
300
400
500
600
700
800
900
1000

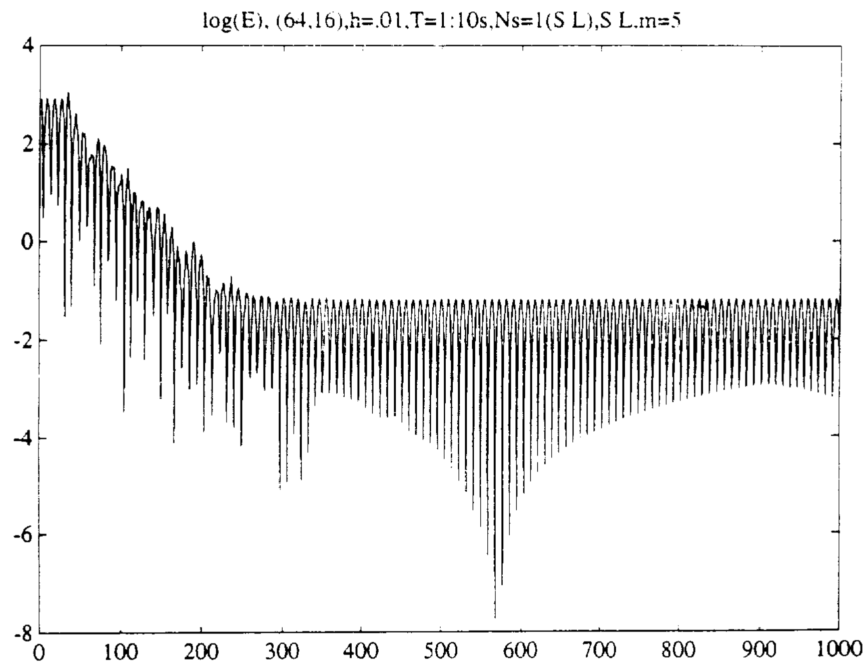

E=X-Xv,(64,16),h=.01,T=1:10s,Ns=1(S L),S L,m=5


log(E), (64,8),h=.01,T=1:10s,Ns=1(S L),S L,m=1

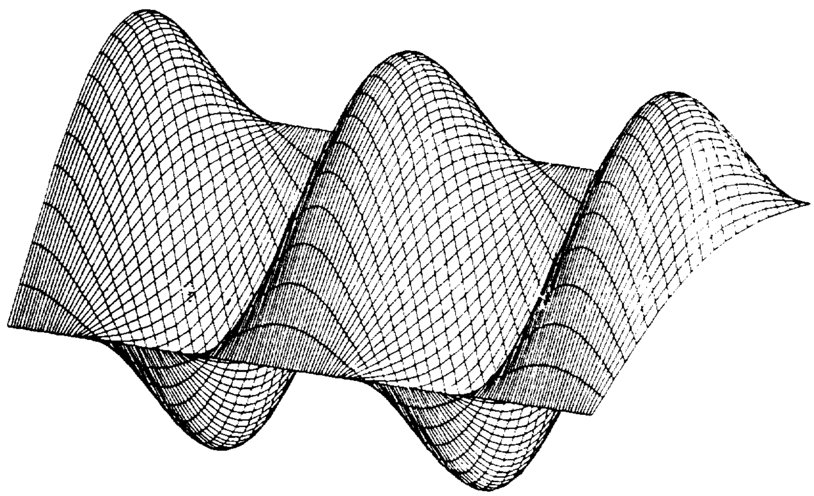

log(E), (64,16),h=.01,T=1:10s,Ns=1(S L),S L,m=2
4
2
0
-2
-4
-6
-8
-10
-12
0
100
200
300
400
500
600
700
800
900
1000


log(E), (64,16),h=.001,T=1:10s,Ns=1(S L),S L,m=2
2
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

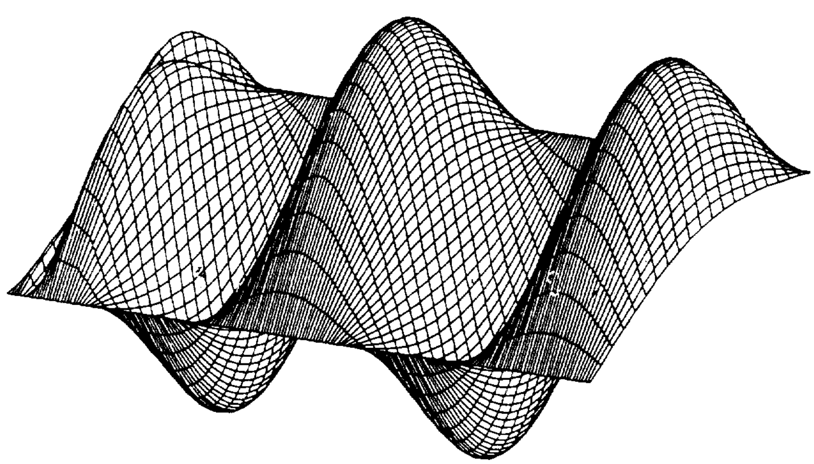

log(E),(64,16),h=.01,Ns=1(S L),S L,m=4
4
2
0
-2
-4
-6
-8
-10
0
100
200
300
400
500
600
700
800
900
1000


log(E), (64,16),h=.01,T=1:10s,Ns=1(S L),S L,m=5
4
2
0
-2
-4
-6
-8
0
100
200
300
400
500
600
700
800
900
1000

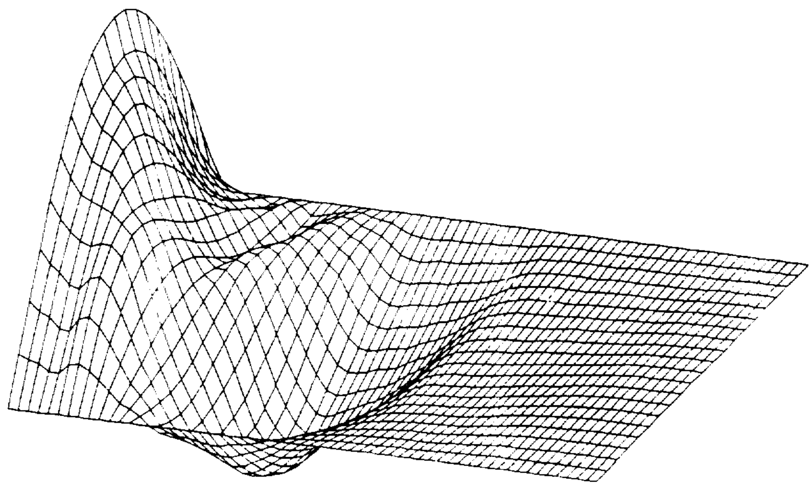
Spatial displacement of beam, NN=64, mode 1



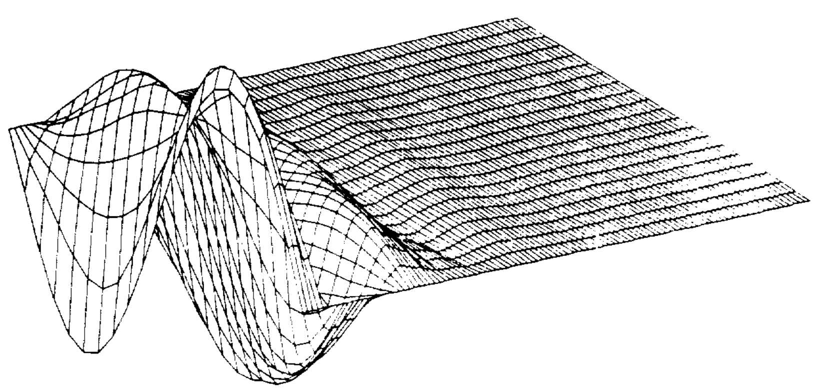
Spatial displacement of observer, NN=4, mode 1



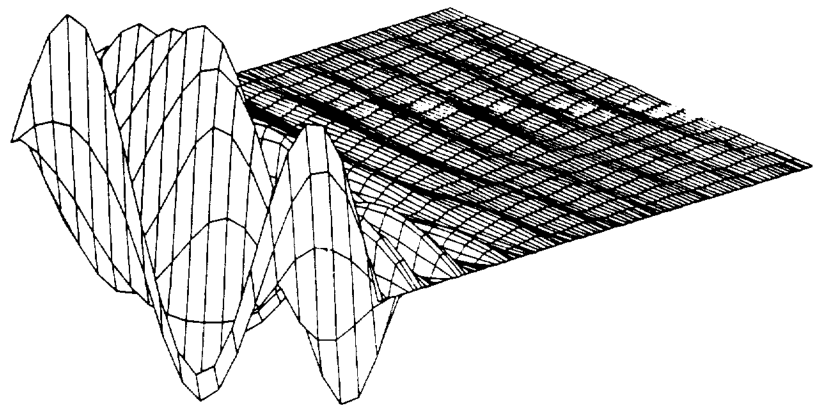
Spatial distribution of observer error, NN=4, e=-2, mode 1



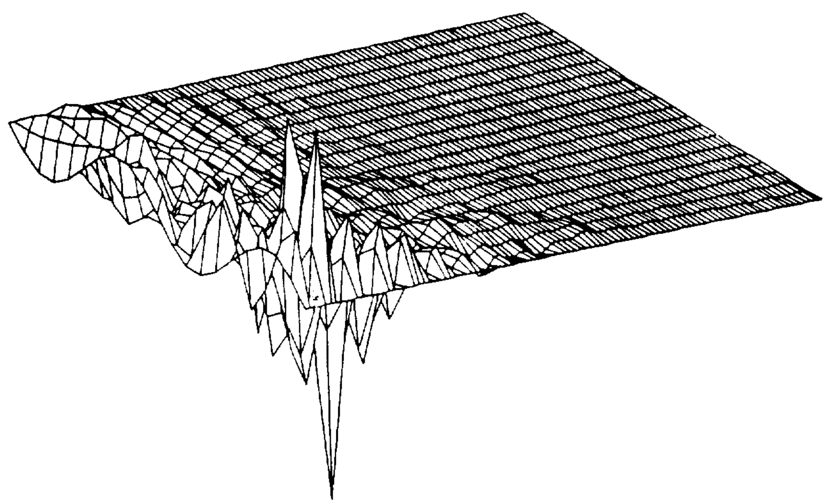
Spatial distribution of observer error,NN=8,e=-2,mode 2,T=5s



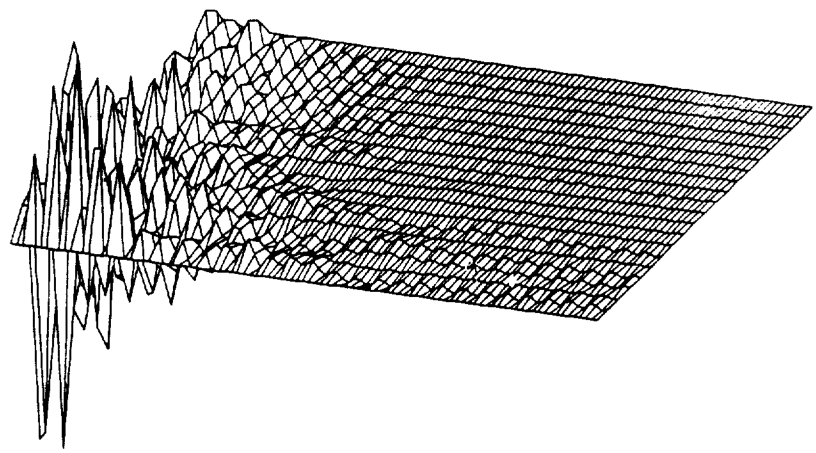
Spatial distribution of observer error, NN=8,e=-2,T=5s,mode 3



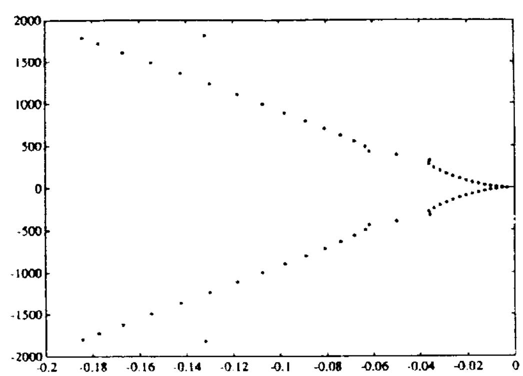
Spatial distribution of observer error, NN=16,e=-2,T=5s,mode 4



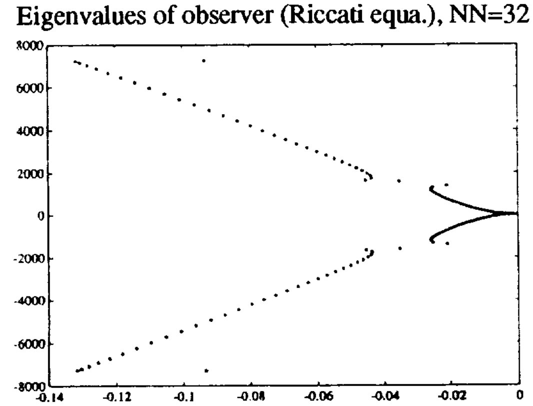
Spatial distribution of observer error, NN=16,e=-2,T=5s,mode 5



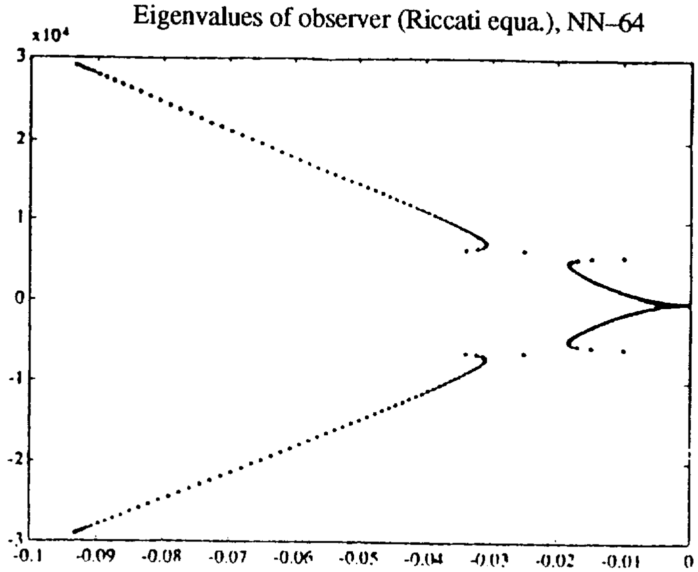
Eigenvalues of observer (Riccati equa.), NN=16



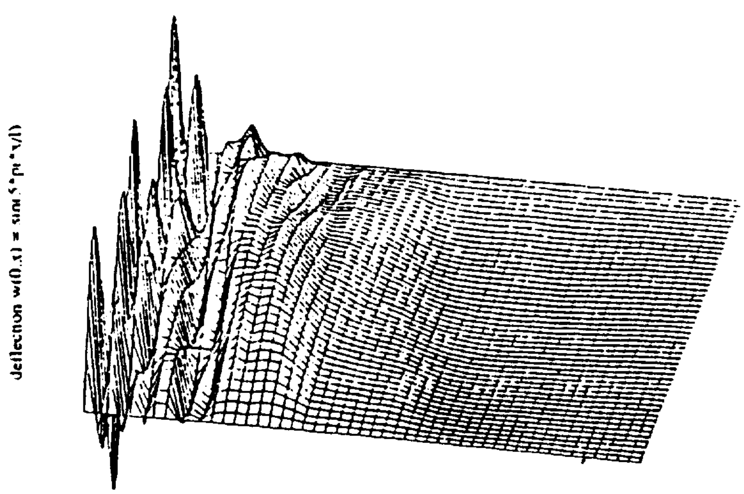
Eigenvalues of observer (Riccati equa.), NN=32



Eigenvalues of observer (Riccati equa.), NN-64



Stabilization of the beam via state feedback. NN=8, m=5, T=5s



Time 0 to 5 sec

Conclusions

1. The computations showed that the dynamically changing shape of the beam can be reconstructed by a finite-element based observer using just one point measurement of the slope at the end. This conclusion is limited to shapes involving only a few first modes. Additional measurements do not improve the process much.

2 In the process of designing the observer, one can either design gain G for a continuous-time model and then discretize in time $\dot{X} = (A - GC)X$ or first discretize in time $\dot{X} = AX$ and then design an observer. The second approach is more accurate.

3. For each mode of the E-B beam there is a steady state periodic error, whose amplitude depends on the mismatch of eigenvalues between the E-B beam and FE model.

The error can be decreased by further shifting eigenvalues of the observer, or by increasing the number of finite elements.

4. The Riccati equation approach yields a conical pattern of eigenvalues. The transients are different, but the steady state periodic error is nearly the same.

References

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