NUMERICAL METHODS FOR MULTIBODY SYSTEMS

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INTRODUCTION

This article gives a brief summary of some results obtained by Nasser (ref. 1) on modeling and simulation of inequality problems in multibody dynamics. In particular, the augmented Lagrangian method discussed here is applied to a constrained motion problem with impulsive inequality constraints. A fundamental characteristic of the multibody dynamics problem is the lack of global convexity of its Lagrangian. The problem is transformed into a convex analysis problem by localization (piecewise linearization), where the augmented Lagrangian has been successfully used [see Glowinski and Le Tallec (ref. 2); Glowinski, Lions, and Tremolières (ref. 3); and Fortin and Glowinski (ref. 4)]. A model test problem is considered, and a set of numerical experiments is presented (Figures 1 through 9).

MATHEMATICAL MODEL

Functional Context

$$X = H^m\left(0, T; \mathbf{R}^N\right), \tag{1}$$

$$H^m\left(0, T; \mathbf{R}^N\right) = \left\{v\colon v \in C^{m-1}\left([0, T]; \mathbf{R}^N\right),\ \frac{d^m v}{dt^m} \in L^2\left(0, T; \mathbf{R}^N\right)\right\}, \tag{2}$$

$\pounds$ is a Lagrangian function, J is a nonlinear functional, and K is a closed subset of X.

Main Problem

$$\boxed{\text{Find } \theta \in K, \text{ for which } J \text{ is stationary}}$$

$$K = K_1 \cap K_2, \tag{3}$$

$$K_1 = \left\{ v \in X : g_j(v(t)) = 0 \quad , \quad j = 1, 2, \ldots, k \quad \text{a.e. on } (0, T) \right\}, \tag{4}$$

$$K_2 = \left\{ v \in X : g_j(v(t)) \leq 0 \quad , \quad j = k+1, \ldots, k+l \quad \text{a.e. on } (0, T) \right\}. \tag{5}$$

The functions $g_i(v(t))$ are real valued; finally, J is defined by

$$J(v) = \int_0^T \pounds(v, \dot{v}, t)\, dt. \tag{6}$$

The stationarity of J at θ can be formulated as shown in the following section.

AUGMENTED LAGRANGIAN FORMULATION

Following a well-known technique [Glowinski and Le Tallec (ref. 2); Glowinski, Lions, and Tremolières (ref. 3); and Fortin and Glowinski (ref. 4)], we associate to (1)-(6) the following problem:

Find θ and λ, with $\theta \in K$ and $\lambda \in \Lambda$, for which the following augmented functional J_r is stationary:

$$J_r(v, \mu) = \int_0^T \pounds_r(v, \dot{v}, \mu)\, dt, \tag{7}$$

where

$$\pounds_r(v, \mu) = \pounds(v, v) + \langle \mu_1, G_1(v) \rangle + \langle \mu_2, G_2(v) \rangle + \frac{r}{2} \left[\langle G_1, G_1 \rangle + \langle G_2^+, G_2^+ \rangle \right], \tag{8}$$

$$G(v) = \left[g_1(v), g_2(v), \ldots, g_{k+l}(v) \right]^T, \tag{9}$$

$$G_1 = \left[g_1, g_2, \dots, g_k\right]^T, \tag{10}$$

$$G_2^+ = \left[g_{k+1}^+, g_{k+2}^+, \dots, g_{k+l}^+\right]^T, \tag{11}$$

$$g_j^+ = \frac{1}{2}\left(g_j + \left| g_j \right|\right), \tag{12}$$

$$\Lambda = \left\{\mu : \left\{\mu_i\right\}_{i=1}^{k+l} \in L^2\left(0, T; \mathbf{R}^{k+l}\right) \;,\; \mu_i \in \mathbf{R} \text{ if } i = 1, 2, \dots, k \;,\; \mu_i \geq 0 \text{ if } i = k+1, \dots, k+l\right\}. \tag{13}$$

There exists a large amount of literature dealing with the case $K_2 = \varnothing$, which leads to index 3 differential algebraic equations. The case when K_2 is nonempty is considerably more difficult from a mathematical point of view, and hence fewer technical papers have been devoted to it. The methodology we shall present includes treatment of both cases.

SOLUTION ALGORITHMS

Given $\theta_k(t_n)$, $\dot{\theta}_k(t_n)$, and $\lambda_k(t_n)$, compute $\theta_{k+1}(t_n)$, $\dot{\theta}_{k+1}(t_n)$, and $\lambda_{k+1}(t_n)$ via the following:

$$\nabla_\theta J_r\left(\theta_{k+1}, \dot{\theta}_{k+1}, \lambda_k\right) = 0, \tag{14}$$

$$\lambda_{k+1} = P_\Lambda\left[\lambda_k + \rho\, G\left(\theta_{k+1}\left(t_n\right)\right)\right]; \tag{15}$$

P_Λ is the projection operator associated with the set Λ. For the choice of r and ρ, see Nasser (ref. 1) and Glowinski, Lions, and Tremolières (ref. 3).

LINEARIZATION AND TIME DISCRETIZATION

Following Nasser (ref. 1), we introduce the perturbation $\delta\theta$, $\delta\dot{\theta}$, $\delta\ddot{\theta}$ of θ, $\dot{\theta}$, $\ddot{\theta}$ to obtain the following system:

$$M(\theta)\,\delta\ddot{\theta} + C(\theta,\dot{\theta})\,\delta\dot{\theta} + K(\theta,\dot{\theta})\,\delta\theta + R(\theta)\,\delta\theta + S(\theta)\,\delta\theta = r.h.s.(\theta)\,. \tag{16}$$

$$S = \sum_{i=1}^{k+l} \lambda_i \nabla_\theta^2 g_i(\theta), \tag{17}$$

$$R = r\left\{ \sum_{i=1}^{k} \left(\nabla_\theta^2 g_i(\theta) \cdot g_i(\theta) + \left[\nabla_\theta g_i(\theta) \right]^2 \right) + \sum_{i=k+1}^{k+l} \left(\nabla_\theta^2 g_i(\theta) \cdot g_i(\theta) + \left[\nabla_\theta g_i^+(\theta) \right]^2 \right) \right\}, \tag{18}$$

$$\begin{aligned} r.h.s.\,(\theta) = - \Bigg\{ & \sum_{i=1}^{k} \lambda_i \nabla_\theta g_i(\theta) + \sum_{i=k+1}^{k+l} \lambda_i \nabla_\theta g_i^+(\theta) + r\Bigg[\sum_{i=1}^{k} g_i(\theta) \cdot \nabla_\theta g_i(\theta) \\ & + \sum_{i=k+1}^{k+l} g_i^+(\theta) \cdot \nabla_\theta g_i^+(\theta) \Bigg] \Bigg\}. \end{aligned} \tag{19}$$

Taylor Series Expansion of θ and $\dot{\theta}$

Using Taylor series expansion for θ and $\dot{\theta}$, we get

$$\theta(t + \Delta t) = \theta(t) + \theta^{(1)}(t)\Delta t + \theta^{(2)}(t)\frac{\Delta t^2}{2} + \theta^{(3)}(t)\frac{\Delta t^3}{6} + \theta^{(4)}(t)\frac{\Delta t^4}{24} + O\left(\Delta t^5\right), \tag{20}$$

$$\dot{\theta}(t + \Delta t) = \theta^{(1)}(t + \Delta t) = \theta^{(1)}(t) + \theta^{(2)}(t)\Delta t + \theta^{(3)}(t)\frac{\Delta t^2}{2} + \theta^{(4)}(t)\frac{\Delta t^3}{6} + O\left(\Delta t^4\right), \tag{21}$$

where dots or superscripts denote the order of the derivatives.

Let

$$\delta\theta(t) = \theta(t + \Delta t) - \theta(t), \tag{22}$$

$$\delta\dot{\theta}(t) = \dot{\theta}(t + \Delta t) - \dot{\theta}(t), \tag{23}$$

$$\theta^{(3)}(t) \approx \frac{\theta^{(2)}(t + \Delta t) - \theta^{(2)}(t)}{\Delta t}, \tag{24}$$

$$\delta\ddot{\theta}(t) = \ddot{\theta}(t + \Delta t) - \ddot{\theta}(t)\,, \tag{25}$$

$$\delta\theta^{(2)} = \theta^{(3)}(t)\Delta t\,. \tag{26}$$

Time Discretization of the Differential Equation (16)

Linear Acceleration Method

This is a widely used scheme in structural dynamics. It consists of assuming that terms involving $\theta^{(4)}(t)$ in equations (20) and (21) are negligible and that the acceleration between t and $t + \Delta t$ varies linearly [i.e., according to equation (26)]. Substituting equation (26) into equation (20), we get

$$\delta\ddot{\theta} = \frac{6}{\Delta t^2}\delta\theta(t) - \frac{6}{\Delta t}\dot{\theta}(t) - 3\ddot{\theta}(t)\,. \tag{27}$$

Taking equations (26) and (27) into account in equation (21), we obtain

$$\delta\dot{\theta} = \frac{3}{\Delta t}\delta\theta(t) - 3\dot{\theta}(t) - \frac{\Delta t}{2}\ddot{\theta}(t)\,. \tag{28}$$

Substituting equations (27) and (28) into equation (16), we get the following linear system (in $\delta\theta$):

$$A\delta\theta = b\,, \tag{29}$$

where

$$A = \frac{6}{\Delta t^2}M + \frac{3}{\Delta t}C + K + R + S\,, \tag{30}$$

$$b = \frac{6}{\Delta t}\dot{\theta}(t)M + 3M\ddot{\theta}(t) + 3C\dot{\theta}(t) + \frac{\Delta t}{2}\ddot{\theta}(t)C\,. \tag{31}$$

We assume that terms involving derivatives of order 5 and higher are negligible and that

$$\theta^{(3)}(t) = \frac{\theta^{(2)}(t + \Delta t) - \theta^{(2)}(t - \Delta t)}{2\Delta t}, \tag{32}$$

$$\theta^{(4)}(t) = \frac{\theta^{(2)}(t + \Delta t) - 2\theta^{(2)}(t) + \theta^{(2)}(t - \Delta t)}{\Delta t^2}, \tag{33}$$

and

$$\delta\ddot{\theta}(t) = \ddot{\theta}(t + \Delta t) - \ddot{\theta}(t). \tag{34}$$

Substituting equations (32), (33), and (34) into (20)-(21) and rearranging terms, we get (analogous to the linear acceleration method):

$$A^{*}\delta\theta = b^{*}, \tag{35}$$

where

$$A^{*} = \frac{8}{\Delta t^2}M + \frac{10}{3\Delta t}C + K + R + S, \tag{36}$$

$$b^{*} = -Md_1 - Cd_2, \tag{37}$$

$$d_1 = \frac{8}{\Delta t^2}\left[\frac{\Delta t^2}{24}\ddot{\theta}(t - \Delta t) - \dot{\theta}(t)\Delta t - \frac{13\Delta t^2}{24}\ddot{\theta}(t)\right], \tag{38}$$

$$d_2 = \frac{13\Delta t}{12}\left[\ddot{\theta}(t) - \frac{\Delta t}{12}\ddot{\theta}(t - \Delta t) + \frac{5\Delta t}{12}d_1\right]. \tag{39}$$

A second algorithm is as follows: Given $\theta(t_n)$, $\dot{\theta}(t_n)$, $\ddot{\theta}(t_n)$, $\ddot{\theta}(t_{n-1})$, and $\lambda(t_n)$, update $\delta\theta(t_n)$, $\theta_{k+1}(t_n)$, and $\lambda_{k+1}(t_n)$ via

$$\delta\theta_k = A^{-1}b\,, \tag{40}$$

$$\lambda_{k+1} = P_{\Lambda^*}\left[\lambda_k + \rho G\left(\theta_{k+1}\right)\right], \tag{41}$$

where

$$\Lambda^* = \left\{\mu : \mu \in \mathbf{R}^{k+l} \;\;,\;\; \mu_i \in \mathbf{R} \;\;,\;\; i = 1,2,\ldots,k \;\;,\;\; \mu \geq 0 \;\;,\;\; i = k+1,\ldots,k+l\right\}. \tag{42}$$

In fact,

$$\Lambda = L^2\left(0,T;\Lambda^*\right). \tag{43}$$

Algorithm (40)-(41) can be used if A and b are replaced by A^* and b^*.

Other integration schemes, such as the ones in Dean, Glowinski, Kuo, and Nasser (ref. 5), may be used, also.

The acceleration $\ddot{\theta}(t)$ may be updated from the solution of equation (14) after convergence on $(\delta\theta(t), \lambda(t))$ has been achieved.

Choice of r and ρ

The parameters r, ρ, and Δt are the variables controlling the stability. For optimal choice of these parameters, refer to Glowinski and Le Tallec (ref. 2) and Glowinski, Lions, and Tremolières (ref. 3).

Using the projection method substantiated and systematically developed in Nasser (ref. 1), the equations of motion of the unconstrained system can be obtained in the following form:

$$M\,\delta\ddot{\theta} + C\,\delta\dot{\theta} + K\,\delta\theta = 0\,. \tag{44}$$

The projection method without piecewise linearization has been used by Keat (ref. 6) and is equivalent to the well-known Kane's method (ref. 7).

TEST PROBLEM

Consider a planar two-body system with a rigid obstacle, as shown in Figure 1. The Cartesian coordinates are related to the Lagrangian coordinates by

$$x_1 = a_1 \sin\theta_1 , \tag{44}$$

$$x_2 = l \sin\theta_1 + a_2 \sin\theta_2 , \tag{45}$$

$$y_1 = a_1 \cos\theta_1 , \tag{46}$$

$$y_2 = l \cos\theta_1 + a_2 \cos\theta_2 , \tag{47}$$

$$2\,KE = \left[m_1\left(\dot{x}_1 + \dot{y}_1\right) + I_1\dot{\theta}_1^2 \right] + \left[m_2\left(\dot{x}_2 + \dot{y}_2\right) + I_2\dot{\theta}_2^2 \right] , \tag{48}$$

$$PE = m_1 g a_1\left(1 - \cos\theta_1\right) + m_2 g \left[l\left(1 - \cos\theta_1\right) + a_2\left(1 - \cos\theta_2\right) \right] . \tag{49}$$

The stationarity of the Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}&\left(I_1 + m_1 a_1 + m_2 l\right)\ddot{\theta}_1 + m_2 l a_2 \cos\left(\theta_2 - \theta_1\right)\ddot{\theta}_2 - m_2 l a_2 \dot{\theta}_2^2 \sin\left(\theta_2 - \theta_1\right) + m_1 g l \sin\theta_1 \\ &+ m_2 g l \sin\theta_1 = 0 ,\end{aligned} \tag{50}$$

$$\left(I_2 + m_2 a_2\right)\ddot{\theta}_2 + m_2 l a_2 \cos\left(\theta_2 - \theta_1\right)\ddot{\theta}_1 - m_2 l a_2 \dot{\theta}_1^2 \sin\left(\theta_2 - \theta_1\right) + m_2 g a_2 \sin\theta_2 = 0 . \tag{51}$$

Data:

m_1 = mass of body 1
m_2 = mass of body 2
l = length
I_1 = moment inertia of body 1
I_2 = moment inertia of body 2
g = acceleration of gravity

Constraints:

$$g_1 = l_1 \sin\theta_1 + d \geq 0$$
$$g_2 = g_1 + l_2 \sin\theta_2 \geq 0$$

For the case $r = 0$, the augmented Lagrangian method reduces to the multiplier method used for the treatment of Coulomb or dry friction problems in Dean, Glowinski, Kuo, and Nasser (ref. 8). For the case $r \neq 0$, $\lambda = 0$, the scheme reduces to the well-known penalty method. The parameter r is the spring stiffness coefficient used in classical contact problems.

CONCLUSION

The augmented Lagrangian method successfully applies to contact/constrained motion problems of multibody dynamics. For constraints involving θ, the technique still applies; however, the details are rather lengthy and were omitted. The case of elastic bodies offers no mathematical difficulty except in the details, and the convergence is influenced by the spatial discretization largest mesh size. For further details, refer to Kikuchi and Oden (ref. 9) and Nasser (ref. 1).

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REFERENCES

1. Nasser, M. G.: Numerical Methods for Multibody Elastic Systems and Contact Problems. Ph.D. diss., University of Houston, Aug. 1992.
2. Glowinski, R.; and Le Tallec, P: *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*. SIAM Studies in Applied Mathematics, vol. 9. Society for Industrial and Applied Mathematics, 1989.

3. Glowinski, R.; Lions, J. L.; and Tremolières, R: *Numerical Analysis of Variational Inequalities.* North-Holland, 1981.

4. Fortin, M.; and Glowinski, R., eds.: *Augmented Lagrangian Methods.* North-Holland, 1983.

5. Dean, E.; Glowinski, R.; Kuo, Y.; and Nasser, M. G.: On the Discretization of Some Second Order in Time Differential Equations: Applications to Nonlinear Wave Problems. *Proceedings of the NASA-UCLA Workshop on Computational Techniques in Identification and Control of Flexible Flight Structures*, Dec. 2-4, 1989.

6. Keat, J.: A Formulation for Simulating Deployment Dynamics of Large Space Structures. *Proceedings of the Workshop on Multibody Simulation*, Jet Propulsion Laboratory, JPLD-5190, vol. I, Apr. 15, 1988.

7. Kane, T. R.; and Levinson, D. A.: *Dynamics: Theory and Applications.* McGraw-Hill, 1985.

8. Dean, E.; Glowinski, R.; Kuo, Y.; and Nasser, M. G.: Multiplier Techniques for Some Dynamical Systems with Dry Friction. *C. R. Acad. Sci. Paris*, t. 314, série I, pp. 153-159, 1992.

9. Kikuchi, N.; and Oden, J. T.: *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods.* SIAM Studies in Applied Mathematics, vol. 8. Society for Industrial and Applied Mathematics, 1989.

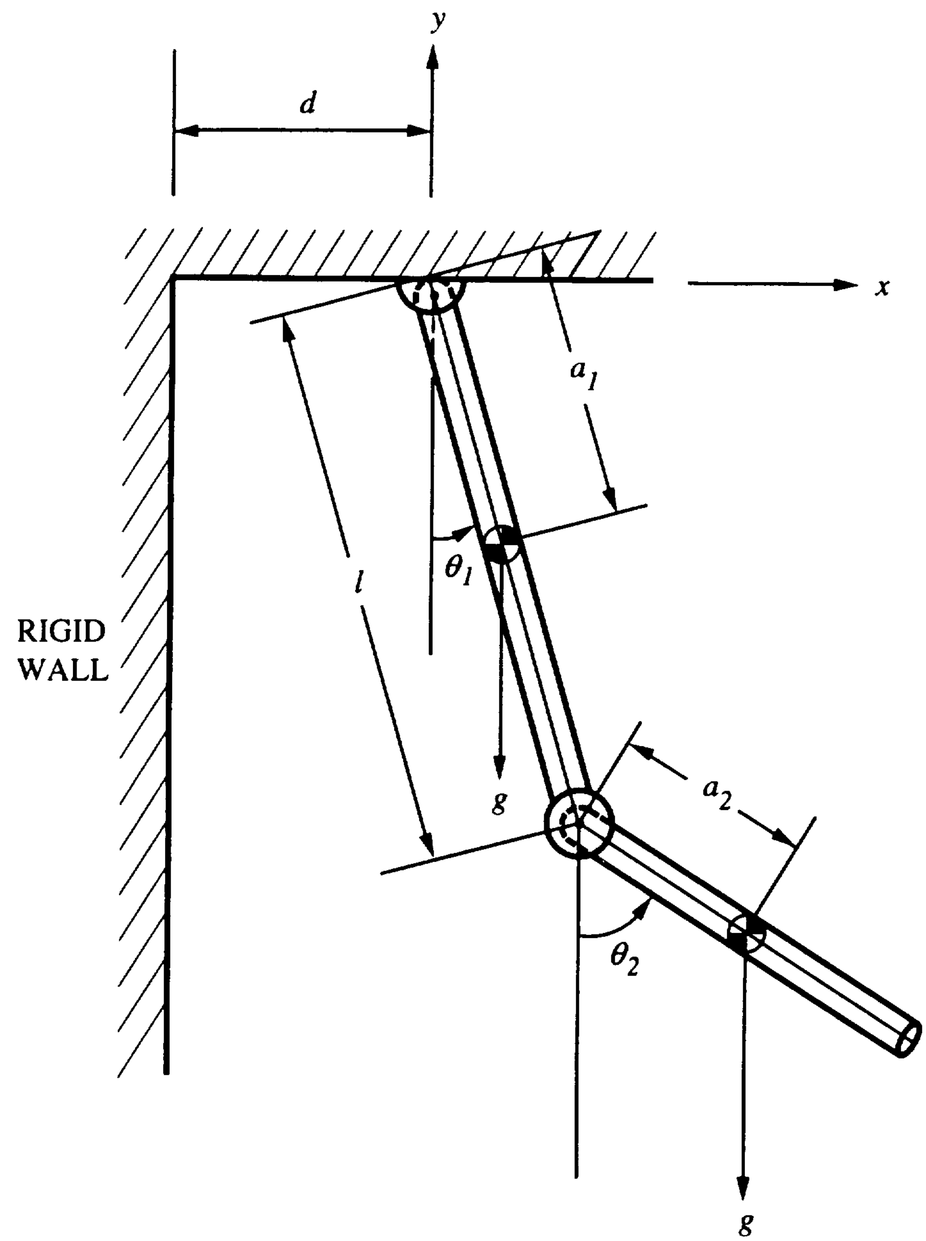
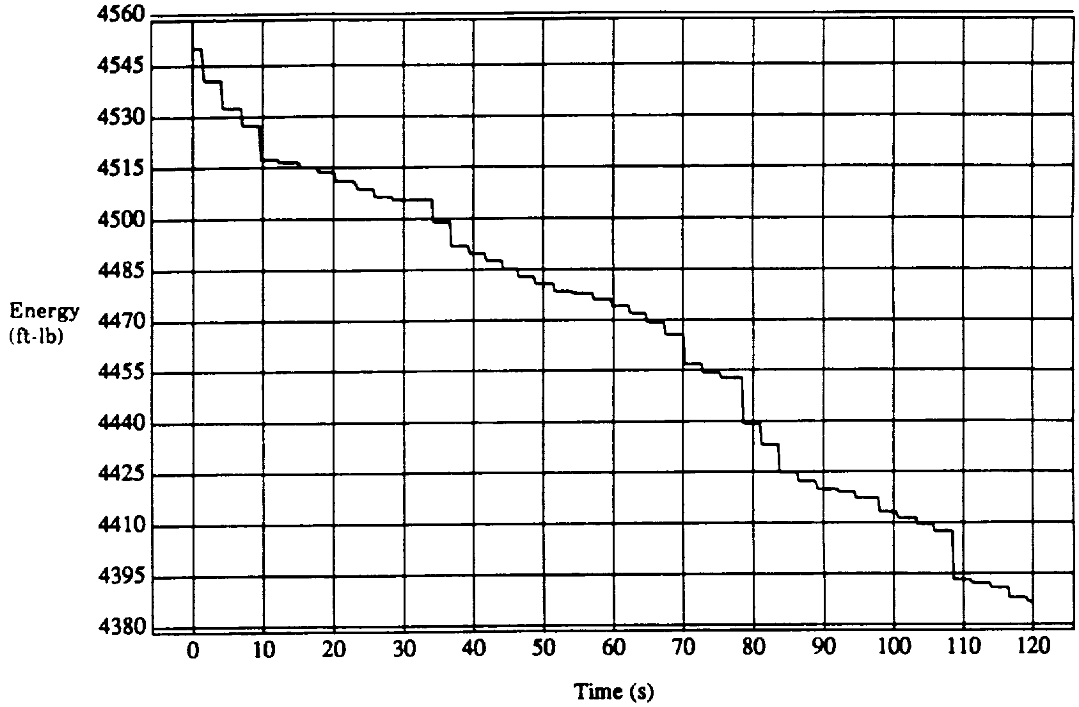
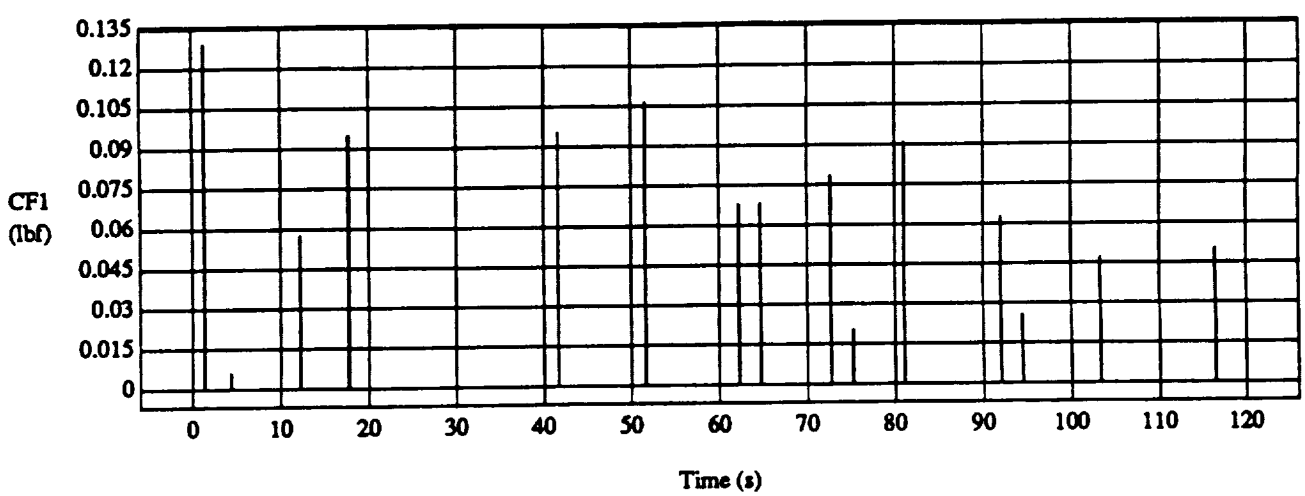


Figure 1. Planar two-body system.



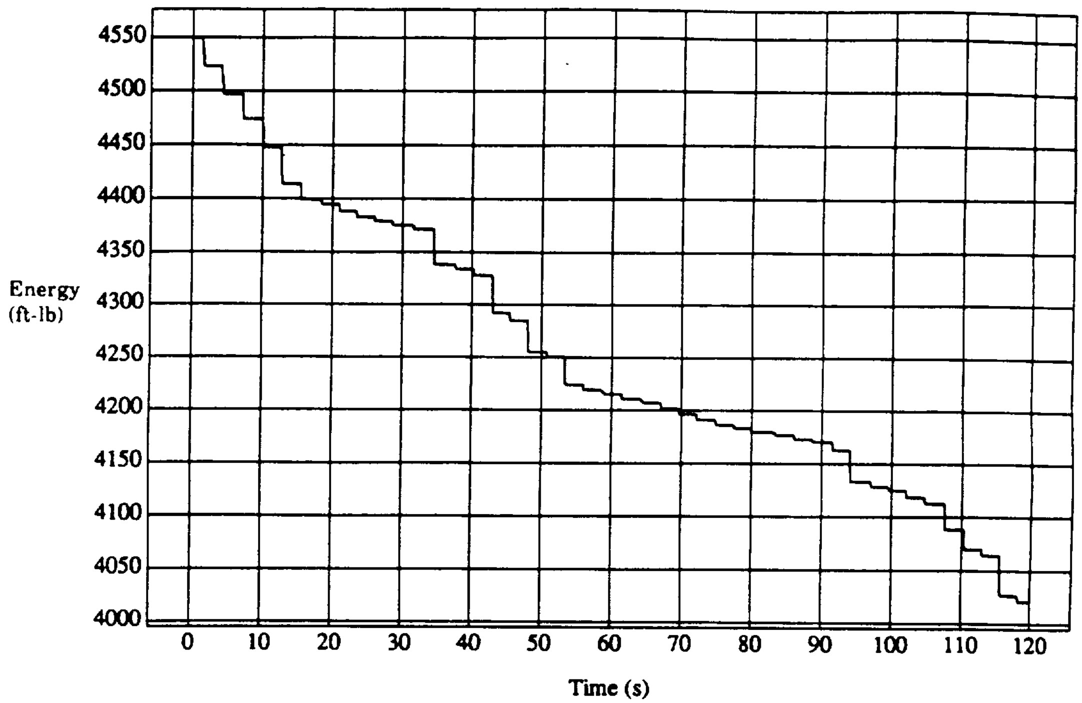
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 2. Multiplier method/high-order scheme – energy.



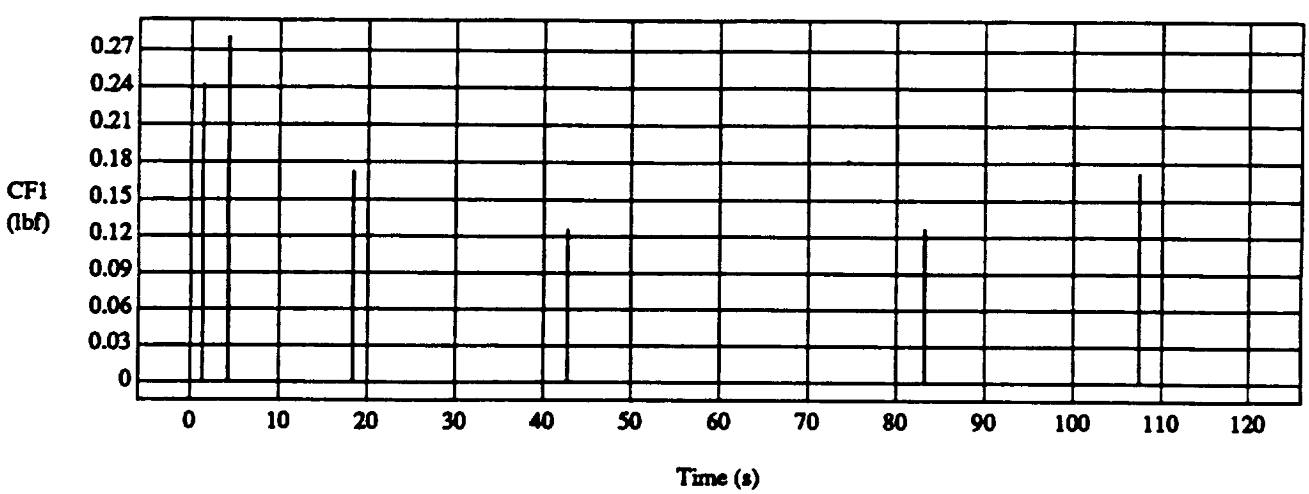
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 3. Multiplier method/high-order scheme – constraint force.



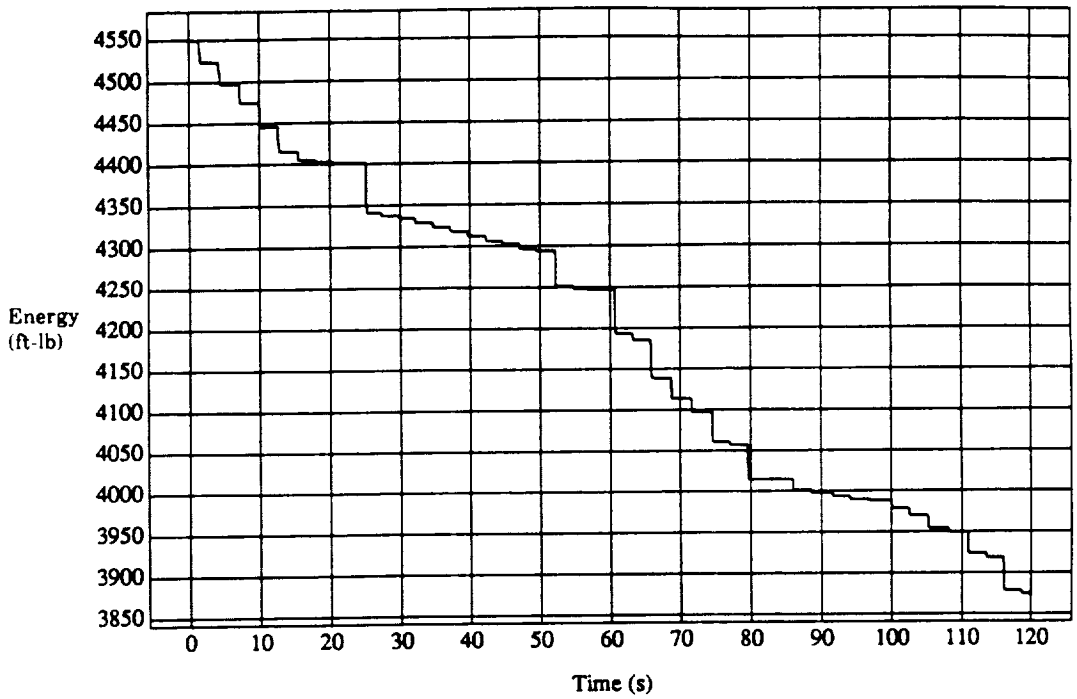
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 4. Augmented Lagrangian/high-order scheme – energy.



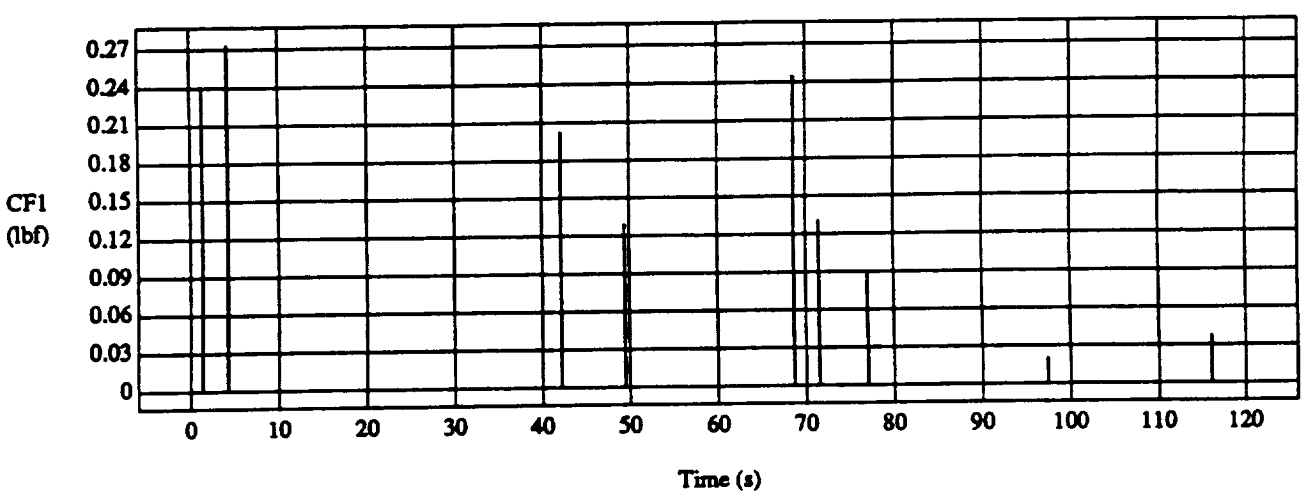
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 5. Augmented Lagrangian/high-order scheme – constraint force.



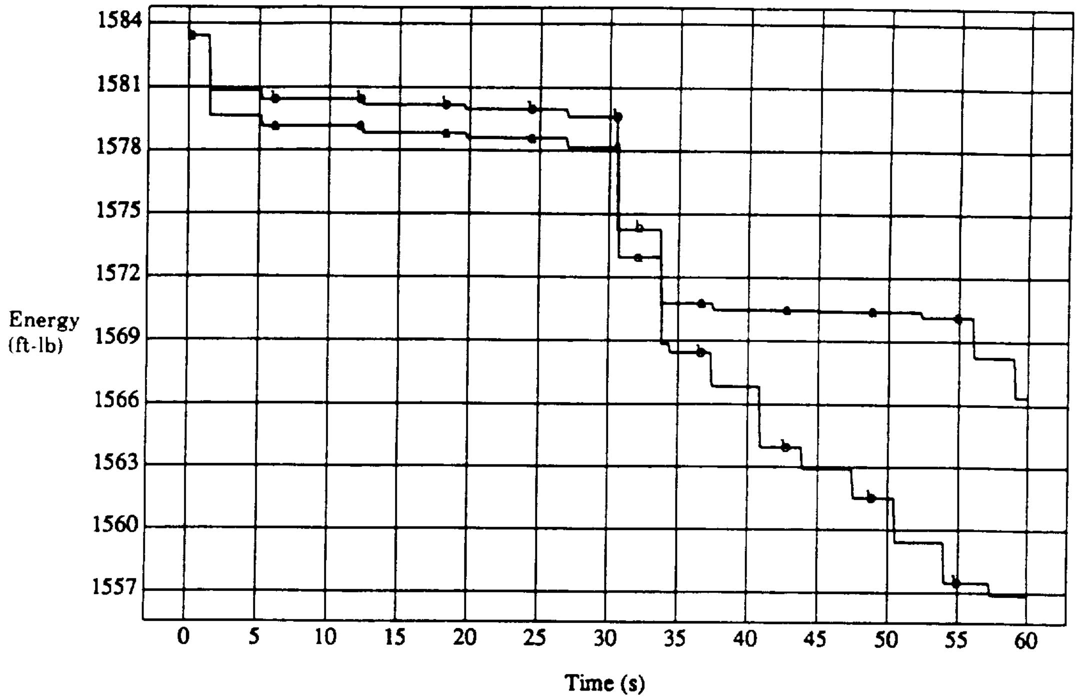
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 6. Penalty method/high-order scheme – energy.



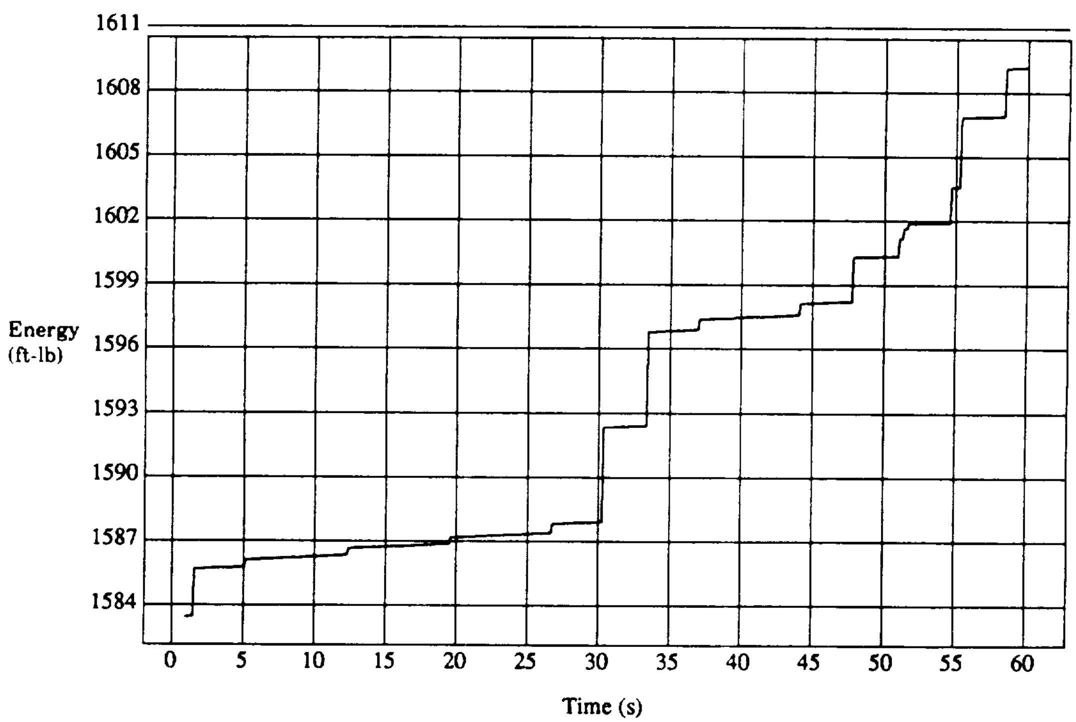
$\theta_1(0) = \pi/3, \theta_2(0) = \pi/2$

Figure 7. Penalty method/high-order scheme – constraint force.



$\theta_1(0) = \pi/3, \theta_2(0) = 0$

Figure 8. Augmented Lagrangian energy comparison for high-order scheme versus the linear acceleration method.



$\theta_1(0) = \pi/3, \theta_2(0) = 0$

Figure 9. Penalty/explicit Euler scheme – energy.