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Approximate Analysis for Repeated Eigenvalue Problems With Applications to Controls-Structures Integrated Design

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Abstract

A method for eigenvalue and eigenvector approximate analysis for the case of repeated eigenvalues with distinct first derivatives is presented. The approximate analysis method developed involves a reparameterization of the multivariable structural eigenvalue problem in terms of a single positive-valued parameter. The resulting equations yield first-order approximations to changes in the eigenvalues and the eigenvectors associated with the repeated eigenvalue problem. This work also presents a numerical technique that facilitates the definition of an eigenvector derivative for the case of repeated eigenvalues with repeated eigenvalue derivatives (of all orders). Examples are given which demonstrate the application of such equations for sensitivity and approximate analysis. Emphasis is placed on the application of sensitivity analysis to large-scale structural and controls-structures optimization problems.

Introduction

Future spacecraft will have increasingly higher performance requirements and enhanced capabilities. Spacecraft such as Space Station *Freedom* and the Earth Observing Satellites will be subject to a wide range of disturbance sources, but these spacecraft will still be required to meet very stringent pointing specifications. Multiple scanning precision instruments and large-angle payload articulation (robotics) are examples of disturbance sources that are expected on proposed future spacecraft. Spacecraft performance requirements in the presence of multiple disturbances present significant technical challenges for spacecraft designers. Researchers are currently investigating spacecraft design techniques that focus on integrating several disciplines to form a common interdisciplinary design environment. Interdisciplinary design offers many advantages over conventional design techniques. The primary advantage of interdisciplinary design is that it provides the framework by which trade-offs, as required in the design process, can be explicitly addressed. Specifically, design trade-offs on subsystem components can be done in such a way as to obtain an optimal design for the entire spacecraft; this process allows the performance requirements to be more easily met.

The integrated design of controlled structures is a specific application of interdisciplinary design techniques. A typical example of integrated design of controlled structures is the simultaneous controls-structures optimization of large flexible space structures. This problem has been addressed from a variety of perspectives. Belvin and Park (ref. 1) have proposed a structural tailoring procedure to increase the system performance and simultaneously decrease the control effort. Maghami et al. (refs. 2 to 4) have developed controls-structures integrated design methodologies to reduce the amount of control energy required to meet pointing requirements while simultaneously decreasing the overall mass of the structure and increasing its disturbance rejection capabilities. Although these techniques are different in form and objective, they are computationally intensive. Most of these computations involve repetitive structural eigensolutions.

The objective of this paper is to present methods for increasing the computational efficiency of structural optimization and controls-structures integrated design methods. Specifically, this paper introduces an eigenvalue/eigenvector first-order approximation technique for the real, symmetric structural eigenproblem for two cases of repeated eigenvalues, and it will demonstrate the advantages of the methodology in the application of eigensensitivity analysis to facilitate the simultaneous controls-structures integrated design process. The two cases of repeated eigenvalue problems to be considered are (1) repeated eigenvalues with distinct eigenvalue derivatives and (2) repeated eigenvalues with repeated eigenvalue derivatives of all orders (herein referred to as "infinitely repeated eigenvalues").

This paper also addresses some of the computational issues related to eigensensitivity analysis for repeated eigenvalue problems. Computational issues are addressed by introducing an alternate eigenvector derivative matrix formulation that has an interesting characteristic (namely, the elimination of the numerical burden related to repeated factorization of design-variable-dependent eigensensitivity matrix equations). This formulation eliminates the need for a matrix factorization of the large-order sensitivity equations for every design variable per eigenvalue/eigenvector pair, and instead it requires only "back-substitution" for the design-variable-dependent right-hand-side vectors.

Nomenclature

A	element cross-sectional area, in^2
b_i	ith design variable
E	Young's modulus, lbf/in^2
I_y	bending moment of inertia about Y-axis, in^4
I_z	bending moment of inertia about Z-axis, in^4
$\mathbf{K}$	global stiffness matrix
$\mathbf{K}_e$	element stiffness matrix
k	design variable linking constraint
L	element length, in.
$\mathbf{M}$	global mass matrix
M_{budget}	total mass budget, slug
$\mathbf{M}_e$	element mass matrix
M_{tot}	total mass, slug
n	number of degrees of freedom
$ndev$	number of structural design variables
r_i	inner radius of tube beam element, in.
r_o	outer radius of tube beam element, in.
rms	root mean square
$\mathbf{x}_i$	ith structural eigenvector
$\mathbf{Y}$	orthonormal transformation matrix
y_{los}	line-of-sight performance measure
λ_i	ith structural eigenvalue
ρ	element mass density, lbf/in.
$\boldsymbol{\phi}_i$	ith differentiable eigenvector

Derivation of Eigenvalue and Eigenvector Sensitivity Equations

In the problem to be considered, sensitivities of real (generalized) eigenvalues λ and eigenvectors $\mathbf{x}$, which lie in the null space of the symmetric matrix $\mathbf{K}(b) - \lambda\mathbf{M}(b)$, are examined with respect to variations in design parameters (b). Eigenvalues of this real, symmetric, structural eigenproblem may be separated into two categories: distinct and repeated. Repeated eigenvalues may be further subdivided into the following three cases:

1. Repeated eigenvalues with distinct eigenvalue derivatives
2. Repeated eigenvalues with some degree of repeated eigenvalue derivatives
3. Infinitely repeated eigenvalues

The simplest case is the distinct eigenvalue problem in which eigenvalues are distinct in the design space of concern, as shown in figure 1. Methods for computing the derivatives of eigenvalues and eigenvectors for the distinct eigenvalue problem have been well developed (ref. 5), and therefore they will not be addressed here.

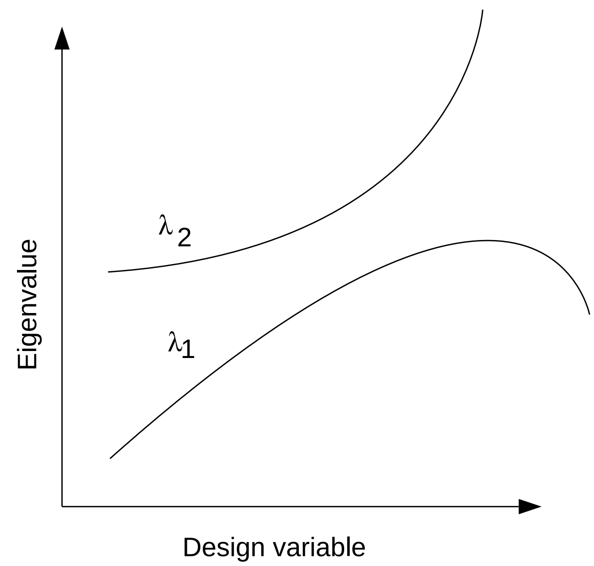

Figure 1. Distinct eigenvalue problem.

Repeated eigenvalues with distinct eigenvalue derivatives often occur at the optimum solution when the fundamental (smallest) eigenvalue of the structure is maximized with respect to some design variable. In this case, the value of the fundamental eigenvalue gradually increases, and at a later stage in the design optimization process, it approaches the value of the second eigenvalue, as shown in figure 2. In this figure, b^0 denotes the optimum design variable solution, and $\lambda_1 \leq \lambda_2$ (by definition). Note that the eigenvalues λ_1 and λ_2 are the same at b^0 and that they are only directionally differentiable.

The computation of eigenvector derivatives in the repeated eigenvalue situation is difficult because the eigenvectors are not unique (i.e., any linear combination of eigenvectors corresponding to a repeated eigenvalue is also an eigenvector). Recent publications, however, such as Ojalvo (ref. 6), Mills-Curran (ref. 7), Dailey (ref. 8), Juang et al.

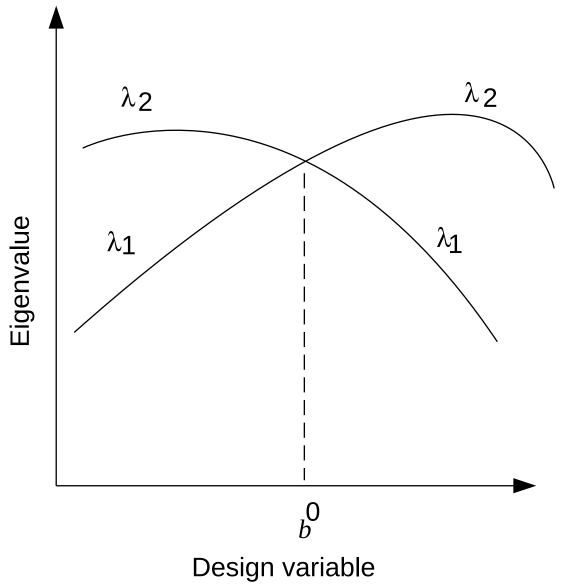

Figure 2. Repeated eigenvalues with distinct first derivatives.

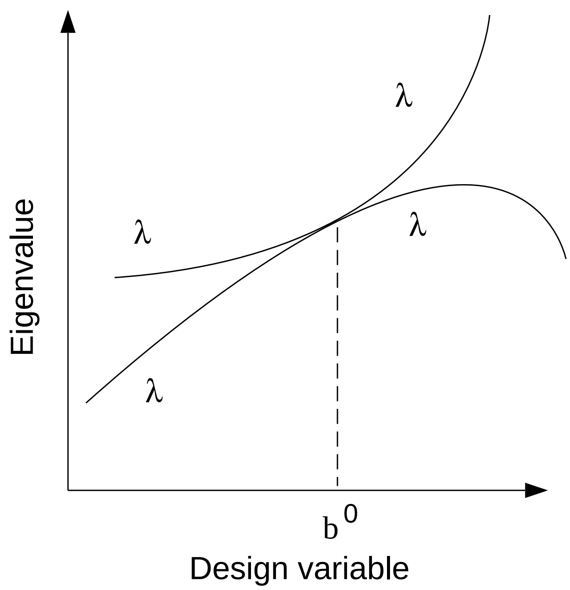

Figure 3. Repeated eigenvalues with repeated first derivatives.

(ref. 9), and Bernard (ref. 10) have proposed numerical procedures to reorient the eigenvectors as a set of differentiable eigenvectors.

An extension of the last type of eigenvalue problem in which not only the eigenvalues but also their derivatives are repeated is discussed in reference 9. However, it is assumed that the eigenvalue derivative is distinct at some finite order. A typical example is shown in figure 3, where $\lambda_1 = \lambda_2$, $\partial\lambda_1/\partial b = \partial\lambda_2/\partial b$, and $\partial^2\lambda_1/\partial b^2 \neq \partial^2\lambda_2/\partial b^2$ at b^0.

The limiting case of reference 9 is the situation in which repeated eigenvalues have derivatives that repeat indefinitely. This case is referred to as infinitely repeated eigenvalues. Here, a pair of eigenvalues and all orders of eigenvalue derivatives are equal within the design space of concern, as shown in figure 4. Dailey (ref. 8) briefly addresses the issue when $\partial\lambda_1/\partial b = \partial\lambda_2/\partial b$. In light of reference 9, this equality should be correctly stated as $\partial^n\lambda_1/\partial b^n = \partial^n\lambda_2/\partial b^n$, where $n \rightarrow \infty$, by noting that eigenvector derivatives corresponding to repeated eigenvalue derivatives are nonunique. Dailey gives an interpretation of nonunique eigenvector derivatives that occur with infinitely repeated eigenvalues analogous to the interpretation of nonunique eigenvectors that occur with simple repeated eigenvalues. Dailey states that any linear combination of eigenvector derivatives corresponding to an infinitely repeated eigenvalue is also an eigenvector derivative. This definition provides an arbitrary parameter, which results in a valid eigenvector derivative (ref. 8).

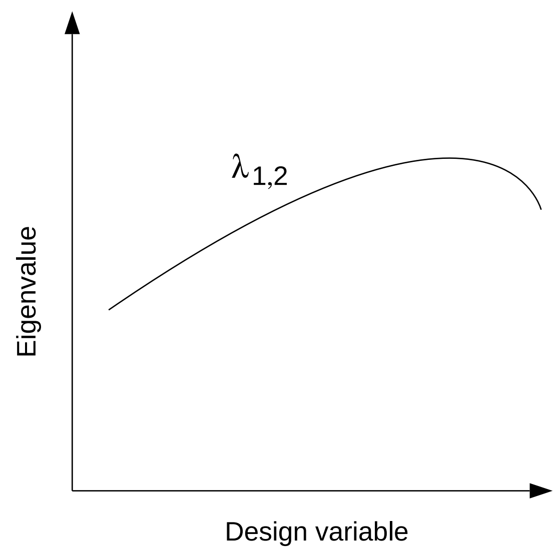

Figure 4. Infinitely repeated eigenvalues.

The rigid-body modes of a structure can be considered as the infinitely repeated case in which the eigenvalues remain zero regardless of design changes. Structural symmetry is another example that typically results in infinitely repeated eigenvalues. Flexible symmetric appendages (such as antennas and solar array masts), which are found on typical space structures, usually exhibit infinitely repeated eigenvalues. The derivatives of these eigenvalues will repeat indefinitely if design changes are made in such a way as to preserve structural symmetry.

In the following two sections, detailed derivations of eigenvalue/eigenvector sensitivity equations for two types of eigenvalues are presented. For the case of repeated eigenvalues with distinct first derivatives, analytically exact eigenvector derivative solution techniques are presented. These techniques are different in form from the modal expansion method (ref. 11) and Nelson's method (ref. 5). The modal expansion method assumes that the eigenvector derivatives can be fully represented as a linear combination of a subset of the eigenvectors. The method presented in this paper does not make this same assumption. Instead, the eigenvector derivatives are exactly represented as linear combinations of all the eigenvectors. In Nelson's method (ref. 6), decisions must be made regarding which components of the solution vector should be constrained. The method presented herein uses the orthogonality properties of the eigenvectors to form a set of linear equality constraints.

For the case of infinitely repeated eigenvalues, numerical assumptions are introduced which allow the definition of eigenvector derivatives. Emphasis is placed upon developing a unified formulation for the matrix equations for both types of repeated behavior. All derivations in this paper are based upon finite-element representations of structural systems.

Repeated Eigenvalues With Distinct First Derivatives

This section outlines the derivations of eigenvalue/eigenvector sensitivity equations for repeated eigenvalue problems with distinct first derivatives. For a more detailed derivation of eigenvalue/eigenvector sensitivity equations, as well as eigenvalue/eigenvector approximate analysis, see Kenny (ref. 12). In the derivation that follows, it is assumed that the original eigenvectors form a nondefective set (i.e., the original set of linearly independent eigenvectors completely spans an n dimensional space).

In the case of repeated eigenvalues, the original eigenvectors are, in general, nondifferentiable. Therefore, the methods derived for simple (distinct) eigenvalue problems are no longer valid. This difficulty can be better explained by investigating the differences between simple and repeated eigenvalue problems. The first and most fundamental difference is that the eigenvectors of repeated eigenvalue problems have an additional degree of nonuniqueness compared with those associated with simple eigenvalues. This degree of nonuniqueness is because any linear combination of the eigenvectors is also a valid eigenvector. The second difference is related to the rank deficiency of the matrix $(\mathbf{K} - \lambda\mathbf{M})$, where $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices, respectively. If repeated eigenvalues occur with a multiplicity of m, then the matrix $(\mathbf{K} - \lambda\mathbf{M})$ will be rank deficient by m, whereas it is deficient by 1 for the simple eigenvalue problem.

To start the derivation, assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the pair of eigenvectors that are associated with the repeated eigenvalue λ. The differentiable eigenvectors (whose existence is established in ref. 9) which are associated with λ can then be represented as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ as follows:

$$\boldsymbol{\phi}_i = \mathbf{X}\mathbf{y}_i \qquad (i = 1, 2) \tag{1}$$

where $\mathbf{X}$ is an $n \times 2$ matrix with $\mathbf{x}_1$ and $\mathbf{x}_2$ as its columns, and $\mathbf{y}_i$ is a 2×1 vector. The differentiable eigenvector should satisfy the n dimensional eigenvalue equation

$$(\mathbf{K} - \lambda\mathbf{M})\boldsymbol{\phi}_i = 0 \tag{2}$$

The literature (refs. 6 to 10 and 12 and 13) shows that the derivatives of the repeated eigenvalues are the solution of the following reduced eigenvalue problem:

$$\left(\widetilde{\mathbf{K}} - \frac{\partial\lambda_i}{\partial b}\widetilde{\mathbf{M}}\right)\mathbf{y}_i = 0 \qquad (i = 1, 2) \tag{3}$$

where

$$\widetilde{\mathbf{K}} = \mathbf{X}^T\left(\frac{\partial\mathbf{K}}{\partial b} - \lambda\frac{\partial\mathbf{M}}{\partial b}\right)\mathbf{X} \tag{4}$$

and

$$\widetilde{\mathbf{M}} = \mathbf{X}^T\mathbf{M}\mathbf{X} \tag{5}$$

The notation $\partial\lambda_1/\partial b$ and $\partial\lambda_2/\partial b$, as used in equation (3), is defined as the unique eigenvalue derivatives of the repeated eigenvalue pair at design point b^0, and it will be used as such throughout this paper.

Note that because $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent and lie within a nondefective set, they can be "forced" to become mass orthonormal by any orthogonalization process (e.g., the Gram-Schmidt process). Therefore, $\widetilde{\mathbf{M}}$ can be set equal to a 2×2 identity matrix without loss of generality. Equation (3) provides a means to find the eigenvalue derivatives and the orthonormal transformation matrix $\mathbf{Y}$ with $\mathbf{y}_1$ and $\mathbf{y}_2$, the eigenvectors of equation (3), as its columns. The matrix $\mathbf{Y}$ will in turn allow the definition of a set of differentiable eigenvectors. If the eigenvalue derivatives are distinct, equation (3) yields two simple eigensolutions: $(\partial\lambda_1/\partial b,\ \mathbf{y}_1)$ and $(\partial\lambda_2/\partial b,\ \mathbf{y}_2)$. The differentiable eigenvectors $\boldsymbol{\phi}_1 = \mathbf{X}\mathbf{y}_1$ and $\boldsymbol{\phi}_2 = \mathbf{X}\mathbf{y}_2$ correspond to the eigenvalue derivatives $\partial\lambda_1/\partial b$ and $\partial\lambda_2/\partial b$, respectively.

To find the eigenvector derivative $\partial \boldsymbol{\phi}_i / \partial b$ for the situation $\lambda_1 = \lambda_2 = \lambda$, one can start with the equation for the eigenvector derivative as given below; this equation can be obtained by simply differentiating equation (2) with respect to the design parameter b such that

$$(\mathbf{K} - \lambda \mathbf{M}) \frac{\partial \boldsymbol{\phi}_i}{\partial b} = \frac{\partial \lambda_i}{\partial b} \mathbf{M} \boldsymbol{\phi}_i - \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_i \qquad (i = 1, 2) \qquad (6)$$

Many methods are available in the literature (refs. 6 to 9) to find the solution of equation (6); however, a different procedure is presented here.

This procedure employs the fact that if a solution to equation (6) exists, then the eigenvector derivative can be rewritten as a linear combination of $(n-2)$ eigenvectors and the products of two undetermined constants with their corresponding differentiable eigenvectors such that

$$\frac{\partial \boldsymbol{\phi}_i}{\partial b} = \mathbf{u}_i + c_{i1} \boldsymbol{\phi}_1 + c_{i2} \boldsymbol{\phi}_2 \qquad (i = 1, 2) \qquad (7)$$

where

$$\mathbf{u}_i = \sum_{j=3}^{n} c_{ij} \boldsymbol{\phi}_j \qquad (8)$$

Note that the particular solution $\mathbf{u}_i$ is mass orthogonal with respect to the complementary solution $(c_{i1} \boldsymbol{\phi}_1 + c_{i2} \boldsymbol{\phi}_2)$ such that

$$\mathbf{u}_i^T \mathbf{M} (c_{i1} \boldsymbol{\phi}_1 + c_{i2} \boldsymbol{\phi}_2) = 0 \qquad (i = 1, 2) \qquad (9)$$

which can be restated as

$$\mathbf{u}_i^T \mathbf{M} \boldsymbol{\phi}_1 = 0 \qquad (i = 1, 2) \qquad (10)$$

$$\mathbf{u}_i^T \mathbf{M} \boldsymbol{\phi}_2 = 0 \qquad (i = 1, 2) \qquad (11)$$

because c_{i1} and c_{i2} are arbitrary in equation (7). (See ref. 12 for the proof regarding the existence of a solution to eq. (6).)

With the aid of equation (7) and the relationship $\partial \lambda_i / \partial b = \boldsymbol{\phi}_i^T (\partial \mathbf{K} / \partial b - \lambda \, \partial \mathbf{M} / \partial b) \boldsymbol{\phi}_i$, equation (6) can be written as

$$(\mathbf{K} - \lambda \mathbf{M}) \mathbf{u}_i = \left[\boldsymbol{\phi}_i^T \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_i \right] (\mathbf{M} \boldsymbol{\phi}_i) - \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_i \qquad (i = 1, 2) \qquad (12)$$

whose solution $\mathbf{u}_i$ has to satisfy the linear equality constraints of equations (10) and (11). According to the theory of Lagrange multipliers (ref. 13), a unique solution of equation (12), subject to the linear constraints equations (10) and (11), can be found as the solution of the following equation:

$$(\mathbf{K} - \lambda \mathbf{M}) \mathbf{u}_i + \mu_{i1} \mathbf{M} \boldsymbol{\phi}_1 + \mu_{i2} \mathbf{M} \boldsymbol{\phi}_2 = - \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_i \qquad (i = 1, 2) \qquad (13)$$

in which $\mathbf{u}_i$, μ_{i1}, and μ_{i2} are unknowns, and μ_{i1} and μ_{i2} are the Lagrange multipliers. Nevertheless, because equation (13) is a function of the differentiable eigenvectors $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ (which are different for each design variable), it has to be factored and solved for every design variable. Therefore, solving equation (13) in a design optimization environment may become computationally expensive. Consequently, deriving an alternate equation for the eigenvector derivatives is desirable.

To eliminate the design variable dependencies from equation (13), a particular solution $\mathbf{u}_i$ is sought by redefining $\mathbf{u}_i$ as a linear combination of $\mathbf{v}_i$, where $i = 1, 2$. In other words, $\mathbf{u}_i = \mathbf{V} \mathbf{y}_i$, where $\mathbf{V}$ is an $n \times 2$ matrix with $\mathbf{v}_i$'s as its columns. With this new definition and the definition of $\boldsymbol{\phi}_i$, equations (10) to (12) can be written as

$$\mathbf{y}_i^T \mathbf{V}^T \mathbf{M} \mathbf{X} \mathbf{y}_1 = 0 \qquad (i = 1, 2) \qquad (14)$$

$$\mathbf{y}_i^T \mathbf{V}^T \mathbf{M} \mathbf{X} \mathbf{y}_2 = 0 \qquad (i = 1, 2) \qquad (15)$$

and

$$(\mathbf{K} - \lambda \mathbf{M}) \mathbf{V} \mathbf{y}_i = \left[\boldsymbol{\phi}_i^T \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_i \right] (\mathbf{M} \mathbf{X} \mathbf{y}_i) - \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) (\mathbf{X} \mathbf{y}_i) \qquad (i = 1, 2) \qquad (16)$$

Because eigenvectors $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent, it can be proved that equations (14) to (15) are equivalent to

$$\mathbf{v}_i^T \mathbf{M} \mathbf{x}_1 = 0 \qquad (i = 1, 2) \qquad (17)$$

$$\mathbf{v}_i^T \mathbf{M} \mathbf{x}_2 = 0 \qquad (i = 1, 2) \qquad (18)$$

To solve equation (16), it is sufficient to solve equation (19) such that

$$(\mathbf{K} - \lambda \mathbf{M}) \mathbf{v}_i = \left[\boldsymbol{\phi}_j^T \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \boldsymbol{\phi}_j \right] (\mathbf{M} \mathbf{x}_i) - \left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b} \right) \mathbf{x}_i \qquad \left\{ \begin{array}{l} (i = 1, 2) \\ (j = 1, 2) \end{array} \right. \qquad (19)$$

Finally, with the aid of the theory of Lagrange multipliers, the solution of equation (19), subjected to the constraints of equations (17) and (18), is equivalent to the solution of the following equation:

$$\begin{bmatrix} \mathbf{K}-\lambda\mathbf{M} & \mathbf{M}\mathbf{x}_1 & \mathbf{M}\mathbf{x}_2 \\ \mathbf{x}_1^T\mathbf{M} & 0 & 0 \\ \mathbf{x}_2^T\mathbf{M} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v}_i \\ \mu_{i1} \\ \mu_{i2} \end{Bmatrix} = \begin{Bmatrix} -\left(\frac{\partial \mathbf{K}}{\partial b} - \lambda \frac{\partial \mathbf{M}}{\partial b}\right)\mathbf{x}_i \\ 0 \\ 0 \end{Bmatrix} \qquad (i=1,2) \quad (20)$$

Note that the left-hand side is independent of design variables and that the above equation can be solved without knowing the differentiable eigenvectors in advance. Recall the eigenvector derivative is defined as

$$\frac{\partial \boldsymbol{\phi}_i}{\partial b} = \mathbf{V}\mathbf{y}_i + c_{i1}\boldsymbol{\phi}_1 + c_{i2}\boldsymbol{\phi}_2 \qquad (i=1,2) \quad (21)$$

To continue the derivation of $\partial\boldsymbol{\phi}_i/\partial b$, equations for the constants c_{i1} and c_{i2} in equation (21) need to be determined. This determination can be partially done by using the first two normalization conditions such that

$$\boldsymbol{\phi}_i^T\mathbf{M}\boldsymbol{\phi}_i = 1 \qquad (i=1,2) \quad (22)$$

whose design derivatives yield

$$c_{11} = -\frac{1}{2}\left(\boldsymbol{\phi}_1^T \frac{\partial \mathbf{M}}{\partial b}\boldsymbol{\phi}_1\right) \quad (23)$$

and

$$c_{22} = -\frac{1}{2}\left(\boldsymbol{\phi}_2^T \frac{\partial \mathbf{M}}{\partial b}\boldsymbol{\phi}_2\right) \quad (24)$$

As for constants c_{12} and c_{21}, equations similar to those given in references 7 and 8 can be derived by taking the second derivative of equation (2) to obtain the following equalities:

$$c_{12} = \left\{ \boldsymbol{\phi}_2^T \left[\frac{\partial^2 \mathbf{K}}{\partial b^2} - 2\left(\frac{\partial \lambda_1}{\partial b}\right)\frac{\partial \mathbf{M}}{\partial b} - \lambda\frac{\partial^2 \mathbf{M}}{\partial b^2}\right]\boldsymbol{\phi}_1 + 2\boldsymbol{\phi}_2^T\left(\frac{\partial \mathbf{K}}{\partial b} - \lambda\frac{\partial \mathbf{M}}{\partial b}\right)\mathbf{u}_1 \right\} \Big/ 2\left(\frac{\partial \lambda_1}{\partial b} - \frac{\partial \lambda_2}{\partial b}\right) \quad (25)$$

and

$$c_{21} = \left\{ \boldsymbol{\phi}_1^T \left[\frac{\partial^2 \mathbf{K}}{\partial b^2} - 2\left(\frac{\partial \lambda_2}{\partial b}\right)\frac{\partial \mathbf{M}}{\partial b} - \lambda\frac{\partial^2 \mathbf{M}}{\partial b^2}\right]\boldsymbol{\phi}_2 + 2\boldsymbol{\phi}_1^T\left(\frac{\partial \mathbf{K}}{\partial b} - \lambda\frac{\partial \mathbf{M}}{\partial b}\right)\mathbf{u}_2 \right\} \Big/ 2\left(\frac{\partial \lambda_2}{\partial b} - \frac{\partial \lambda_1}{\partial b}\right) \quad (26)$$

It is worthwhile mentioning here that the derivative of the orthogonal condition between $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_1^T\mathbf{M}\boldsymbol{\phi}_2 = 0$, should be satisfied by the eigenvector derivatives $\partial\boldsymbol{\phi}_1/\partial b$ and $\partial\boldsymbol{\phi}_2/\partial b$. The eigenvector derivatives should preserve the following identity:

$$\frac{\partial \boldsymbol{\phi}_1^T}{\partial b}\mathbf{M}\boldsymbol{\phi}_2 + \boldsymbol{\phi}_1^T\mathbf{M}\frac{\partial \boldsymbol{\phi}_2}{\partial b} + \boldsymbol{\phi}_1^T\frac{\partial \mathbf{M}}{\partial b}\boldsymbol{\phi}_2 = 0 \quad (27)$$

which results in an equality with c_{12} and c_{21} as

$$c_{12} + c_{21} + \boldsymbol{\phi}_1^T\frac{\partial \mathbf{M}}{\partial b}\boldsymbol{\phi}_2 = 0 \quad (28)$$

Thus, once c_{12} is found by using equation (25), c_{21} can be simply obtained via equation (28); this process is computationally more efficient than using equation (26).

Note that, although eigenvalue/eigenvector derivatives associated with repeated eigenvalues can be determined by equation (3) and equations (20) to (28), they cannot be used directly to estimate the change in the eigensolution. This estimation will be the subject of discussion in the next section. Examples are presented subsequently to demonstrate the application of the sensitivity equations derived above.

Infinitely Repeated Eigenvalues

For any structure, the rigid-body modes can be found separately from the rest of flexible modes; therefore, the discussion that follows focuses on the infinitely repeated eigenvalues among the flexible modes.

Similar to the case studied in the previous section, the eigenvalue equation for the infinitely repeated eigenvalues fails to specify the eigenvector uniquely. Any linear combination of the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ for the eigenvalue λ will also be an eigensolution. However, for the case of infinitely repeated eigenvalues, the eigenvalue derivatives also repeat. Therefore, the "eigenvectors" of

$$\left(\widetilde{\mathbf{K}} - \frac{\partial \lambda_i}{\partial b_j}\widetilde{\mathbf{M}}\right)\mathbf{y}_i = 0 \qquad \begin{cases} (i=1,2) \\ (j=1,ndev) \end{cases} \quad (29)$$

are themselves nonunique (i.e., any linear combination of $\mathbf{y}_1$ and $\mathbf{y}_2$ will also satisfy the above reduced eigenvalue problem, and the transformation from $\mathbf{x}_i$ to $\boldsymbol{\phi}_i$ is no longer unique). Note that the limit on subscript $j, ndev$, is defined as the number of independent design variables.

In this study, the eigenvectors themselves are assumed differentiable as follows:

$$\boldsymbol{\phi}_1 = \mathbf{x}_1 \tag{30}$$

$$\boldsymbol{\phi}_2 = \mathbf{x}_2 \tag{31}$$

Recall that the relationship between $\boldsymbol{\phi}_i$ and $\mathbf{x}_i$ is given by $\boldsymbol{\phi}_i = \mathbf{X}\mathbf{y}_i$. Therefore, the transformation matrix for the above assumption is given as

$$[\mathbf{y}_1|\mathbf{y}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

Furthermore, it is assumed that the derivative of one of the infinitely repeated eigenvectors is orthogonal to the other repeated eigenvectors. In other words, the constants $c_{ij} (i \neq j)$ in equation (7) are assumed to be zero. Therefore, the derivatives of the infinitely repeated eigenvectors can be expressed as

$$\frac{\partial \boldsymbol{\phi}_1}{\partial b} = \mathbf{v}_1 + c_{11}\mathbf{x}_1 \tag{33}$$

$$\frac{\partial \boldsymbol{\phi}_2}{\partial b} = \mathbf{v}_2 + c_{22}\mathbf{x}_2 \tag{34}$$

where $\mathbf{v}_i (i = 1, 2)$ can be solved by equation (20), and c_{11} and c_{22} can be determined by differentiating the following normalization conditions:

$$\boldsymbol{\phi}_1^T \mathbf{M} \boldsymbol{\phi}_1 = 1 \tag{35}$$

$$\boldsymbol{\phi}_2^T \mathbf{M} \boldsymbol{\phi}_2 = 1 \tag{36}$$

$$\boldsymbol{\phi}_1^T \mathbf{M} \boldsymbol{\phi}_2 = 0 \tag{37}$$

as given by equations (23) and (24). Nevertheless, the orthogonal condition indicated by equation (37) imposes an extra constraint on the eigenvector derivatives such that

$$\boldsymbol{\phi}_1^T \mathbf{M} \frac{\partial \boldsymbol{\phi}_2}{\partial b} + \boldsymbol{\phi}_2^T \mathbf{M} \frac{\partial \boldsymbol{\phi}_1}{\partial b} + \boldsymbol{\phi}_1^T \frac{\partial \mathbf{M}}{\partial b} \boldsymbol{\phi}_2 = 0 \tag{38}$$

Substituting equations (33) and (34) for $\partial \boldsymbol{\phi}_1/\partial b$ and $\partial \boldsymbol{\phi}_2/\partial b$, the preceding equation simply implies that

$$\boldsymbol{\phi}_1^T \frac{\partial \mathbf{M}}{\partial b} \boldsymbol{\phi}_2 = 0 \tag{39}$$

It is obvious that equation (39) is satisfied exactly if $\partial \mathbf{M}/\partial b$ is proportional to $\mathbf{M}$. For the Earth Pointing System (EPS) studied in this paper, numerical results show that the values of $\boldsymbol{\phi}_1^T (\partial \mathbf{M}/\partial b) \boldsymbol{\phi}_2$ are indeed negligibly small, and equations (33) and (34) are valid for the infinitely repeated eigenvectors considered.

In summary, it is proposed that the derivatives of infinitely repeated eigenvectors be expressed by equations (33) and (34), provided that the condition of equation (39) is not violated. Note that the above computational procedure is by no means rigorous, although it performs well for the problems studied in this work. Future research is required in this regard.

Stiffness and Mass Matrix Derivatives

In the eigenvalue/eigenvector sensitivity equations derived previously, all equations require derivatives of the stiffness and mass matrices. The two most popular methods are the analytical method and the semianalytical method. The analytical method, sometimes referred to as the direct differentiation method, uses explicit functional relationships between the design variables and the defining matrices.

Direct or analytical method. For the case in which the design variables are nonlinearly related to the stiffness and mass matrices, a chain rule for partial differentiation is required to determine matrix design derivatives. As an example, assume that $\mathbf{K}$ and $\mathbf{M}$ for beam elements have the following general functional form:

$$\left.\begin{aligned} \mathbf{K} &= \mathbf{K}(E, G, A, I_y, I_z, L) \\ \mathbf{M} &= \mathbf{M}(\rho, A, I_y, I_z, L) \end{aligned}\right\} \tag{40}$$

where G is the shear modulus. Further assume that the generic beam elements defined above are of a tubular cross section and that E, G, ρ, and L are not functions of design parameters (i.e., E, G, ρ, and L are constants). The terms $\mathbf{K}$ and $\mathbf{M}$ are then of the following form:

$$\left.\begin{aligned} \mathbf{K} &= \mathbf{K}(A, I) \\ \mathbf{M} &= \mathbf{M}(A, I) \end{aligned}\right\} \tag{41}$$

where

$$\left.\begin{aligned} A &= A(r_o, r_i) \\ I &= I(r_o, r_i) \end{aligned}\right\} \tag{42}$$

By the chain rule of partial differentiation, the derivative of $\mathbf{K}$ and $\mathbf{M}$, with respect to a general radius term b_j (where b_j can be either r_o or r_i, which is the outer and inner radius of the tube element, respectively), is

$$\left.\begin{aligned} \frac{\partial \mathbf{K}}{\partial b_j} &= \frac{\partial \mathbf{K}}{\partial A}\left(\frac{\partial A}{\partial b_j}\right) + \frac{\partial \mathbf{K}}{\partial I}\left(\frac{\partial I}{\partial b_j}\right) \\ \frac{\partial \mathbf{M}}{\partial b_j} &= \frac{\partial \mathbf{M}}{\partial A}\left(\frac{\partial A}{\partial b_j}\right) + \frac{\partial \mathbf{M}}{\partial I}\left(\frac{\partial I}{\partial b_j}\right) \end{aligned}\right\} \tag{43}$$

The second derivatives are

$$\left.\begin{aligned} \frac{\partial^2 \mathbf{K}}{\partial b_j^2} &= \left[\frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{K}}{\partial A}\right)\right]\frac{\partial A}{\partial b_j} + \frac{\partial \mathbf{K}}{\partial A}\left(\frac{\partial^2 A}{\partial b_j^2}\right) + \left[\frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{K}}{\partial I}\right)\right]\frac{\partial I}{\partial b_j} + \frac{\partial \mathbf{K}}{\partial I}\left(\frac{\partial^2 I}{\partial b_j^2}\right) \\ \frac{\partial^2 \mathbf{M}}{\partial b_j^2} &= \left[\frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{M}}{\partial A}\right)\right]\frac{\partial A}{\partial b_j} + \frac{\partial \mathbf{M}}{\partial A}\left(\frac{\partial^2 A}{\partial b_j^2}\right) + \left[\frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{M}}{\partial I}\right)\right]\frac{\partial I}{\partial b_j} + \frac{\partial \mathbf{M}}{\partial I}\left(\frac{\partial^2 I}{\partial b_j^2}\right) \end{aligned}\right\} \tag{44}$$

Because the stiffness and mass matrices are linearly related to (A, I), the following terms are zero:

$$\left.\begin{aligned} \frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{K}}{\partial A}\right) &= 0 \\ \frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{K}}{\partial I}\right) &= 0 \\ \frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{M}}{\partial A}\right) &= 0 \\ \frac{\partial}{\partial b_j}\left(\frac{\partial \mathbf{M}}{\partial I}\right) &= 0 \end{aligned}\right\} \tag{45}$$

The second derivatives then can be rewritten as

$$\left.\begin{aligned} \frac{\partial^2 \mathbf{K}}{\partial b_j^2} &= \frac{\partial \mathbf{K}}{\partial A}\left(\frac{\partial^2 A}{\partial b_j^2}\right) + \frac{\partial \mathbf{K}}{\partial I}\left(\frac{\partial^2 I}{\partial b_j^2}\right) \\ \frac{\partial^2 \mathbf{M}}{\partial b_j^2} &= \frac{\partial \mathbf{M}}{\partial A}\left(\frac{\partial^2 A}{\partial b_j^2}\right) + \frac{\partial \mathbf{M}}{\partial I}\left(\frac{\partial^2 I}{\partial b_j^2}\right) \end{aligned}\right\} \tag{46}$$

The area and bending inertia relationships for a tube beam element are

$$\left.\begin{aligned} A &= \pi\left(r_o^2 - r_i^2\right) \\ I &= \frac{\pi\left(r_o^4 - r_i^4\right)}{4} \end{aligned}\right\} \tag{47}$$

where r_o and r_i are, respectively, the outer and inner radii of an element. The first and second derivatives of the area and bending inertia terms, with respect to the inner and outer radii, are then

$$
\left.\begin{aligned}
\frac{\partial A}{\partial r_o} &= 2\pi r_o & \frac{\partial A}{\partial r_i} &= -2\pi r_i \\
\frac{\partial^2 A}{\partial r_o^2} &= 2\pi & \frac{\partial^2 A}{\partial r_i^2} &= -2\pi \\
\frac{\partial I}{\partial r_o} &= \pi r_o^3 & \frac{\partial I}{\partial r_i} &= -\pi r_i^3 \\
\frac{\partial^2 I}{\partial r_o^2} &= 3\pi r_o^2 & \frac{\partial^2 I}{\partial r_i^2} &= -3\pi r_i^2
\end{aligned}\right\} \tag{48}
$$

These area and bending inertia derivative terms can be substituted into equations (43) and (46) to complete the expressions for the derivatives of the stiffness and the mass with respect to an inner and outer radius design parameter.

A reduction of the number of independent design variables can be obtained by developing relationships between design variables. This practice of design variable reduction is commonly referred to as design variable linking. A simple example of design variable linking would be to consider the inner radius of a tube a function of the outer radius (i.e., $r_i = kr_o$, where k is a constant that defines the ratio of the inner radius to the outer radius, and it must satisfy $0 \leq k < 1$). The area and the bending inertia then take the following functional forms:

$$
\left.\begin{aligned}
A &= A[r_o, r_i(r_o)] \\
I &= I[r_o, r_i(r_o)]
\end{aligned}\right\} \tag{49}
$$

With the above functional forms, expressions for the full derivatives of the mass and stiffness matrices with respect to the outer radius of a tube beam element can be obtained. Applying the chain rule to equation (41), one may obtain as first derivatives

$$
\left.\begin{aligned}
\frac{d\mathbf{K}}{dr_o} &= \frac{\partial \mathbf{K}}{\partial r_o} + \frac{\partial \mathbf{K}}{\partial r_i}\left(\frac{dr_i}{dr_o}\right) \\
\frac{d\mathbf{M}}{dr_o} &= \frac{\partial \mathbf{M}}{\partial r_o} + \frac{\partial \mathbf{M}}{\partial r_i}\left(\frac{dr_i}{dr_o}\right)
\end{aligned}\right\} \tag{50}
$$

and as second derivatives

$$
\left.\begin{aligned}
\frac{d^2\mathbf{K}}{dr_o^2} &= \frac{\partial^2 \mathbf{K}}{\partial r_o^2} + \frac{\partial^2 \mathbf{K}}{\partial r_i\,\partial r_o}\left(\frac{dr_i}{dr_o}\right) + \left[\frac{\partial^2 \mathbf{K}}{\partial r_i\,\partial r_o} + \frac{\partial^2 \mathbf{K}}{\partial r_i^2}\left(\frac{dr_i}{dr_o}\right)\right]\frac{dr_i}{dr_o} + \frac{\partial \mathbf{K}}{\partial r_i}\left(\frac{d^2 r_i}{dr_o^2}\right) \\
\frac{d^2\mathbf{M}}{dr_o^2} &= \frac{\partial^2 \mathbf{M}}{\partial r_o^2} + \frac{\partial^2 \mathbf{M}}{\partial r_i\,\partial r_o}\left(\frac{dr_i}{dr_o}\right) + \left[\frac{\partial^2 \mathbf{M}}{\partial r_i\,\partial r_o} + \frac{\partial^2 \mathbf{M}}{\partial r_i^2}\left(\frac{dr_i}{dr_o}\right)\right]\frac{dr_i}{dr_o} + \frac{\partial \mathbf{M}}{\partial r_i}\left(\frac{d^2 r_i}{dr_o^2}\right)
\end{aligned}\right\} \tag{51}
$$

Because the partial derivatives of the stiffness and the mass (with respect to the inner and outer radii) are functions of only the inner and outer radii, respectively, this implies that the cross derivative terms are zero. The above expressions for the second derivatives then reduce to

$$
\left.\begin{aligned}
\frac{d^2\mathbf{K}}{dr_o^2} &= \frac{\partial^2 \mathbf{K}}{\partial r_o^2} + \frac{\partial^2 \mathbf{K}}{\partial r_i^2}\left(\frac{dr_i}{dr_o}\right)^2 \\
\frac{d^2\mathbf{M}}{dr_o^2} &= \frac{\partial^2 \mathbf{M}}{\partial r_o^2} + \frac{\partial^2 \mathbf{M}}{\partial r_i^2}\left(\frac{dr_i}{dr_o}\right)^2
\end{aligned}\right\} \tag{52}
$$

Although the analytical method is an exact method, it can sometimes be too complicated to implement if the design variables have complex nonlinear relationships to the stiffness and the mass. However, if design variables are linearly related to the stiffness and the mass, the analytical derivatives can then be obtained simply by setting the design variable to unity or zero; this setting depends on whether first or higher order derivatives are required.

For example, if

$$\mathbf{K}_e = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \tag{53}$$

then

$$\frac{\partial \mathbf{K}_e}{\partial A} = \begin{bmatrix} \frac{E}{L} & -\frac{E}{L} \\ -\frac{E}{L} & \frac{E}{L} \end{bmatrix} = \mathbf{K}_e\Big|_{A=1} \tag{54}$$

and

$$\frac{\partial^n \mathbf{K}_e}{\partial A^n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{K}_e\Big|_{A=0} \qquad (n > 1) \tag{55}$$

Semianalytical method. The semianalytical method (ref. 14), which is commonly referred to as the finite-difference method for determining stiffness and mass matrix derivatives, is a technique that greatly simplifies the implementation of eigenvalue/eigenvector sensitivity analysis. The essence of this method is a replacement of the exact stiffness and mass matrix derivatives (eqs. (50) and (52)) by linearly approximated derivatives. Therefore, nonlinear design variables can be treated with almost the same ease as linear design variables. An example of stiffness and mass matrix derivatives formed by a forward finite difference is as follows:

$$\left.\begin{aligned} \frac{\partial \mathbf{K}}{\partial b_j} &\approx \frac{\Delta \mathbf{K}}{\Delta b_j} \equiv \frac{\mathbf{K}^* - \mathbf{K}^0}{b_j^* - b_j^0} \\ \frac{\partial \mathbf{M}}{\partial b_j} &\approx \frac{\Delta \mathbf{M}}{\Delta b_j} \equiv \frac{\mathbf{M}^* - \mathbf{M}^0}{b_j^* - b_j^0} \end{aligned}\right\} \tag{56}$$

where the superscript 0 indicates the quantity defined at b^0 (the current design), and the asterisk denotes the quantity defined at the new design. Specifically, $\mathbf{K}^*$ and $\mathbf{M}^*$ are the stiffness and mass matrices that are formed by a positive perturbation of the independent design variable b_j, and $\mathbf{K}^0$ and $\mathbf{M}^0$ are the nominal stiffness and mass matrices.

The major disadvantage of the semianalytical method is that it is an approximate technique, and hence the eigenvalue/eigenvector derivatives are no longer exact with respect to the finite-element model. Therefore, when considering implementation of the semianalytical method, one must weigh the advantage of simplicity versus the disadvantage of decreased accuracy. In the applications to follow, the analytical method was chosen because of the high numerical accuracy desired.

Approximate Eigenvalue/Eigenvector Analysis Techniques

The majority of the computational cost associated with any analysis or optimization procedure requiring iterative eigensolutions is related to the eigensolutions. This situation is true for even relatively small problems (<100 degrees of freedom). The undeniable fact is that eigensolutions are extremely time consuming and computationally burdensome. To alleviate some of this burden, development of methods to efficiently approximate eigensolutions becomes desirable. One such method is the Taylor's series expansion in which the new eigenvalues/eigenvectors are linearly approximated as

$$\lambda_i^*(b^*) \approx \lambda_i^0\left(b^0\right) + \sum_{j=1}^{ndev} \frac{\partial \lambda_i^0}{\partial b_j} \Delta b_j \tag{57}$$

$$\mathbf{x}_i^*(b^*) \approx \mathbf{x}_i^0\left(b^0\right) + \sum_{j=1}^{ndev} \frac{\partial \mathbf{x}_i^0}{\partial b_j} \Delta b_j \tag{58}$$

where b^* and b^0 are the new and current design variables, respectively, and Δb_j is the change in design variable b_j. The maximum allowable design variable perturbation is governed by the linearity of the function with respect to that variable. In general, information regarding the linearity of eigenvalues/eigenvectors is not available prior to the selection of design variable perturbation sizes. Consequently, it is virtually impossible to determine general guidelines for their selection. A conservative approach, therefore, must be taken to ensure the accuracy and the stability of the approximated eigensolutions. Based upon the numerical experience of this study, the term conservative refers to design variable perturbations of ≤ 5 percent of nominal values (i.e., $\Delta b \leq 0.05 b^0$).

Approximate analysis of eigenvalues/eigenvectors in the presence of repeated eigenvalues is complicated by the apparent discontinuous nature of the eigensolution. As mentioned earlier, repeated eigenvalues are only directionally differentiable, and therefore the use of equations (57) and (58) must be severely restricted. Restrictions on equations (57) and (58) are required partly because the reduced eigenvalue problem (eq. (3)) fails to indicate the correspondence between the eigenvalue derivatives and the distinct eigenvalues after the design has changed. Specifically, to implement equation (57), one must have a priori knowledge of the transformation from repeated eigenvalues to distinct eigenvalues after the design has changed. Once this transformation is known, the eigenvalue derivatives from equation (3) can be properly ordered to yield $\lambda_1^* < \lambda_2^* < \ldots < \lambda_n^*$, as required after the design has changed. A multidesign variable implementation of equation (58) may not be possible because equation (58) requires the eigenvector derivatives in the summation $\sum_{j=1}^{ndev} (\partial \mathbf{x}_i^0 / \partial b_j) \Delta b_j$ to be associated with the same eigenvector. As seen earlier, differentiable eigenvectors are design variable dependent, and therefore equation (58) may not be a valid approximation. A different set of equations, as suggested by references 6 and 14, should be derived to account for the directional dependencies of the eigenvalues and eigenvectors.

A method to reparameterize the multivalued eigenproblem into one that is in terms of a single positive-valued design parameter will be introduced to avoid these directional dependencies. This reparameterization will effectively eliminate the directional dependencies of the original multivalued eigenproblem, thus allowing the eigenvalues/eigenvectors to be approximated using conventional series expansion techniques (i.e., a specific form of eqs. (57) and (58)).

Let a repeated eigenvalue problem formulated at the current design b^0 be represented as

$$\mathbf{K}\left(b^0\right)\mathbf{x}^0 = \lambda^0 \mathbf{M}\left(b^0\right)\mathbf{x}^0 \tag{59}$$

Next, introducing a design change Δb, the new design variable b^* is then given as

$$b^* = b^0 + \Delta b \tag{60}$$

which yields an eigenvalue problem as

$$\mathbf{K}(b^*)\mathbf{x}^* = \lambda^* \mathbf{M}(b^*)\mathbf{x}^* \tag{61}$$

Introducing an intermediate design variable $b(\epsilon) = b^0 + \epsilon \Delta b$, the above two eigenvalue equations can be collectively represented by a single equation such that

$$\mathbf{K}(b(\epsilon))\mathbf{x}(\epsilon) = \lambda(\epsilon)\mathbf{M}(b(\epsilon))\mathbf{x}(\epsilon) \tag{62}$$

where ϵ, a real parameter, ranges from 0 to 1. Note that the current and new eigenvalue equations can be realized by substituting the value ϵ by 0 and 1, respectively. Letting ϵ, which is always positive, be assigned as the only design variable, the sensitivity equation derived in the previous section can be applied here to find the eigenvalue/eigenvector derivatives of equation (62) with respect to ϵ at $\epsilon = 0$. Note that equation (62) entertains a pair of repeated eigenvalues at $\epsilon = 0$.

The eigenvalue derivatives of equation (62) at $\epsilon = 0$ are the eigenvalues of the following reduced eigenvalue problem:

$$\left(\widetilde{\mathbf{K}} - \gamma_i \widetilde{\mathbf{M}}\right)\mathbf{y}_i = 0 \tag{63}$$

where $\widetilde{\mathbf{K}} = \mathbf{X}^T(\mathbf{K}' - \lambda \mathbf{M}')\mathbf{X}$, $\widetilde{\mathbf{M}} = \mathbf{I}$ (where $\mathbf{I}$ is the identity matrix), and γ_i equals $\partial \lambda_i / \partial b$.

The detailed representations of $\mathbf{K}'$ and $\mathbf{M}'$ are given as

$$\left.\begin{aligned}
\mathbf{K}' &= \left.\frac{d\mathbf{K}\left(b^0 + \epsilon \Delta b\right)}{d\epsilon}\right|_{\epsilon=0} \\
\mathbf{K}' &= \frac{\partial \mathbf{K}}{\partial b}\Delta b \\
\mathbf{K}' &= \sum_{i=1}^{ndev} \frac{\partial \mathbf{K}}{\partial b_i}\Delta b_i
\end{aligned}\right\} \tag{64}$$

and similarly,

$$\mathbf{M}' = \sum_{i=1}^{ndev} \frac{\partial \mathbf{M}}{\partial b_i}\Delta b_i \tag{65}$$

The eigenvector derivative $\boldsymbol{\phi}_i'$, corresponding to the eigenvalue derivative γ_i, is now defined as

$$\boldsymbol{\phi}_i' = \mathbf{V}\mathbf{y}_i + c_{i1}\boldsymbol{\phi}_1 + c_{i2}\boldsymbol{\phi}_2 \qquad (i = 1, 2) \quad (66)$$

where here $\mathbf{y}_i$ is the eigenvector of equation (63), and $\mathbf{v}_i'$ are the solutions of the following equation:

$$\begin{bmatrix} \mathbf{K} - \lambda\mathbf{M} & \mathbf{M}\mathbf{x}_1 & \mathbf{M}\mathbf{x}_2 \\ \mathbf{x}_1^T\mathbf{M} & 0 & 0 \\ \mathbf{x}_2^T\mathbf{M} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v}_i \\ \mu_{i1} \\ \mu_{i2} \end{Bmatrix} = \begin{Bmatrix} -(\mathbf{K}' - \lambda\mathbf{M}')\mathbf{x}_i \\ 0 \\ 0 \end{Bmatrix} \qquad (i = 1, 2)\ (67)$$

Note that the above equation is similar to equation (20), in which $\partial\mathbf{K}/\partial b$ and $\partial\mathbf{M}/\partial b$ are substituted by $\mathbf{K}'$ and $\mathbf{M}'$ (defined by eqs. (64) and (65)). Furthermore, the constants c_{i1} and c_{i2} (where $i = 1, 2$) in equation (66) can be obtained by using equations (23) and (24) and any two of the three equations (25) or (26) and (28). In these equations, $\partial\mathbf{K}/\partial b$ and $\partial\mathbf{M}/\partial b$ should be replaced by $\mathbf{K}'$ and $\mathbf{M}'$, and $\partial^2\mathbf{K}/\partial b^2$ and $\partial^2\mathbf{M}/\partial b^2$ should be replaced by $\mathbf{K}''$ and $\mathbf{M}''$, which are defined as

$$\begin{aligned} \mathbf{K}'' &= \left.\frac{d^2\mathbf{K}\left(b^0 + \epsilon\,\Delta b\right)}{d^2\epsilon}\right|_{\epsilon=0} \\ &= \sum_{i=1}^{ndev}\sum_{j=1}^{ndev} \frac{\partial^2\mathbf{K}}{\partial b_i\,\partial b_j}\Delta b_i\,\Delta b_j \qquad (68) \end{aligned}$$

and

$$\mathbf{M}'' = \sum_{i=1}^{ndev}\sum_{j=1}^{ndev} \frac{\partial^2\mathbf{K}}{\partial b_i\,\partial b_j}\Delta b_i\,\Delta b_j \qquad (69)$$

The first-order approximation of the new eigenvalue, at $\epsilon = 1$, can then be done by using the Taylor's series expansion with respect to ϵ about $\epsilon = 0$ as

$$\lambda_i^* \approx \lambda_i^0 + \gamma_i \qquad (i = 1, 2) \qquad (70)$$

where γ_1 is less than γ_2. Therefore, as a result, λ_1^* is always less than λ_2^*. Similarly, the first-order approximation of eigenvectors can be obtained by

$$\boldsymbol{\phi}_i^* \approx \boldsymbol{\phi}_i^0 + \boldsymbol{\phi}_i' \qquad (i = 1, 2) \qquad (71)$$

where $\boldsymbol{\phi}_i^0$ is the differentiable eigenvector at $\epsilon = 0$ determined by $\mathbf{y}_i$, which is the eigenvector corresponding to γ_i in equation (63).

Numerical Verification of Approximate Analysis and Applications to Structural Optimization

Several examples will be presented to verify the approximate analysis derived for repeated eigenvalue problems. In addition, how sensitivity information can be applied in a design optimization loop to approximate eigensolutions will be demonstrated. These examples will address issues related to successful implementation of design sensitivity analysis in a design optimization environment.

Stepped Cantilever Beam

The goals of this section are to present a simple finite-element model with well-understood characteristics and to use this model to validate the formulation for eigenvalue/eigenvector approximate analysis in the presence of repeated eigenvalues. Note that a priori knowledge regarding the characteristics of repeated eigenvalues is rarely available. The lack of knowledge of this behavior is particularly true when considering complex models with multiple design variables. With more than two design variables, the transformation from repeated to distinct eigenvalues becomes difficult, if not impossible, to visualize. Fortunately, a priori knowledge is not required for the success of the methods developed in this paper. Essentially, all that is required is knowledge of the particular design variable changes.

The first part of this validation example will be used to introduce the characteristics of the particular model being considered. This introduction will be done by carefully monitoring the design variable perturbations and examining the resulting change in the eigensolutions. To facilitate this, only one design variable at a time is allowed to be varied.

The model used in this study is a stepped cantilever beam, which is shown in figure 5. The finite-element model consists of 21 nodes, each with three degrees of freedom (two translational and one rotational). The model has been divided into 20 tubular beam elements. The ratios between the inner and outer radii for both sections are fixed in this study. The elements have been subdivided into two groups of design variables. The outer radius of the first 10 elements is design variable 1 (b_1), and the outer radius of the remaining elements is design variable 2 (b_2).

The initial design parameters for the beam are given in table I. At the initial (nominal) design, $b_1 = 0.22699$, $b_2 = 0.36259$, and the frequency of bending mode 3 is equal to longitudinal mode 1. Bending mode 3 and longitudinal mode 1 correspond

Table I. Stepped Cantilever Beam Initial Design Parameters

Beam parameters	Design variable 1	Design variable 2
Length, in.	5	5
Area, in^2	.07082	.18071
Unit weight, lbf/in.	.92999	.92999
I_y, in^4	1.4255×10^{-3}	9.2807×10^{-3}
Young's modulus, lbf/in^2 . . .	10.300×10^3	10.300×10^3

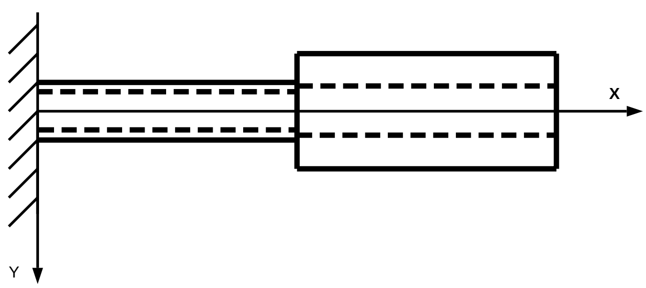

Figure 5. Stepped cantilever beam.

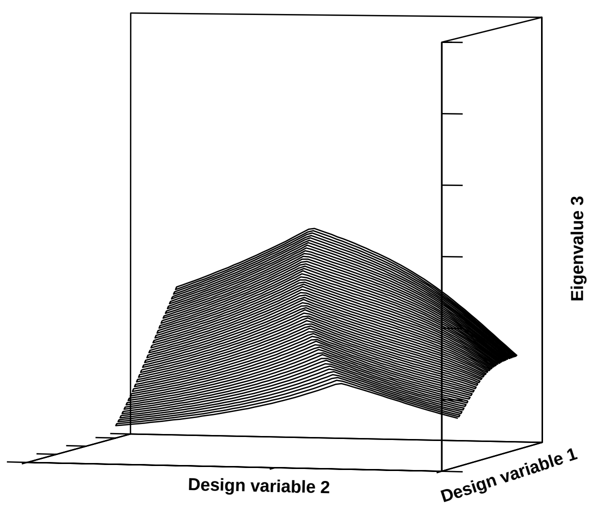

Figure 6. Variations of eigenvalue 3 versus design variables 1 and 2.

to eigenvalues 3 and 4, respectively. In this study, eigenvalues are considered repeated if their absolute difference is < 0.01.

Figures 6 and 7 individually display how eigenvalues 3 and 4 vary with respect to progressive changes in design variables 1 and 2. The combined eigenvalue variations are shown in figure 8. These figures clearly show that the slopes (i.e., eigenvalue derivatives) are discontinuous across the "ridges" where the eigenvalues are repeated.

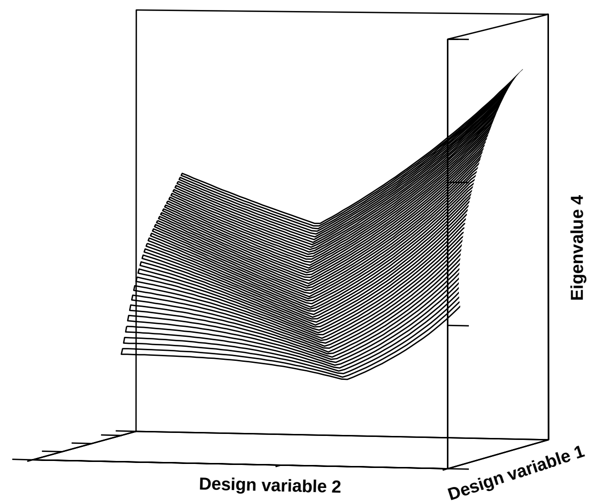

Figure 7. Variations of eigenvalue 4 versus design variables 1 and 2.

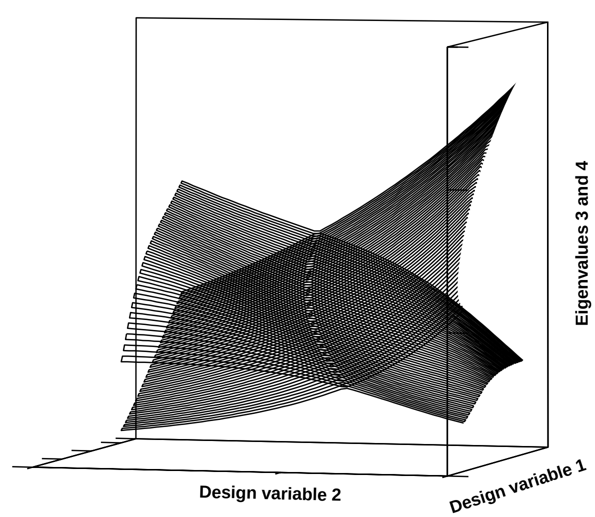

Figure 8. Eigenvalue variations versus design variables 1 and 2.

Tables II and III present eigenvectors 3 and 4 at their nominal design (column 3) and with

Table II. Eigenvector 3 With Positive Perturbations in Design Variables 1 and 2

Node	dof	$\mathbf{x}$	$\mathbf{x}^*_{b_1}$	$\mathbf{x}^*_{b_2}$
1	1	0	0	0
1	2	0	0	0
5	1	0	0	7.590
5	2	37.190	37.125	0
9	1	0	0	14.802
9	2	20.007	20.000	0
13	1	0	0	19.307
13	2	−13.803	−13.807	0
17	1	0	0	20.879
17	2	−7.861	−7.873	0
21	1	0	0	21.412
21	2	25.418	25.439	0

Table III. Eigenvector 4 With Positive Perturbations in Design Variables 1 and 2

Node	dof	$\mathbf{x}$	$\mathbf{x}^*_{b_1}$	$\mathbf{x}^*_{b_2}$
1	1	0	0	0
1	2	0	0	0
5	1	7.596	7.595	0
5	2	0	0	37.217
9	1	14.814	14.810	0
9	2	0	0	20.000
13	1	19.323	19.319	0
13	2	0	0	−13.785
17	1	20.899	20.898	0
17	2	0	0	−7.842
21	1	21.433	21.433	0
21	2	0	0	25.371

independent positive perturbations in design variables 1 and 2 (columns 4 and 5, respectively). In these tables, the column headings labeled $\mathbf{x}^*_{b_i}$ are components of eigenvectors 3 and 4, which are evaluated when design variable i is perturbed by a positive Δb_i. Also, the column heading labeled "dof" refers to either an X degree of freedom (axial deformation) denoted by 1 or a Y degree of freedom (lateral deformation) denoted by 2 in the corresponding eigenvector. These tables implicitly show the transformation that must be performed to define a set of differentiable eigenvectors.

To better understand this transformation, one can examine the eigenvectors after independent positive perturbations in design variables 1 and 2. Tables II and III show that with a positive Δb_1 (with $\Delta b_2 = 0$ (column 4)), the corresponding differentiable eigenvectors are $\boldsymbol{\phi}_3 = \mathbf{X}\mathbf{y}_1 = \mathbf{x}_3$ and $\boldsymbol{\phi}_4 = \mathbf{X}\mathbf{y}_2 = \mathbf{x}_4$. Thus, for this type of design change, switching modes is not observed. As indicated in column 4 of tables II and III, the newly perturbed eigenvectors (3 and 4) remain as bending and longitudinal eigenvectors that have nonzero lateral and axial displacement components, respectively. However, when a positive Δb_2 (with $\Delta b_1 = 0$ (column 5)) is introduced, the corresponding differentiable eigenvectors are $\boldsymbol{\phi}_3 = \mathbf{X}\mathbf{y}_1 = \mathbf{x}_4$ and $\boldsymbol{\phi}_4 = \mathbf{X}\mathbf{y}_2 = \mathbf{x}_3$. This transformation states that the longitudinal mode becomes the bending mode, and the bending mode becomes the longitudinal mode (i.e., an exact switch has taken place). As indicated in column 5 of table II, the newly perturbed eigenvector 3 now becomes a longitudinal mode that shows no lateral displacement. The same situation can be observed in column 5 of table III, in which the newly perturbed eigenvector 4 is switched to a bending mode (i.e., zero axial displacement).

With knowledge of the transformation from repeated eigenvalues to distinct eigenvalues, the solution of the reduced eigenvalue problem (eq. (63)) can be properly ordered (i.e., the correspondence between eigenvalue derivatives at the repeated eigenvalue design point and the distinct eigenvalues after the perturbation is known). For a positive perturbation in design variable 1, the correct ordering of the solution of equation (63) is given as follows:

$$\frac{\partial \lambda_3}{\partial b_1} = 195191.2 \qquad (\mathbf{y}_1 = (1,0)^T)$$

and

$$\frac{\partial \lambda_4}{\partial b_1} = 378836.8 \qquad (\mathbf{y}_2 = (0,1)^T)$$

Similarly, for a positive Δb_2,

$$\frac{\partial \lambda_3}{\partial b_2} = -237167.5 \qquad (\mathbf{y}_1 = (0,1)^T)$$

and

$$\frac{\partial \lambda_4}{\partial b_2} = 149976.9 \qquad (\mathbf{y}_2 = (1,0)^T)$$

Note that these unitary diagonal transformation matrices, $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2]$, are special cases of the more general situation that occurs when the eigenvectors are expressed as linear combinations of the pure bending and longitudinal modes. In the above discussion, design variable perturbations have been limited to single-variable positive perturbations. A similar analysis can be performed using negative perturbations. The results will be exactly opposite (i.e., a negative perturbation in design variable 1 will cause the two modes to switch, whereas a negative perturbation in design variable 2 will not cause the modes to switch).

The characteristics of this problem can be described by examining the ratio of the design variables b_2 to b_1. When the eigenvalues are equal, the ratio b_2/b_1 has a particular value C_r, and when changes in either b_2 or b_1 increase C_r, the original modes will switch. Alternatively, when changes in b_2 or b_1 decrease C_r, the modes will not switch.

It should be stressed that the numerical validation results that follow do not rely on the knowledge gained from this perturbation exercise. The above perturbations were given only to provide the reader with a better understanding of the characteristics of this problem.

Numerical validation of the method presented for eigenvalue/eigenvector approximate analysis is accomplished by investigating two types of design variable perturbations. The first perturbation contains both design variables in such a way as to increase C_r, thereby causing the original bending and longitudinal modes to switch. The second perturbation also permits both design variables to vary simultaneously; however, the perturbation is such that the ratio C_r decreases. Regardless of the design change, the methods developed in this paper can be used to accurately predict first-order changes in both the eigenvalues and the eigenvectors associated with repeated eigenvalues.

Results are presented in the following format. The design variable perturbation is presented first. The tables that contain the differentiable eigenvectors and their first-order approximations and exact values at the perturbed design are presented second. Next, a first-order approximation of the eigenvalues and their exact values at the newly perturbed design are presented.

Perturbation results for eigenvectors 3 and 4 are presented in tables IV to VII. In these tables, column 2 is the differentiable eigenvector, column 3 is the first-order approximation of the change in the eigenvectors as given in equation (66), and columns 4 and 5 give the exact eigenvector obtained by solving equation (2) and the approximate eigenvector as given by equation (71), respectively.

Table IV. First-Order Approximation for Eigenvector 3 (Perturbation 1)

X dof at node	ϕ^0	ϕ'	ϕ^*_{exact}	$\phi^*_{\text{approximated}}$
2	1.9140	-0.09726×10^{-2}	1.9128	1.9130
6	9.4505	$-.4420 \times 10^{-2}$	9.4452	9.4461
10	16.5160	$-.6138 \times 10^{-2}$	16.5080	16.5099
14	19.8110	-1.2712×10^{-2}	19.7970	19.7983
18	21.1320	-1.8087×10^{-2}	21.1120	21.1139

Table V. First-Order Approximation for Eigenvector 4 (Perturbation 1)

Y dof at node	ϕ^0	ϕ'	ϕ^*_{exact}	$\phi^*_{\text{approximated}}$
2	4.8653	1.6680×10^{-2}	4.8789	4.8820
6	41.7220	.1165	41.8120	41.8385
10	7.0410	-2.7260×10^{-2}	7.0083	7.0137
14	−16.0760	2.8072×10^{-2}	−16.0380	−16.0479
18	−.8723	1.1553×10^{-2}	−.8602	−.8607

Table VI. First-Order Approximation for Eigenvector 3 (Perturbation 2)

Y dof at node	ϕ^0	ϕ'	ϕ^*_{exact}	$\phi^*_{\text{approximated}}$
2	4.8653	-1.6684×10^{-2}	4.8517	4.8486
6	41.7220	−.1165	41.6320	41.6055
10	7.0410	2.7260×10^{-2}	7.0726	7.0683
14	−16.0760	-2.8072×10^{-2}	−16.1150	−16.1041
18	−.8723	-1.1553×10^{-2}	−.8844	−.8838

Table VII. First-Order Approximation for Eigenvector 4 (Perturbation 2)

X dof at node	ϕ^0	ϕ'	ϕ^*_{exact}	$\phi^*_{\text{approximated}}$
2	1.9140	0.9725×10^{-2}	1.9151	1.9150
6	9.4505	$.4420 \times 10^{-2}$	9.4559	9.4549
10	16.5160	$.6137 \times 10^{-2}$	16.5240	16.5221
14	19.8110	1.2712×10^{-2}	19.8260	19.8237
18	21.1320	1.8087×10^{-2}	21.1520	21.1501

For perturbation 1 ($b_2/b_1 > C_r$), the arbitrarily selected perturbations of -0.1 percent in design variable 1 and 0.1 percent in design variable 2 were chosen to satisfy the above constraint $b_2/b_1 > C_r$ as

$$\Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} = \begin{pmatrix} -2.2699 \times 10^{-4} \\ 3.625959 \times 10^{-4} \end{pmatrix}$$

The results of perturbation 1 and eigenvalues 3 and 4, respectively, are as follows:

$$\lambda_3^* = \lambda^0 + \lambda_3' = 53488.06 + (-171.99)$$

$$\lambda_3^* = 53316.07 \text{ (approximated)}$$

$$\lambda_3^* = 53316.28 \text{ (exact)}$$

$$\lambda_4^* = \lambda^0 + \lambda_4' = 53488.06 + 10.07$$

$$\lambda_4^* = 53498.13 \text{ (approximated)}$$

$$\lambda_4^* = 53497.89 \text{ (exact)}$$

For perturbation 2 ($b_2/b_1 < C_r$), arbitrarily selected perturbations of 0.1 percent in design variable 1 and -0.1 percent in design variable 2 were chosen to satisfy the above constraint $b_2/b_1 < C_r$ such that

$$\Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} = \begin{pmatrix} 2.2699 \times 10^{-4} \\ -3.625959 \times 10^{-4} \end{pmatrix}$$

The results for eigenvalues 3 and 4, respectively, are as follows:

$$\lambda_3^* = \lambda^0 + \lambda_3' = 53488.06 + (-10.07)$$

$$\lambda_3^* = 53477.99 \text{ (approximated)}$$

$$\lambda_3^* = 53477.75 \text{ (exact)}$$

$$\lambda_4^* = \lambda^0 + \lambda_4' = 53488.06 + 171.99$$

$$\lambda_4^* = 53660.05 \text{ (approximated)}$$

$$\lambda_4^* = 53660.27 \text{ (exact)}$$

The results presented above clearly show that the methods developed in this paper can be used to accurately predict first-order changes in eigenvalues and eigenvectors for repeated eigenvalue problems with distinct eigenvalue derivatives.

Earth Pointing System Optimization

The Earth Pointing System (EPS), shown in figure 9, is composed of three major component groups,

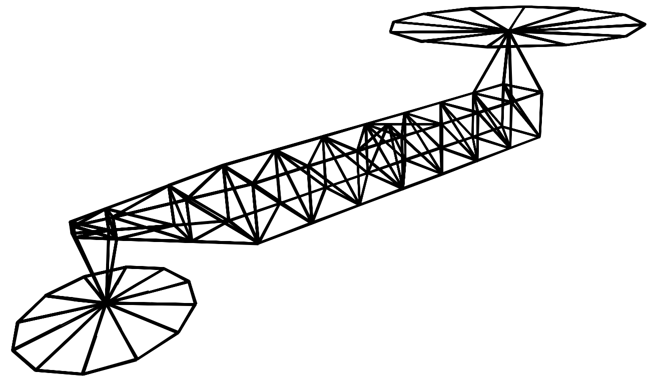

Figure 9. Earth Pointing System (EPS) structure.

a 25-m truss, two antennas (15 m and 7.5 m), and two antenna supports. Note that the EPS is a hypothetical structure that was used for early controls-structures-interaction (CSI) studies. The finite-element model consists of 95 nodes, 223 Euler-type beam elements, and 3 design variables. Each node has 6 degrees of freedom (dof) for a total of 570 dof. The beam elements have tubular cross sections and have been divided into three groups of design variables (one for each of the major components). The outer radius of each group of elements is considered a design variable. A design variable linking has been chosen such that the inner radius remains proportional to the outer radius, thereby eliminating the need to consider the inner radius as an independent design variable. Specifically, the design variable linking is of the form $r_i = kr_o$, where k is a constant that satisfies $0 \leq k < 1$. The design variables for this study are as follows:

- Design variable 1: outer radius of truss elements
- Design variable 2: outer radius of antenna elements
- Design variable 3: outer radius of antenna support elements

The characteristics of the structure at the initial design are such that it exhibits local vibration modes associated with the 15-m and 7.5-m antennas. This localized response is a result of the main truss section being very rigid at the initial design. Symmetric substructures that exhibit local response are almost always responsible for the occurrence of infinitely repeated eigenvalues. As an example, the first 20 frequencies of the EPS are tabulated in table VIII. The first six frequencies corresponding to rigid-body modes are clearly infinitely repeated eigenvalues. Moreover, the three pairs of flexible

Table VIII. Tabulation of Frequencies for EPS

Mode	Frequency, Hz	Description
1 to 6	0	Rigid body
7	.2422	Torsion 15-m antenna
8	.4056	Z-rotation 15-m antenna
9	.5652	X-rotation 15-m antenna
10	.6562	Torsion 7.5-m antenna
11	.8882	15-m antenna mode
12	.8882	15-m antenna mode
13	1.4377	Z-rotation 7.5-m antenna
14	1.5360	X-rotation 7.5-m antenna
15	1.7762	15-m antenna mode
16	1.7762	15-m antenna mode
17	3.0258	15-m antenna mode
18	3.0258	15-m antenna mode
19	3.5132	15-m antenna mode
20	3.5312	15-m antenna mode

modes (11 and 12, 15 and 16, and 17 and 18) are repeated eigenvalues caused by structural symmetry of the 15-m antenna. If design changes are made in such a way as to maintain the symmetry of the 15-m antenna, those repeated eigenvalues will remain repeated. Note that the 7.5-m antenna also exhibits infinitely repeated eigenvalues, but they are not shown in table VIII because they occur outside the 20-mode range that is presented.

This optimization study will demonstrate the computational efficiency of using eigensensitivity for eigenvalue approximate analysis in the presence of infinitely repeated eigenvalues. The objective of this problem is to "place" the first eigenvalue of the 15-m antenna as high in the frequency range as possible with constraints on the total mass of the structure and on the first flexible mode of the structure. The idea is to leave the flexible modes of the structure basically unchanged and "push" the local antenna modes higher in the frequency spectrum to avoid adverse interactions with the control system. To accomplish this, eigenvalue tracking must be enforced (i.e., the repeated eigenvalue associated with the first antenna mode must be continually identified as design variables are varied as a result of the optimization process). Tracking of the first antenna mode is accomplished by selecting the first pair of eigenvalues which exhibits infinitely repeated characteristics.

Mathematically, the optimization problem can be stated as follows:

maximize

$$\lambda_j \qquad \text{(first infinitely repeated mode)}$$

subject to

$$M_{\text{tot}} \leq 72.0 \text{ slugs} \qquad \text{(total structural mass)}$$

$$\lambda_7 \leq 3.0 \text{ (rad/sec)}^2 \qquad \text{(first flexible mode)}$$

$$\lambda_7 \geq 1.0 \text{ (rad/sec)}^2$$

The initial design has the following conditions:

$$\lambda_{11} = 31.144 \text{ (rad/sec)}^2 \qquad \text{(first infinitely repeated mode)}$$

$$M_{\text{tot}} = 60.2 \text{ slugs}$$

$$\lambda_7 = 2.32 \text{ (rad/sec)}^2$$

$$r_1 = r_2 = r_3 = 1.0021 \text{ in.}$$

For a description of the software used to conduct this optimization study and the specific issues related to implementing approximate eigensolutions within an optimization loop, see appendix A. The optimization is performed using the Automated Design Synthesis (ADS) software package. The optimization technique selected was a linear, extended interior penalty function method with a Broyden, Fletcher, Goldfarb, and Shanno (BFGS) variable metric method for the unconstrained subproblem, followed by a polynomial interpolation for the one-dimensional search.

The optimal design has the following conditions:

$$\lambda_{12} = 61.054 \text{ (rad/sec)}^2 \qquad \text{(first infinitdy repeatedmode)}$$

$$M_{\text{tot}} = 71.9 \text{ slugs}$$

$$\lambda_7 = 2.002 \text{ (rad/sec)}^2$$

$$r_1 = 0.3930 \text{ in. (lower bound)}; r_2 = 1.4030 \text{ in.}; r_3 = 1.0033 \text{ in.}$$

The results of this study are presented in table IX. Comparisons are based upon using varying levels of allowable design variable perturbations for the Taylor series eigensolution approximations. An allowable design variable perturbation is one in which the new design variables remain within a prescribed domain of assumed linearity. Figure 10 presents the optimization process and graphically shows how approximate eigensolutions are used. In this figure, the shaded circles represent the domains of assumed linearity or "linear" neighborhoods. Any design point that falls within the domain of assumed linearity may be approximated; points outside this region must be solved for exactly. Column 2 of table IX shows that all solution methods converged to the same optimal design. Table IX presents a performance comparison of four different techniques. Of these four, the 0-percent-allowable design variable perturbation

Table IX. Performance Comparison of Various Levels of Design Variable Perturbations

Perturbation technique, percent approximation	Objective at optimal design, $(\text{rad/sec})^2$	Function evaluations requested	Function approximations	CPU time,* sec	Reduction in CPU time, percent
0	61.054	444	0	24 643.6	
.5	61.054	447	120	19 395.2	21
2.0	61.054	448	170	16 729.6	32
5.0	61.054	456	296	11 545.7	53

*Sun SPARC 2.

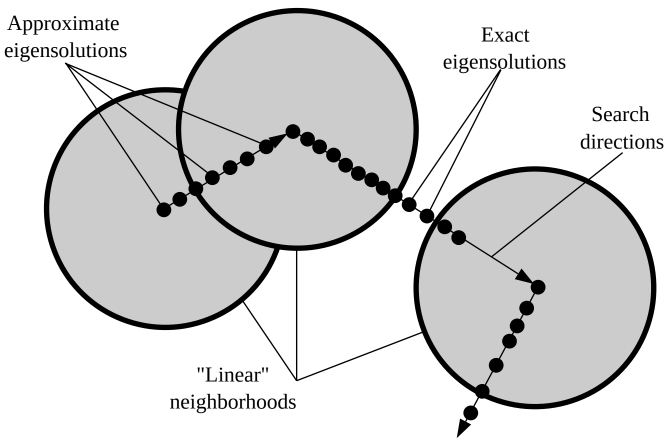

Figure 10. Optimization with approximate eigensolutions.

technique was by far the most CPU (central processor unit) intensive; the 0.5-percent-allowable design variable perturbation was second. A 0-percent-allowable design variable perturbation technique is one in which all eigensolutions are calculated exactly (i.e., no approximate analysis is allowed because the area of assumed linearity is zero). The reduction of CPU time followed a logical progression; that is, as the allowable design variable perturbation increased, the CPU time decreased. Note that the decreased CPU time is a direct result of the increased number of eigensolution approximations (column 4) compared with the total number of requested function evaluations (column 3). As the allowable perturbation increases, the number of "new" designs that fall within the small neighborhood around b^0 increases, and therefore more functions can be approximated. With every eigensolution approximation, a costly exact eigensolution is avoided, which in turn makes the entire optimization process more efficient. In this study, a 5-percent-allowable design variable perturbation in the eigensolution approximations produced the best results. This technique yielded more than a 50-percent reduction in the CPU time compared with the baseline 0-percent technique.

Finally, note that the baseline 0-percent technique would have been much more costly if it were not for the particular eigensolver employed. A subspace iteration technique that uses the previous eigenvectors as the trial vectors for the current iteration has been employed. This method of using the previous eigenvectors greatly decreases the convergence time of the new eigensolution.

Applications of Approximate Analysis to Integrated Design of Phase-Zero CSI Evolutionary Model

In the previous example, eigenvalue/eigenvector sensitivity analysis was successfully applied in ways to significantly reduce the computational burden of structural optimization of large finite-element models. The computational burden is greatly magnified when considering the integrated design of controlled structures. For the integrated design techniques presented in references 2 to 4, control gains are added as design variables to the set of structural design variables to form one large simultaneous controls-structures optimization problem. One of the most difficult tasks in integrated design, or any optimization problem, is the proper selection of realistic objective and constraint functions. This selection or learning phase adds an additional loop to the design process. Therefore, in a design environment that is as computationally intensive as integrated design, every effort must be taken to make the optimization as efficient as possible.

Integrated Design Formulation

The Phase-Zero CSI Evolutionary Model (CEM) (ref. 15) was a laboratory test bed designed and constructed at Langley Research Center for experimental validation of control design methods and integrated design methodology as part of the NASA Controls-Structures Interaction Technology Program (ref. 16). The Phase-Zero CEM, shown in figure 11, consists of a 62-bay central truss (with each bay

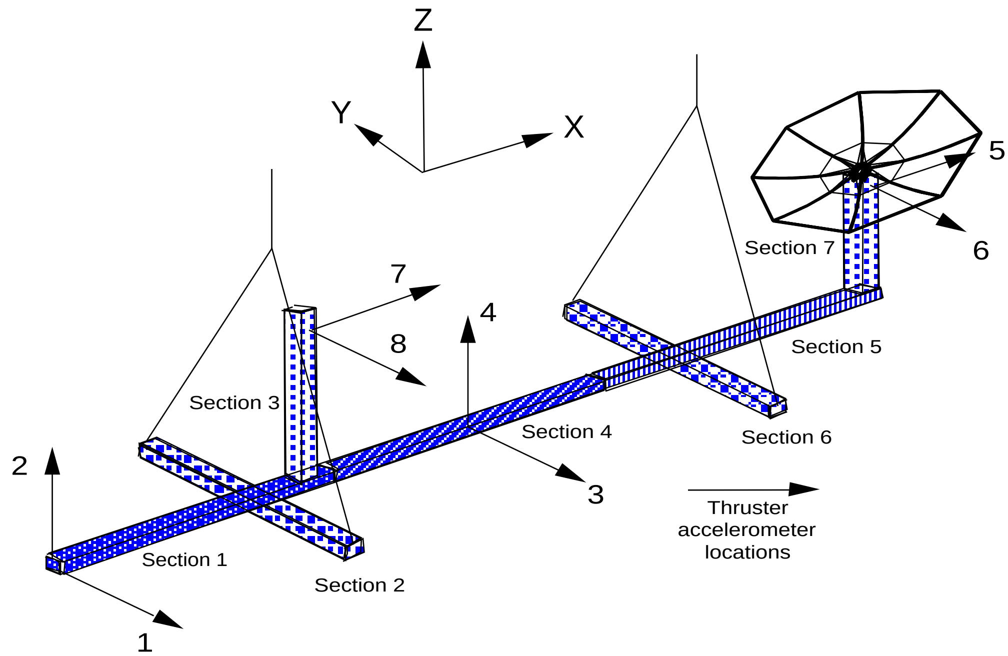

Figure 11. Phase-Zero CEM.

10 in. long), two vertical towers, and two horizontal booms. The structure is suspended using two cables, as shown. A laser source is mounted at the top of one of the towers, and a reflector with a mirrored surface is mounted on the other tower. The laser beam is reflected by the mirrored surface onto a detector surface 660 in. above the reflector. Eight proportional, bidirectional, gas thrusters provide the input actuation, while collocated servoaccelerometers provide output measurements. The first 20 frequencies of the Phase-Zero CEM are tabulated in table X. An open-loop damping ratio of 0.5 percent is assumed for all the modes.

Table X. Frequencies of Phase-Zero CEM

Mode	Frequency, Hz	Mode	Frequency, Hz
1	0.122	11	3.972
2	.126	12	4.127
3	.173	13	4.172
4	.680	14	5.788
5	.704	15	6.456
6	1.159	16	6.568
7	1.572	17	6.777
8	1.676	18	7.806
9	2.085	19	8.732
10	3.867	20	9.396

To perform the integrated design, the structure was divided into seven sections (three sections in the main bus and one section each for the two horizontal booms and the two vertical towers). Three structural design variables were used in each section (namely, effective cross-sectional area of the longerons, the battens, and the diagonals), thus making a total of 21 structural design variables.

The static (or constant-gain) dissipative controller, which is used for feedback control, employs collocated and compatible actuators and sensors and consists of feedbacks of the measured attitude vector $\mathbf{y}_p$ and the attitude rate vector $\mathbf{y}_r$ using constant, positive-definite gain matrices $\mathbf{G}_p$ and $\mathbf{G}_r$. This controller is robust in the presence of parametric uncertainties, unmodelled dynamics, and certain types of actuator and sensor nonlinearities (ref. 17). Here, two of the eight available actuators were used to generate persistent white-noise disturbances, and the remaining six actuators were used for feedback control. The static dissipative controller uses a 6×6 diagonal rate-gain matrix with no position feedback. (Because this system has no zero eigenvalues, position feedback is not necessary for asymptotic stability.) Thus, in the integrated design with the static dissipative controller, the total number of design variables was 27 (21 structural plus 6 control design variables).

An integrated controls-structures design was obtained by minimizing the steady-state average control power $\lim_{t\to\infty}\left\{Tr\left[E\left(uu^T\right)\right]\right\}$, where $Tr(\ )$ denotes the trace of ( ), in the presence of white-noise input disturbances with unit intensity (i.e., standard

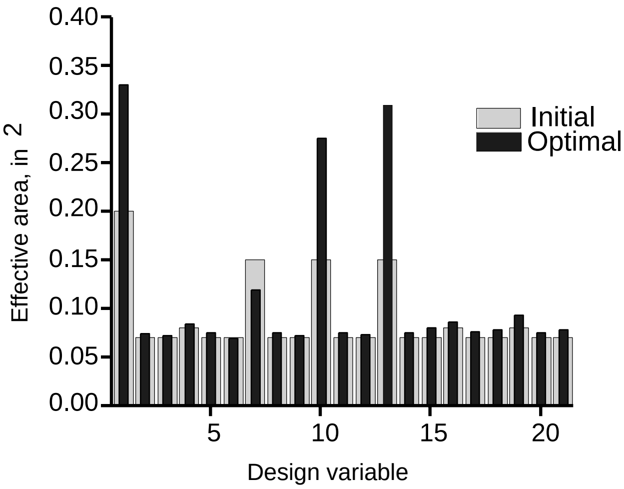

Figure 12. Structural design variables (initial and final for 0-percent case).

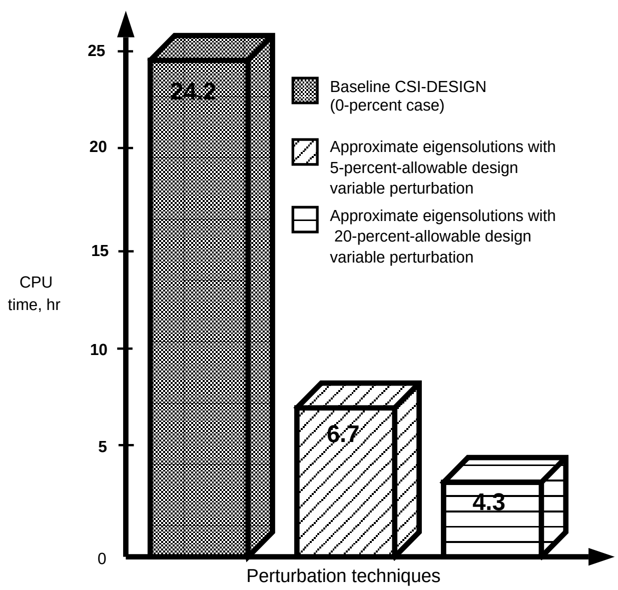

Figure 13. CPU time performance comparisons of 0-percent, 5-percent, and 20-percent eigensolution approximation techniques.

deviation intensity equals 1 lbf), at actuators Number 1 and Number 2. (These actuators are located at the end of the main bus nearest the laser tower.) A constraint was placed on the steady-state line-of-sight rms position error $\lim_{t\to\infty}\left\{Tr\left[E\left(y_{\text{los}}y_{\text{los}}^T\right)\right]\right\}$, at the laser detector (above the structure), for reasonable steady-state pointing performance. Additionally, the total mass of the structure M_{tot} was constrained to facilitate a fair comparison with the Phase-Zero CEM design. The six remaining actuators and the velocity signals (required for feedback by the dissipative controllers) obtained by processing

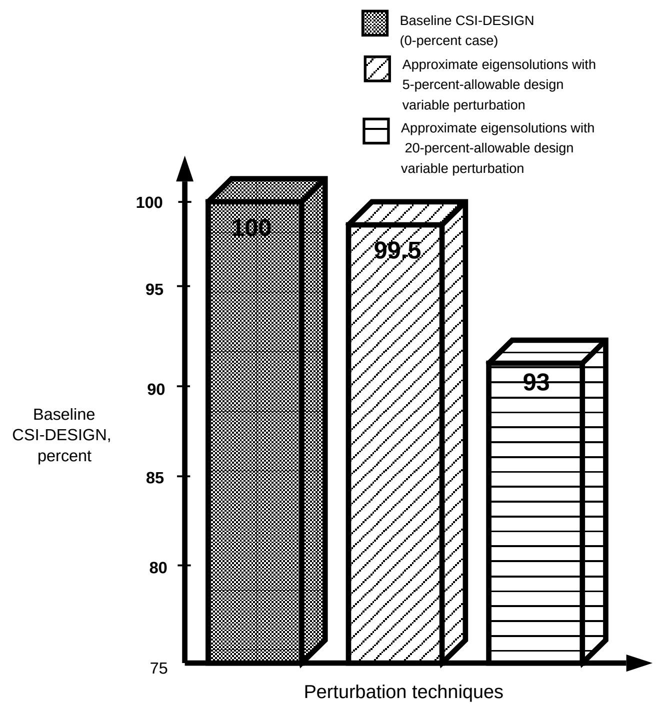

Figure 14. Effect of eigensolution approximations on solution integrity.

the accelerometer outputs were used in the control design. Upper and lower bounds were also placed on the structural design variables for safety and practicality concerns. Lower bound values were placed on these variables to satisfy structural integrity requirements against buckling and stress failures. Conversely, upper bound values were placed on these variables to accommodate design and fabrication limitations.

Mathematically, the integrated design problem can be stated as follows:

minimize

$$J \equiv \lim_{t\to\infty}\left\{Tr\left[E\left(uu^T\right)\right]\right\}$$

subject to

$$\lim_{t\to\infty}\left\{Tr\left[E\left(y_{\text{los}}y_{\text{los}}^T\right)\right]\right\} \leq \epsilon$$

and

$$M_{\text{tot}} \leq M_{\text{budget}}$$

The controls-structures integrated design results were obtained using the ADS software package. The solutions were obtained using a linear, extended interior penalty function method with a BFGS variable metric method for the unconstrained subproblem, followed by a polynomial interpolation for the one-dimensional search.

Results of the controls-structures integrated design are presented in figures 12 to 14. A performance comparison is based upon using varying levels of allowable structural design variable perturbations. The perturbations considered in this study are 5-percent and 20-percent variations in the effective areas of the structural members. Solution times obtained using these perturbations are compared with the baseline CSI-DESIGN code in figure 13. All CPU times were obtained using a Sun SPARC 2. Results show reductions in CPU times of 72 and 82 percent for the 5-percent- and 20-percent-allowable design variable perturbations, respectively. The effect of structural eigenvalue/eigenvector approximate analysis on the objective function for the integrated design problem is shown in figure 14. This figure indicates that approximate analysis with allowable design variable perturbations as large as 20 percent produces an optimal objective function that is within 7 percent of the baseline solution.

Concluding Remarks

Detailed theoretical derivations of eigenvalue/eigenvector sensitivity derivatives for two cases of the real symmetric structural eigenvalue problem have been developed in this paper. The two cases addressed were the repeated eigenvalues with distinct first eigenvalue derivatives and the infinitely repeated eigenvalues. An emphasis was placed upon developing efficient unified solution techniques for both cases.

The idea of a differentiable eigenvector was introduced for the case in which repeated eigenvalues with distinct first eigenvalue derivatives existed. A set of differentiable eigenvectors was obtained via linear transformation of the original eigenvectors. The transformation matrix and the eigenvalue derivatives were obtained by solving a reduced eigenvalue problem. The additional conditions required to uniquely define the eigenvector derivative were obtained by taking the second derivative of the structural eigenvalue problem.

A numerical technique has been proposed for the case of infinitely repeated eigenvalues. In this proposed technique, the assumption is made that the eigenvector derivative corresponding to the first eigenvector is orthogonal to the second eigenvector for a repeated eigenvalue with a multiplicity of two. An additional assumption was introduced regarding the "rotation" that must be performed to obtain a set of differentiable eigenvectors. More specifically, it was assumed that the original set of eigenvectors are themselves differentiable eigenvectors. The accuracy of the proposed technique was contingent upon satisfying an additional binormalization condition of the eigenvectors with respect to the derivative of the mass matrix. Approximate analysis results for the Earth Pointing System (EPS) model show that the proposed technique performs well for this type of problem.

An eigenvalue/eigenvector approximate analysis method was then derived in which the perturbed eigenvalue problem was continuously transformed to an eigenvalue problem with a single positive-valued design variable ϵ. The first-order Taylor's series expansion was then applied to estimate the change in the repeated eigenvalues and their corresponding eigenvectors caused by the changes in design variables. In this technique, the multivariable structural eigenvalue problems defined in the neighborhood of the repeated eigenvalues are reparameterized in terms of the single parameter ϵ, and they range from 0 to 1. Consequently, the design space becomes one dimensional and positive. Therefore, the reordering of the eigenvectors was solely dependent upon the magnitudes of the eigenvalue derivatives. This technique has been validated by numerical examples.

The uncontrolled EPS structure was used as an example to demonstrate the computational efficiency of eigenvalue/eigenvector approximate analysis as applied to a structural design optimization environment. The formulation was based upon a frequency objective function that was subject to a mass constraint and additional frequency constraints. The goal of this formulation was to demonstrate the efficiency of using a first-order approximation scheme in the presence of repeated eigenvalues and to demonstrate the versatility of eigensensitivity analysis by implementing an eigenvalue tracking technique based on eigenvalue derivative information. The results of this study show that more than a 50-percent reduction in CPU time can be obtained using a first-order approximation scheme.

Finally, eigenvalue/eigenvector approximate analysis was applied to the integrated design of a typical large flexible space structure. In this application, approximate analysis was used to facilitate the controls-structures optimization of the Phase-Zero CEM. The results show that the computational efficiency of the design process can be dramatically increased by employing the eigenvalue/eigenvector approximate analysis.

NASA Langley Research Center
Hampton, VA 23681-0001
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Appendix A

Controlled Structures Integrated Design Software and Implementation of Approximate Analysis

This appendix consists of two sections. The first section describes the software that was developed to conduct research in the integrated design of controlled structures. The second section deals with specific implementation issues of incorporating structural eigenvalue and eigenvector approximate analysis within the Automated Design Synthesis (ADS) optimization loop.

Controlled Structures Simulation Software (CS^3)

The Controlled Structures Simulation Software (CS^3) test bed (ref. 18) will be used as a research tool to investigate various applications of eigensensitivity analysis to controls-structures optimization. The baseline version of CS^3 consists of three modules. These modules include a finite-element modeling and structural analysis module, a control synthesis module, and an optimization module. The CS^3 test bed was developed to take advantage of efficient in-core intermodule data transfer and sparse matrix storage techniques. For a detailed description of the baseline CS^3 package, see reference 18. The baseline CS^3 package has evolved to include many new capabilities and increased functionality. Most of the improvements to CS^3 have been directed toward increasing its finite-element modeling and control synthesis capabilities. Along with its new capabilities, the CS^3 has also been given a new name. This new name, CSI-DESIGN, is more consistent with its focus of controls-structures integrated design. An eigenvalue/vector sensitivity analysis software (ESAS) module within CSI-DESIGN has been developed to implement the eigensensitivity equations derived in this paper. The ESAS, which follows the format of the CSI-DESIGN package, has been developed to utilize efficient in-core data transfer and sparse matrix storage techniques. The ESAS consists of three sections: a decision section, a sensitivity analysis section, and an approximate eigenvalue/eigenvector analysis section. The purpose of the decision section is to evaluate the current status of the eigenvalues by determining if they are distinct, repeated with distinct first eigenvalue derivatives, or infinitely repeated. Eigenvalue tracking in a generalized format is not enforced. The sensitivity analysis section calculates derivatives of the stiffness and mass matrices and solves the system of equations required to determine the eigenvector derivatives. The types of eigenvalue problems considered are distinct, repeated with distinct first eigenvalue derivatives, and infinitely repeated. Options are provided for either eigenvalue or both eigenvalue/eigenvector sensitivity analysis. The types of design variables include structural sizing (tube members only) and actuator masses (fixed locations). The approximate eigenvalue/eigenvector analysis section uses eigenvalue/eigenvector derivatives in a truncated Taylor series expression to provide linear approximations of the eigenvalues and eigenvectors. The approximate eigenvalue/eigenvector section provides an optimization interface so that eigensensitivity can be used to improve the efficiency of both structure and controls-structures optimization.

Implementation of Approximate Analysis in ADS Optimization Loop

Automated Design Synthesis (ADS) is a general-purpose numerical optimization software package (ref. 19). The ADS is composed of three main levels: the strategy, the optimizer, and the one-dimensional search. In the strategy level, the constrained problem is successively reduced to a series of linearized subproblems that can be constrained or unconstrained. The unconstrained subproblems can be formed by any of a variety of penalty function techniques that range from interior/exterior penalty function methods to the augmented Lagrange multiplier method. The constrained subproblem, conversely, can be reparameterized in either linear or quadratic form. The optimizer and the one-dimensional search are then used to find the minimum of the subproblem. The optimizer and the one-dimensional search also have a wide variety of options.

Usually, many function evaluations are required to solve the subproblem before convergence is obtained. In addition to function evaluations, gradient information is required to determine search directions. The ADS allows gradients to be either user supplied or internally calculated by finite differences. However, ADS does not provide an automatic utility to handle simultaneously both user-supplied and internally calculated gradients. In some situations, this limitation may present less than optimal efficiency. As an example, consider the mass minimization of a structure subject to nodal point displacement constraints. User-supplied (analytic) gradients of the objective function, namely derivatives of the mass with respect to sizing design variables, can be determined very easily, but gradients of the nodal point displacements require extensive program development. If these displacement gradients are not available analytically, the ADS finite-difference option must be chosen. In this case, ADS requires all gradients to be internally calculated by finite differences, and the analytic mass derivatives cannot be employed.

A simple technique has been developed to take advantage of the efficiency of approximate analysis and to alleviate the problem of only partial analytic gradient information. The technique is outlined below as follows:

1. Allow ADS to calculate finite-difference gradients
2. Approximate, via the Taylor series method described in the section entitled "Approximate Eigenvalue/Eigenvector Analysis Techniques," the perturbed function values for those functions whose analytic gradients are available
3. Perform exact function evaluations for those functions whose analytic gradients are not available

To elaborate on the above procedure, a finite-difference gradient is calculated as

$$\left.\frac{\partial \tilde{F}_i}{\partial b_j}\right|_{b^0} \approx \frac{F_i(b) - F_i\left(b^0\right)}{\Delta b} \tag{A1}$$

where the tilde implies that the expression is approximate. As described in item 2 above, the function may be approximated as

$$\tilde{F}_i(b) \approx F_i\left(b^0\right) + \left.\frac{\partial F_i}{\partial b_j}\right|_{b^0} \Delta b \tag{A2}$$

Substituting equation (A2) into equation (A1) gives

$$\frac{F_i\left(b^0\right) + \left(\partial F_i/\partial b_j\right)\big|_{b^0}\Delta b - F_i\left(b^0\right)}{\Delta b} = \left.\frac{\partial F_i}{\partial b_j}\right|_{b^0} \tag{A3}$$

where the exact expression in equation (A1), $F_i(b)$, has been substituted by an approximation given by equation (A2). Therefore, the finite-difference calculated derivative has "recovered" the analytic derivative. This technique is outlined in the flowchart given in figure A1.

To use approximate analysis in the one-dimensional line search, a way must exist to ensure that the perturbations are within allowable limits. This procedure can be accomplished by either setting limits on the allowable step size for each unconstrained subproblem or by allowing the optimizer to take its normal step and approximating the function only when this step size is within a given tolerance. In this research, the second approach has been chosen as the criteria for approximating the eigensolutions. A flowchart of this technique is given in figure A2.

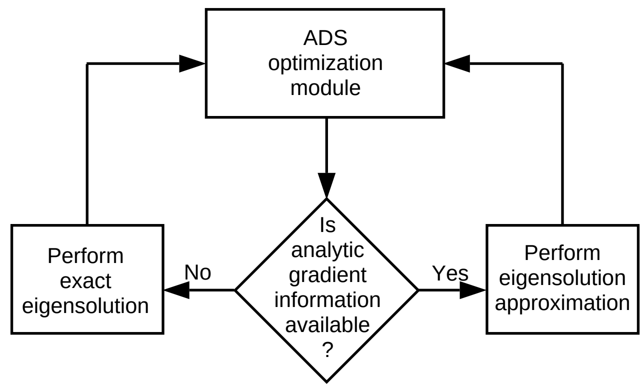

Figure A1. Implementation of partial analytic gradient information.

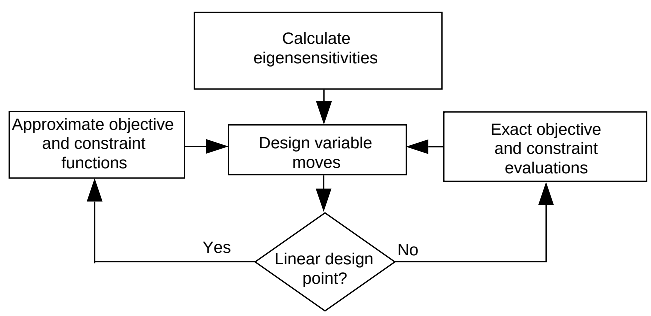

Figure A2. Implementation of approximate analysis in ADS loop.

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13. ABSTRACT *(Maximum 200 words)*
A method for eigenvalue and eigenvector approximate analysis for the case of repeated eigenvalues with distinct first derivatives is presented. The approximate analysis method developed involves a reparameterization of the multivariable structural eigenvalue problem in terms of a single positive-valued parameter. The resulting equations yield first-order approximations to changes in the eigenvalues and the eigenvectors associated with the repeated eigenvalue problem. This work also presents a numerical technique that facilitates the definition of an eigenvector derivative for the case of repeated eigenvalues with repeated eigenvalue derivatives (of all orders). Examples are given which demonstrate the application of such equations for sensitivity and approximate analysis. Emphasis is placed on the application of sensitivity analysis to large-scale structural and controls-structures optimization problems.

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