NONLINEAR EFFECTS ON THE NATURAL MODES OF OSCILLATION OF A FINITE LENGTH INVISCID FLUID COLUMN

Supplement II

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CONTENTS

Chapter 1 Introduction and Literature Review 1

Chapter 2 Governing Equations and Formulations 9

Chapter 3 Solutions for the Linear Order 20

Chapter 4 Solutions of the Order of $O(\varepsilon^2)$ 30

Chapter 5 Results at the Second Order of $O(\varepsilon^2)$:
The Shape Function and Velocity Potential Corrections 41

Chapter 6 Derivation at Order of $O(\varepsilon^3)$ 42

Chapter 7 Results of $O(\varepsilon^3)$:
Corrections to the Frequency, ω 52

References 53

Appendix A 57

Appendix B 65

Chapter 1 Introduction and Literature Review

1.1. General Background.

The proposed study is an investigation into the nonlinear oscillations of liquid bridges. The liquid bridge is a finite length fluid column which is held by both wetting forces (capillarity) and surface tension between two solid end disks which in this study are taken to be coaxial.

In addition to begin of fundamental interest, the liquid bridge configuration serves to model floating zones in crystal growth applications. Moreover, recent studies of Marangoni convection (including numerical studies) have been performed in this configuration. Although recent attention has been focused on the fundamental issues of liquid column formation and the possible paths to breakage (Padday, 1992, Martinez, 1984), the technical applications also provide impetus for investigation into the nonlinear dynamics of the liquid bridge.

The focus of this investigation is on aspects of the nonlinear behavior of the finite length liquid column. The emphasis is on bridge dynamics. This is in contrast to the quite interesting studies which look at possible equilibrium shapes and their (static) stability, as illustrated by the recent work of Langbein (1992) into the static stability of liquid bridges held between parallel plates. Of course, the initial interface shape of the liquid bridge in the dynamical studies will have to satisfy the capillary equation and be statically stable.

The proposed investigation will concentrate on nonlinear fluid dynamics. Thermal and solutal fields, so necessary to actual crystal growth process (Brown, 1988), will be absent. However, the floating zone milieu in which crystal growth would occur does involve fluid dynamics; and results of the proposed investigation should yield insight into nonlinear effects which would impact crystal growth.

In the microgravity environment provided by space shuttle, residual accelerations which could affect the stability limits of the liquid bridge configuration exist. In one space experiment, an amphora type liquid bridge configuration resulted instead of the c-mode-type that the experiment was designed to excite (Martinez, 1984). The interface response to periodic residual accelerations oriented parallel to the longitudinal axis of the bridge has been studied in a series of theoretical investigations (Zhang & Alexander, 1990, Lyell, 1991, Meseguer & Perals, 1991). Only the work of Zhang and

Alexander incorporated any nonlinear effects; however, their formation used a simplified one-dimensional slice model for the liquid bridge. With regard to the response of the liquid column to accelerations in alternate orientations relative to the column, the work of Lyell (1993) investigates interface stability in the presence of periodic forcing aligned normal to the longitudinal axis. However, the column was taken to be infinite in extent.

However, before any detailed investigations into the full nonlinear dynamics of the liquid bridge (with resonance effects and external forcing) can be undertaken, it is necessary to investigate the effect of nonlinearity on the oscillation frequencies of the system.

With regard to liquid bridge dynamics, it is only recently that the natural oscillation frequencies of the liquid bridge have been calculated in an inviscid linear analysis (Sanz, 1985 (axisymmetric); Sanz & Lopez-Diez, 1989 (non-axisymmetric)). A preliminary attempt has been performed by Eidel and Bauer (1988) to try to determine the effect of the nonlinearity upon the natural oscillation frequencies of the liquid bridge. However, they did not include the anchored triple contact line boundary condition in their formulation, and so have not modeled the finite liquid bridge correctly. (in essence, their results have application to an infinite column.)

The task of the proposed investigation is to determine the effect of nonlinearity upon the natural oscillation frequencies of the liquid bridge; to ascertain whether the liquid bridge system exhibits softening (or hardening) oscillations for a range of bridge slenderness parameters as well as higher order frequencies. Moreover, results of such an analysis will yield additional information on nonlinear corrections to the interface shape.

The liquid bridge configuration has both the fluid interface as well as solid boundaries provided by the end disks. It can be viewed as a configuration lying between the containerized fluid for which only the top boundary is that of a free surface (in a 1 gravity environment) and the free liquid drop, which involves no solid boundaries.

1.2. Literature Review.

1.2.1. Fluid Physics Literature.

Early work on the oscillation, dynamics, and breakage of liquid bridges utilized the one-dimensional inviscid slice model, which is valid in the limit of slender liquid bridges. Such a model assumes that axial velocity in the bridge is independent of the radial coordinate. Oscillatory frequencies for the slender limit case were determined in a linearized analysis by Meseguer (1983), in a study which investigated the dynamics of slender liquid bridges using the one-dimensional slice model. Additional efforts which utilized the one-dimensional slice model include that of Rivas and Meseguer (1984) and Meseguer and Sanz (1985). A report on liquid bridge breakage aboard Spacelab-D1 was given by Meseguer et al (1986).

It has been roughly a decade since the oscillation frequencies of the infinite length cylindrical liquid column were determined (Bauer, 1982). Such calculations were extended to include the oscillation frequencies of viscoelastic infinite length cylindrical columns (Bauer, 1986). The determination of the natural oscillation frequencies of the liquid bridge without the restriction to the slender limit has been performed more recently. A three dimensional linear model was developed by Sanz (1985); axisymmetric modes (Sanz 1985) and non-axisymmetric modes (Sanz and Lopez-Diez, 1989) have been investigated and oscillation frequencies determined.

In addition to fundamental interest, information regarding oscillation models is important for applications such as crystal growth via float-zone process. Also, recent experiments and numerical studies have investigated thermocapillary flow in the liquid bridge configuration. (See, for example, Preisesser et al, 1990, and Velten et al, 1991).

An attempt to evaluate the nonlinear effects on the frequencies of oscillation of a liquid bridge has been made by Eidel and Bauer (1988). However, their analysis did not impose the restriction of an anchored triple contact line at end disks, and so did not characterize correctly the finite length fluid bridge. (In essence, their analysis refers to the infinite length cylindrical fluid column).

It is this problem which the proposed effort will solve. Nonlinear corrections to the oscillation frequencies will be determined over a wide range of slenderness parameters (no restriction to the slender limit). Whether the system exhibits hardening or softening characteristics will be ascertained. The nonlinear correction to the interface shape will be found. The proposed methodology will utilize a Lindstedt-Poincare expansion of the frequency coupled with domain perturbation techniques.

1.2.2. Mathematical Formulation Literature Review.

The investigation of the nonlinear corrections to the frequencies of oscillation as well as to the interface shape will proceed via utilization of a Lindstedt-Poincare expansion for the frequency, and domain perturbation techniques. As the solution is required to be periodic in time, the Lindstedt-Poincare technique is applicable. Since the domain in which the solution is to be obtained is not stationary, domain perturbation techniques are quite useful.

Domain perturbation techniques were developed in the context of non-linear water wave theory. Early contributions include the work of Tadjbakhsh and Keller (1960), and Verma and Keller (1963) in their investigations of standing finite amplitude surface waves (in two and three dimensions, respectively). A formal discussion of the methodology awaited the contribution of Joseph (1973). However, it was not until the 1982 article by Lebovitz that a formal equivalence between expansions produced via domain perturbation techniques and those derived from more classical techniques (see Wehausen and Laitone, 1960) was exhibited. Recently Gu et al. (1988) used domain perturbation techniques in their study of the resonant surface waves in a rectangular container subject to periodic forcing oriented in the vertical direction.

For a further discussion on the genesis and details of the method, see the aforementioned references. Application of the domain perturbation techniques results in the development of a hierarchy of equations at successive orders in a given expansion parameter. Each of these sets of equations (including governing equations and boundary/interface conditions) is to be solved on a fixed domain. This fixed domain is achieved via a transformation of the oscillating interface boundary.

1.3. Objectives.

The primary objectives of this work are:

(1) to determine the nonlinear corrections to the interface shape of a naturally oscillating finite length liquid column, and

(2) to determine the nonlinear corrections to the oscillation frequencies for various modes of oscillation. The modes of oscillations themselves may be quickly characterized physically by the number of half-waves present upon the free surface.

The work will investigate the nonlinear characteristics of free oscillations only. This is not only a very demanding task, it is also the first task which must be accomplished in any rational plan of investigation into the nonlinear behavior of the liquid bridge.

In order to accomplish the objective of the proposed investigation, several subtasks become critical to the overall effort. First among these is the selection of a methodology which may be applied to the governing equations and allow for the incorporation of nonlinear efforts. Moreover, the methodology should be such that known linear results may also be recovered.

It is planned that the approach be theoretical and analytical (as opposed to computational). A methodology capable of achieving the objectives has been selected. It is discussed in Chapter II.

Application of the methodology (Lindstedt-Poincare expansion in conjunction with the domain perturbation technique) results in an hierarchical system of equations. The system discussed in Chapter III represents a recovery of known linear results.

Nonlinear corrections to the interface shape are achieved by solution of the second order system given in Chapter IV. Graphical results are presented in Chapter V.

In order to ascertain the nonlinear correction to the natural frequencies of oscillation and thereby satisfy the final objective of this proposed investigation, it is necessary to develop a solvability condition at third order in the hierarchical sets of equations. Theoretical results are presented in Chapter VI. Preliminary numerical results are given in Chapter VII.

Chapter 2 Governing Equations and Formulations

2.1. Governing Equations.

The time-periodic, irrotational and incompressible motion of an inviscid fluid column of finite length is considered. The following nondimensionalized quantities are used:

$$R\underline{X} = \underline{\tilde{X}}, \quad \frac{\sigma}{R}P = \tilde{P}, \quad \left(\frac{\sigma}{\rho R}\right)^{\frac{1}{2}} \underline{U} = \underline{\tilde{U}},$$
$$\left(\frac{\rho R^3}{\sigma}\right)^{\frac{1}{2}} \omega^{-1} t = \tilde{t}, \quad \left(\frac{\sigma}{\rho R^3}\right)^{\frac{1}{2}} \omega = \tilde{\omega},$$

where

$\underline{X}$ – spatial coordinates

$\underline{U}$ – velocity field

P – pressure

t – time

ω — angular frequency

R — radius of the column

σ — interface tension

ρ — density of the fluid comprising column

and where the tildes indicate corresponding dimensional quantities.

The volume of the undisturbed column is $\tilde{V} = \pi R^2 L$, where L is the length of the column. The surface of the column during axisymmetric oscillations is described by $RF(z,t)$, where $F(z,t)$ is the dimensionless shape function of the column.

The nondimensional slenderness of the column is defined as

$$\Lambda = \frac{L}{2R}.$$

Using the nondimensionalizations, the equations governing the inviscid time periodic motion are listed below, where $\Phi(r,z,t)$ denotes the velocity potential function.

$$\nabla^2\Phi(r,z,t) = 0, \ (0 \le r \le F(z,t), \ -\Lambda \le z \le \Lambda), \tag{2.1}$$

which results from using the potential form of the velocity field in the conservation of mass equation.

The conservation of momentum equation is given by

$$\nabla \left[\omega \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \Phi}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi}{\partial z}\right)^2 \right] = -\nabla P, \ (0 \le r \le F, \ -\Lambda \le z \le \Lambda). \tag{2.2}$$

The boundary/interface conditions are

$$\frac{\partial \Phi}{\partial r} = 0, \; (r = 0, \; -\Lambda \leq z \leq \Lambda). \tag{2.3a}$$

$$-\omega \frac{\partial F(z,t)}{\partial t} + \frac{\partial \Phi}{\partial r} - \frac{\partial \Phi}{\partial z}\frac{\partial F}{\partial z} = 0, \; (r = F(z,t)). \tag{2.3b}$$

$$\Delta P = \frac{\frac{1}{F}\left[1 + \left(\frac{\partial F}{\partial z}\right)^2\right] - \frac{\partial^2 F}{\partial z^2}}{\left[1 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{\frac{3}{2}}}, \; (r = F(z,t)). \tag{2.3c}$$

$$\nabla \Phi(r, z, t + 2\pi) = \nabla \Phi(r, z, t). \tag{2.3d}$$

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=\pm\Lambda} = 0. \tag{2.3e}$$

$$\int_{-\Lambda}^{\Lambda} F^2(z,t)dz = 2\Lambda. \tag{2.3f}$$

$$F(\pm\Lambda, t) = 1. \tag{2.3g}$$

Equation (2.1) is the Laplace equation governing the flow; (2.2) is the Euler equation; (2.3a) is the condition for a zero radial velocity at the center of the column, required for the restriction to axisymmetry; (2.3b) and (2.3c) are the kinematic condition and the normal force equation, respectively, at the interface; (2.3d) is the condition for periodicity in time of the velocity field; (2.3e) is the condition for zero normal velocity at the end disks; (2.3f) is the conservation of the volume condition and (2.3g) is the anchored triple contact line condition.

2.2. Linstedt-Poincare Expansion with Domain Perturbation Method.

The unknowns of the equations (2.1) - (2.3) are the shape function $F(z,t)$, the velocity potential function $\Phi(r,z,t)$, and the frequency ω. These variables will be calculated as the terms in expansion of the amplitude of the motion by the Linstedt-Poincare method (see Nayfeh & Mook 1979). The dependence of the velocity potential $\Phi(r,z,t)$ on the shape of the mathematical domain which is given by the moving boundary $r = F(z,t)$ is very complicated. The domain perturbation technique as detailed by Joseph (1973) will be applied.

To immobilize the boundary shape, introduce the change of variables $r = \mu F(z,t)$. Let ϵ denote a small positive real number. The expansions of the dependent variables can be written in terms of ϵ as follows:

$$\begin{pmatrix} F(z,t;\epsilon) \\ \Phi(r,z,t;\epsilon) \\ P(r,z,t;\epsilon) \\ \omega(\epsilon) \end{pmatrix} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \begin{pmatrix} F^{(k)}(z,t) \\ \Phi^{[k]}(\mu,z,t) \\ P^{[k]}(\mu,z,t) \\ \omega^{(k)} \end{pmatrix}, \tag{2.4}$$

where

$$\begin{cases} F^{(k)}(z,t) & \equiv & \dfrac{d^k F(z,t;0)}{d\epsilon^k}, \\ \Phi^{[k]}(\mu,z,t) & \equiv & \dfrac{d^k \Phi(\mu,z,t;0)}{d\epsilon^k}, \\ P^{[k]}(\mu,z,t) & \equiv & \dfrac{d^k P(\mu,z,t;0)}{d\epsilon^k}, \\ \omega^{(k)} & \equiv & \dfrac{d^k \omega(0)}{d\epsilon^k} \end{cases}.$$

The static cylindrical column is recovered as the zeroth order solution of the equation, with:

$$F^{(0)}(z,t) = 1,\ \Phi^{(0)}(\mu,z,t) = 0,\ P^{(0)}(\mu,z,t) = 0. \tag{2.5}$$

Using the chain rule for differentiation, each term $\Phi^{[k]}(\mu,z,t)$ and $P^{[k]}(\mu,z,t)$ in the expansion for the potential and the pressure can be written as a sum of contributions evaluated on the cylindrical domain $(0 \le \mu \le 1)$, and $(-\Lambda \le z \le \Lambda)$. Let $\dfrac{\partial F}{\partial \epsilon} = F^{(1)}(z,t)$. Then the first few relationships are:

$$\Phi^{[0]}(\mu,z,t;0) \equiv \Phi^{(0)}(\mu,z,t)$$

$$\Phi^{[1]}(\mu,z,t;0) \equiv \Phi^{(1)}(\mu,z,t) + F^{(1)}(z,t)\frac{\partial\Phi^{(0)}}{\partial\mu}$$

$$\Phi^{[2]}(\mu,z,t;0) \equiv \Phi^{(2)}(\mu,z,t) + 2F^{(1)}(z,t)\frac{\partial\Phi^{(1)}}{\partial\mu} + F^{(2)}(z,t)\frac{\partial\Phi^{(0)}}{\partial\mu} + (F^{(1)}(z,t))^2\frac{\partial^2\Phi^{(0)}}{\partial\mu^2}$$

$$\Phi^{[3]}(\mu,z,t;0) \equiv \Phi^{(3)}(\mu,z,t) + 3F^{(1)}(z,t)\frac{\partial\Phi^{(2)}}{\partial\mu} + 3F^{(2)}(z,t)\frac{\partial\Phi^{(1)}}{\partial\mu} + 3(F^{(1)}(z,t))^2\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} + 3F^{(1)}(z,t)F^{(2)}(z,t)\frac{\partial^2\Phi^{(0)}}{\partial\mu^2} + (F^{(1)}(z,t))^3\frac{\partial^3\Phi^{(0)}}{\partial\mu^3} + F^{(3)}(z,t)\frac{\partial\Phi^{(0)}}{\partial\mu}$$

Similarly, the following results are obtained.

$$P^{[0]}(\mu,z,t;0) \equiv P^{(0)}(\mu,z,t)$$

$$P^{[1]}(\mu,z,t;0) \equiv P^{(1)}(\mu,z,t) + F^{(1)}(z,t)\frac{\partial P^{(0)}}{\partial\mu}$$

$$P^{[2]}(\mu,z,t;0) \equiv P^{(2)}(\mu,z,t) + 2F^{(1)}(z,t)\frac{\partial P^{(1)}}{\partial\mu} + F^{(2)}(z,t)\frac{\partial P^{(0)}}{\partial\mu} + (F^{(1)}(z,t))^2\frac{\partial^2 P^{(0)}}{\partial\mu^2}$$

$$P^{[3]}(\mu,z,t;0) \equiv P^{(3)}(\mu,z,t) + 3F^{(1)}(z,t)\frac{\partial P^{(2)}}{\partial\mu} + 3F^{(2)}(z,t)\frac{\partial P^{(1)}}{\partial\mu} + 3(F^{(1)}(z,t))^2\frac{\partial^2 P^{(1)}}{\partial\mu^2} + 3F^{(1)}(z,t)F^{(2)}(z,t)\frac{\partial^2 P^{(0)}}{\partial\mu^2}$$

$$+ \quad (F^{(1)}(z,t))^3 \frac{\partial^3 P^{(0)}}{\partial \mu^3} + F^{(3)}(z,t) \frac{\partial P^{(0)}}{\partial \mu}$$

where $\Phi^{(k)}(\mu, z, t) \equiv \frac{\partial^k \Phi}{\partial \epsilon^k}$, $P^{(k)}(\mu, z, t) \equiv \frac{\partial^k P}{\partial \epsilon^k}$ are always defined in the cylindrical coordinates system $(0 \le \mu \le 1,\ -\Lambda \le z \le \Lambda)$.

2.3. Hierarchical Systems of Equations.

In this section, the complete system of equations occurring at each order in ϵ are displayed.

Linear System.

For $O(\epsilon)$, which represents the linear order, we obtain the following governing system of equations:

The Laplace equation:

$$\nabla^2 \Phi^{(1)}(\mu, z, t) = 0,\ (0 \le \mu \le 1,\ -\Lambda \le z \le \Lambda).$$

Condition on radial velocity required for axisymmetry:

$$\frac{\partial \Phi^{(1)}}{\partial \mu} = 0,\ (\mu = 0,\ -\Lambda \le z \le \Lambda).$$

Kinematic condition:

$$-\omega^{(0)} \frac{\partial F^{(1)}}{\partial t} + \frac{\partial \Phi^{(1)}}{\partial \mu} = 0,\ (\mu = 1).$$

Normal force balance:

$$\omega^{(0)}\frac{\partial\Phi^{(1)}}{\partial t} - F^{(1)} - \frac{\partial^2 F^{(1)}}{\partial z^2} = BN^{(1)},\ (\mu = 1).$$

Periodicity in time:

$$\nabla\Phi^{(1)}(\mu, z, t + 2\pi) = \nabla\Phi^{(1)}(\mu, z, t).$$

Zero normal velocity required at end disks:

$$\left.\frac{\partial\Phi^{(1)}}{\partial z}\right|_{z=\pm\Lambda} = 0.$$

Conservation of volume:

$$\int_{-\Lambda}^{\Lambda} F^{(1)}(z,t)dz = 0.$$

Triple contact line condition:

$$F^{(1)}(\pm\Lambda, t) = 0.$$

System at Nonlinear Order ϵ^2.

For $\mathcal{O}(\epsilon^2)$, which involves the first nonlinear contributions, the following governing system of equations has been obtained:

The Laplace equation:

$$\nabla^2\Phi^{(2)}(\mu, z, t) = 0,\ (0 \le \mu \le 1,\ -\Lambda \le z \le \Lambda).$$

Condition on radial velocity required for axisymmetry:

$$\frac{\partial \Phi^{(2)}}{\partial \mu} = 0, \ (\mu = 0, \ -\Lambda \leq z \leq \Lambda).$$

Kinematic condition:

$$-\omega^{(0)}\frac{\partial F^{(2)}}{\partial t} + \frac{\partial \Phi^{(2)}}{\partial \mu}$$

$$= \ 2\omega^{(1)}\frac{\partial F^{(1)}}{\partial t} - 2F^{(1)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu^2} + 2\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial F^{(1)}}{\partial z}, \quad (\mu = 1).$$

Normal force balance:

$$\omega^{(0)}\frac{\partial \Phi^{(2)}}{\partial t} - F^{(2)} - \frac{\partial^2 F^{(2)}}{\partial z^2}$$

$$= \ -2\omega^{(1)}\frac{\partial \Phi^{(1)}}{\partial t} - 2\omega^{(0)}F^{(1)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial t} - \left(\frac{\partial \Phi^{(1)}}{\partial \mu}\right)^2 - \left(\frac{\partial \Phi^{(1)}}{\partial z}\right)^2$$

$$-2(F^{(1)})^2 + \left(\frac{\partial F^{(1)}}{\partial z}\right)^2 + BN^{(2)}, \quad (\mu = 1).$$

Periodicity in time:

$$\nabla \Phi^{(2)}(\mu, z, t + 2\pi) = \nabla \Phi^{(2)}(\mu, z, t).$$

Zero normal velocity required at end disks:

$$\left.\frac{\partial \Phi^{(2)}}{\partial z}\right|_{z=\pm\Lambda} = 0.$$

Conservation of volume:

$$\int_{-\Lambda}^{\Lambda} \left[F^{(2)} + \left(F^{(1)}\right)^2\right] dz = 0.$$

Triple contact line condition:

$$F^{(2)}(\pm\Lambda, t) = 0.$$

Nonlinear System at Order ϵ^3.

For $O(\epsilon^3)$, we obtain the following governing system of equations:

The Laplace equation:

$$\nabla^2\Phi^{(3)}(\mu, z, t) = 0, \ (0 \leq \mu \leq 1, \ -\Lambda \leq z \leq \Lambda).$$

Condition on radial velocity required for axisymmetry:

$$\frac{\partial \Phi^{(3)}}{\partial \mu} = 0, \ (\mu = 0, \ -\Lambda \leq z \leq \Lambda).$$

Kinematic condition:

$$-\omega^{(0)}\frac{\partial F^{(3)}}{\partial t} + \frac{\partial \Phi^{(3)}}{\partial \mu}$$

$$= 3\omega^{(2)}\frac{\partial F^{(1)}}{\partial t} + 3\omega^{(1)}\frac{\partial F^{(2)}}{\partial t} - 3F^{(1)}\frac{\partial^2 \Phi^{(2)}}{\partial \mu^2}$$

$$-3F^{(2)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu^2} - 3(F^{(1)})^2\frac{\partial^3 \Phi^{(1)}}{\partial \mu^3} + 3\frac{\partial \Phi^{(2)}}{\partial z}\frac{\partial F^{(1)}}{\partial z}$$

$$+6F^{(1)}\frac{\partial F^{(1)}}{\partial z}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial z} + 3\frac{\partial F^{(2)}}{\partial z}\frac{\partial \Phi^{(1)}}{\partial z}, \quad (\mu = 1).$$

Normal force balance:

$$\omega^{(0)}\frac{\partial\Phi^{(3)}}{\partial t} - F^{(3)} - \frac{\partial^2 F^{(3)}}{\partial z^2}$$

$$= -3\omega^{(2)}\frac{\partial\Phi^{(1)}}{\partial t} - 3\omega^{(1)}\frac{\partial\Phi^{(2)}}{\partial t} - 6\omega^{(1)}F^{(1)}\frac{\partial^2\Phi^{(1)}}{\partial\mu\partial t}$$

$$-3\omega^{(0)}F^{(1)}\frac{\partial^2\Phi^{(2)}}{\partial\mu\partial t} - 3\omega^{(0)}F^{(2)}\frac{\partial^2\Phi^{(1)}}{\partial\mu\partial t} - 3\omega^{(0)}(F^{(1)})^2\frac{\partial^3\Phi^{(1)}}{\partial\mu^2\partial t}$$

$$-3\frac{\partial\Phi^{(1)}}{\partial\mu}\frac{\partial\Phi^{(2)}}{\partial\mu} - 6F^{(1)}\frac{\partial\Phi^{(1)}}{\partial\mu}\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} - 3\frac{\partial\Phi^{(1)}}{\partial z}\frac{\partial\Phi^{(2)}}{\partial z}$$

$$-6F^{(1)}\frac{\partial\Phi^{(1)}}{\partial z}\frac{\partial^2\Phi^{(1)}}{\partial\mu\partial z} - 6F^{(1)}F^{(2)} + 6(F^{(1)})^3 - 3F^{(1)}\left(\frac{\partial F^{(1)}}{\partial z}\right)^2$$

$$+3\frac{\partial F^{(1)}}{\partial z}\frac{\partial F^{(2)}}{\partial z} - 9\left(\frac{\partial F^{(1)}}{\partial z}\right)^2\frac{\partial^2 F^{(1)}}{\partial z^2} + BN^{(3)}, \quad (\mu = 1).$$

Periodicity in time:

$$\nabla\Phi^{(3)}(\mu, z, t + 2\pi) = \nabla\Phi^{(3)}(\mu, z, t).$$

Zero normal velocity required at end disks:

$$\left.\frac{\partial\Phi^{(3)}}{\partial z}\right|_{z=\pm\Lambda} = 0.$$

Conservation of volume:

$$\int_{-\Lambda}^{\Lambda}\left[\frac{1}{3}F^{(3)} + F^{(1)}F^{(2)}\right]dz = 0.$$

Triple contact line condition:

$$F^{(3)}(\pm\Lambda, t) = 0.$$

Chapter 3 Solutions for the Linear Order

In this chapter, the linearized problem is solved. Functional results from the linear order will appear as nonlinear forcing terms in the higher order systems. The major results were first proved by Sanz (1985). In this chapter, Sanz's results are recovered in the notation of the present effort.

In the previous chapter, the governing equations for the linear order $O(\epsilon)$ were listed. They are repeated here:

The Laplace equation:

$$\nabla^2\Phi^{(1)}(\mu, z, t) = 0, \ (0 \le \mu \le 1, \ -\Lambda \le z \le \Lambda). \tag{3.1}$$

Condition on radial velocity required for axisymmetry:

$$\frac{\partial \Phi^{(1)}}{\partial \mu} = 0, \ (\mu = 0, \ -\Lambda \le z \le \Lambda). \tag{3.2a}$$

Kinematic condition:

$$-\omega^{(0)}\frac{\partial F^{(1)}}{\partial t}+\frac{\partial \Phi^{(1)}}{\partial \mu}=0,\ (\mu=1). \tag{3.2b}$$

Normal force balance:

$$\omega^{(0)}\frac{\partial \Phi^{(1)}}{\partial t}-F^{(1)}-\frac{\partial^2 F^{(1)}}{\partial z^2}=B\mathcal{N}^{(1)},\ (\mu=1). \tag{3.2c}$$

Periodicity in time:

$$\nabla\Phi^{(1)}(\mu,z,t+2\pi)=\nabla\Phi^{(1)}(\mu,z,t). \tag{3.2d}$$

Zero normal velocity at end disks:

$$\left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=\pm\Lambda}=0. \tag{3.2e}$$

Conservation of volume:

$$\int_{-\Lambda}^{\Lambda}F^{(1)}(z,t)dz=0. \tag{3.2f}$$

Triple contact line condition:

$$F^{(1)}(\pm\Lambda,t)=0. \tag{3.2g}$$

For axisymmetric problems, $\dfrac{\partial}{\partial\theta}=0$, and using cylindrical coordinates, Laplace equation becomes

$$\nabla^2\Phi^{(1)}=\frac{1}{\mu}\frac{\partial}{\partial\mu}\left(\mu\frac{\partial}{\partial\mu}\right)\Phi^{(1)}+\frac{\partial^2}{\partial z^2}\Phi^{(1)} \tag{3.3}$$

Write $\Phi^{(1)} = T(t)R(\mu)Z(z)$ and substitute it into the Laplace equation to obtain

$$\frac{1}{\mu}\frac{\partial}{\partial\mu}\left(\mu\frac{\partial R}{\partial\mu}\right) - \lambda R = 0 \tag{3.4}$$

$$Z'' + \lambda Z = 0 \tag{3.5}$$

where λ is a constant.

Using the change of variables $\alpha = \sqrt{\lambda}$ and $\xi = \alpha\mu$ in (3.4), one obtains the modified Bessel's equation:

$$\mu^2\frac{\partial^2 R}{\partial\mu^2} + \mu\frac{\partial R}{\partial\mu} - \xi^2 R = 0. \tag{3.6}$$

Thus for $\lambda > 0$, the general solution for (3.6) is

$$R = AI_0(\xi) + BK_0(\xi),$$

where I_0 and K_0 are the modified Bessel's functions of the zeroth order (indicated by the subscript) of the first kind. Since the z-axis ($r = 0$) is part of the domain, $B = 0$, in order to preserve finiteness, and so

$$R = AI_0(\xi) = AI_0(\alpha\mu). \tag{3.7}$$

From (3.5), it is clear that

$$Z = E\cos(\sqrt{\lambda}z + \Delta) = E\cos(\alpha z + \Delta). \tag{3.8}$$

Thus the solution of $\Phi^{(1)}$ is

$$\Phi^{(1)} = AI_0(\alpha\mu)\cos(\alpha z + \Delta)T(t). \tag{3.9}$$

By the condition of zero normal velocity at the end disks,

$$\left.\frac{\partial\Phi^{(1)}}{\partial z}\right|_{z=\pm\Lambda} = AI_0(\alpha\mu)(-\alpha)T(t)\sin(\alpha z + \Delta)\Big|_{z\pm\Lambda} = 0,$$

which implies that $\Delta = \frac{n\pi}{2}$, for some integer n.

Let $\alpha_n = l_n = \frac{n\pi}{2\lambda}$. It follows

$$\Phi^{(1)} = \sum_{n=0}^{\infty} A_n I_0(l_n\mu)\cos l_n(z+\Lambda)T(t). \tag{3.10}$$

Note that $I_0'(\xi) = I_1(\xi)$. Therefore,

$$\frac{\partial\Phi^{(1)}}{\partial\mu} = \sum_{n=0}^{\infty} A_n l_n I_1(l_n\mu)\cos l_n(z+\Lambda)T(t).$$

As $I_1(0) = 0$,

$$\left.\frac{\partial\Phi^{(1)}}{\partial\mu}\right|_{\mu=0} = 0,$$

and so the radial velocity condition is satisfied. Since $\sin(2l_n\Lambda) = \sin(n\pi) = 0$, the condition of normal velocity on the end disks is also satisfied. Therefore, the spatial form of the solution of $\Phi^{(1)}$ has been obtained without knowing the specific t-dependence.

The initial potential is assumed to be zero. Using the periodicity in time condition, one can define the time dependence of $\Phi^{(1)}$ as $\sin t$. Hence

$$\Phi^{(1)}(\mu,z,t) = \sin t\sum_{n=0}^{\infty} A_n I_0(l_n\mu)\cos l_n(z+\Lambda). \tag{3.11}$$

By the kinematic condition and normal force balance equations,

$$-\omega^{(0)}\frac{\partial F^{(1)}}{\partial t} + \frac{\partial\Phi^{(1)}}{\partial\mu} = 0, \quad (\mu = 1), \tag{3.12}$$

and

$$\omega^{(0)}\frac{\partial\Phi^{(1)}}{\partial t} - F^{(1)} - \frac{\partial^2 F^{(1)}}{\partial z^2} - BN^{(1)} = 0 \quad (\mu = 1). \tag{3.13}$$

Assume that the solution of $F^{(1)}$ is in the form of

$$F^{(1)} = Q_n(z)\cos t. \tag{3.14}$$

Substituting (3.14) in the kinematic condition (3.12) yields

$$\omega^{(0)}Q_n(z) + \sum_{n=0}^{\infty} A_n I_0'(l_n) l_n \cos l_n(z+\Lambda) = 0. \tag{3.15}$$

Substituting (3.14) in the normal force balance equation (3.13), we have

$$Q_n''(z) + Q_n(z) = \omega^{(0)} \sum_{n=0}^{\infty} A_n^{(1)} I_0(l_n)\cos l_n(z+\Lambda) - BN^{(1)}. \tag{3.16}$$

Solving (3.16) yields

$$Q_n = \alpha^{(1)}\cos z + \beta^{(1)}\sin z - BN^{(1)} + \sum_{n=0}^{\infty} \frac{\omega^{(0)}}{1-l_n^2} A_n I_0(l_n)\cos l_n(z+\Lambda). \tag{3.17}$$

To study the two expressions of $Q_n(z)$ in (3.15) and in (3.17), expand

$$\cos z = \sum_{n=0}^{\infty} C_n \cos l_n(z+\Lambda)$$

to obtain

$$\cos z \doteq \sum_{k=1}^{\infty}\left[\frac{2\sin\Lambda}{\Lambda(1-l_{2k}^2)}\cos l_{2k}(z+\Lambda)\right] + \frac{\sin\Lambda}{\Lambda}. \tag{3.18}$$

Similarly,

$$\sin z = \sum_{k=1}^{\infty} \frac{2\cos\Lambda}{\Lambda(1-l_{2k-1}^2)}\cos l_{2k-1}(z+\Lambda). \tag{3.19}$$

Therefore (3.17) becomes

$$Q_n = \alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \alpha^{(1)}\frac{2\sin\Lambda}{\Lambda}\sum_{k=1}^{\infty}\frac{1}{1-l_{2k}^2}\cos l_{2k}(z+\Lambda)$$

$$+ \beta^{(1)}\frac{2\cos\Lambda}{\Lambda}\sum_{k=1}^{\infty}\frac{1}{1-l_{2k-1}^2}\cos l_{2k-1}(z+\Lambda) - BN^{(1)}$$

$$+ \sum_{n=0}^{\infty}\frac{\omega^{(0)}}{1-l_n^2}A_n I_0(l_n)\cos l_n(z+\Lambda). \tag{3.20}$$

For $n = 0$, (z-independent terms), that $I_1(0) = 0$ and $I_0(0) = 1$ give

$$0 = \alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \omega^{(0)}A_0 - BN^{(1)}.$$

Physically, $BN^{(1)}$ adjusts the pressure. No adjustment at this linear order is needed for physical consistency. Therefore, the value of $BN^{(1)}$ is selected to be zero, and so

$$0 = \alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \omega^{(0)}A_0. \tag{3.21}$$

Similarly, for $n = 2k > 0$,

$$A_{2k} = -2\alpha^{(1)}\omega^{(0)}\frac{\sin\Lambda}{\Lambda}[(\omega^{(0)})^2 I_0(l_{2k}) + l_{2k}(1-l_{2k}^2)I_0'(l_{2k})]^{-1}; \tag{3.22}$$

and for $n = 2k-1 > 0$,

$$A_{2k-1} = -2\beta^{(1)}\omega^{(0)}\frac{\cos\Lambda}{\Lambda}[(\omega^{(0)})^2 I_0(l_{2k-1}) + l_{2k-1}(1-l_{2k-1}^2)I_0'(l_{2k-1})]^{-1}. \tag{3.23}$$

This gives the solution of $F^{(1)}(z,t) = \cos t\, Q_n(z)$ as follows:

$$F^{(1)}(z,t) = \cos t\left\{-\alpha^{(1)}\frac{\sin\Lambda}{\Lambda}\right.$$

$$\left. +\alpha^{(1)}\cos z + \beta^{(1)}\sin z + \sum_{n=1}^{\infty}\frac{\omega^{(0)}}{1-l_n^2}A_n I_0(l_n)\cos l_n(z+\Lambda)\right\}. \tag{3.24}$$

with $l_n = \frac{n\pi}{2\Lambda}$, or

$$F^{(1)}(z,t) = \frac{\cos t}{\omega^{(0)}} \sum_{n=1}^{\infty} A_n l_n I_0'(l_n) \cos l_n(z+\Lambda). \tag{3.25}$$

Substitute (3.24) into the condition of conservation of volume (3.2f) to obtain

$$\begin{aligned}
&\int_{-\Lambda}^{\Lambda} F^{(1)}(z,t)dz \\
=\ & \cos t \left\{ \alpha^{(1)} \frac{2\sin\Lambda}{\Lambda} \sum_{k=1}^{\infty} \frac{1}{(1-l_{2k}^2)l_{2k}} \sin l_{2k}(z+\Lambda) \Big|_{-\Lambda}^{\Lambda} \right. \\
+\ & \beta^{(1)} \frac{2\cos\Lambda}{\Lambda} \sum_{k=1}^{\infty} \frac{1}{(1-l_{2k-1}^2)l_{2k-1}} \sin l_{2k-1}(z+\Lambda) \Big|_{-\Lambda}^{\Lambda} \\
+\ & \sum_{n=1}^{\infty} \frac{\omega^{(0)}}{1-l_n^2} A_n I_0(l_n) \frac{1}{l_n} \sin l_n(z+\Lambda) \Big|_{-\Lambda}^{\Lambda} \\
=\ & 0
\end{aligned}$$

and so (3.24) satisfies the conservation of mass for an incompressible fluid column.

Substituting (3.24) into the triple contact line condition (3.2g), for $z = \Lambda$, the anchored triple contact line requirement yields

$$\begin{aligned}
&F^{(1)}(\Lambda,t) = \cos t\, Q_n(\Lambda) = \cos t \left\{ -\alpha^{(1)} \frac{\sin\Lambda}{\Lambda} \right. \\
&\left. +\alpha^{(1)} \cos\Lambda + \beta^{(1)} \sin\Lambda + \sum_{n=1}^{\infty} \frac{\omega^{(0)}}{1-l_n^2} A_n I_0(l_n) \cos l_n(2\Lambda) \right\} = 0;
\end{aligned} \tag{3.26}$$

and for $z = -\Lambda$,

$$F^{(1)}(-\Lambda,t) = \cos t\, Q_n(-\Lambda) = \cos t \left\{ -\alpha^{(1)} \frac{\sin\Lambda}{\Lambda} \right.$$

$$+ \alpha^{(1)} \cos\Lambda - \beta^{(1)} \sin\Lambda + \sum_{n=1}^{\infty} \frac{\omega^{(0)}}{1-l_n^2} A_n I_0(l_n)\cos(0) \Big\} = 0. \qquad (3.27)$$

Add (3.26) and (3.27), and use the fact that $\cos(n\pi) = (-1)^n$ to obtain

$$-\alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \alpha^{(1)}\cos\Lambda + \sum_{k=1}^{\infty} \frac{\omega^{(0)}}{1-l_{2k}^2} A_{2k} I_0(l_{2k}) = 0 \qquad (3.28)$$

Substitute (3.22) into (3.28) to obtain

$$\frac{\Lambda - \tan\Lambda}{2\tan\Lambda} - \sum_{k=1}^{\infty} \frac{(\omega^{(0)})^2}{(1-l_{2k}^2)\left[(\omega^{(0)})^2 + l_{2k}(1-l_{2k}^2)\frac{I_0'(l_{2k})}{I_0(l_{2k})}\right]} = 0 \qquad (3.29)$$

Similarly, by subtracting (3.27) from (3.26), and by using (3.23),

$$\frac{1}{2}\Lambda\tan\Lambda + \sum_{k=1}^{\infty} \frac{(\omega^{(0)})^2}{(1-l_{2k-1}^2)\left[(\omega^{(0)})^2 + l_{2k-1}(1-l_{2k-1}^2)\frac{I_0'(l_{2k-1})}{I_0(l_{2k-1})}\right]} = 0 \qquad (3.30)$$

Equations (3.29) and (3.30) represent the dispersion relations.

Conclusions for the Solutions of the Order $O(\epsilon)$. For the order $O(\epsilon)$,

$$\Phi^{(1)}(\mu, z, t) = \sin t \sum_{n=0}^{\infty} A_n I_0(l_n\mu)\cos l_n(z+\Lambda)$$

and

$$\begin{aligned} &\mathcal{F}^{(1)}(z,t) \\ = \;& \cos t \left\{ -\alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \alpha^{(1)}\cos z + \beta^{(1)}\sin z + \sum_{n=1}^{\infty}\frac{\omega^{(0)}}{1-l_n^2} A_n I_0(l_n)\cos l_n(z+\Lambda)\right\}. \end{aligned}$$

or

$$F^{(1)}(z,t) = \frac{\cos t}{\omega^{(0)}} \sum_{n=1}^{\infty} A_n l_n I_0'(l_n)\cos l_n(z+\Lambda).$$

where

$$0 = \alpha^{(1)}\frac{\sin\Lambda}{\Lambda} + \omega^{(0)}A_0,$$

$$A_{2k} = -2\alpha^{(1)}\omega^{(0)}\frac{\sin\Lambda}{\Lambda}[(\omega^{(0)})^2 I_0(l_{2k}) + l_{2k}(1-l_{2k}^2)I_0'(l_{2k})]^{-1},$$

$$\begin{aligned} A_{2k-1} &= -2\beta^{(1)}\omega^{(0)}\frac{\cos\Lambda}{\Lambda}[(\omega^{(0)})^2 I_0(l_{2k-1}) \\ &+ l_{2k-1}(1-l_{2k-1}^2)I_0'(l_{2k-1})]^{-1}; \end{aligned}$$

and, setting $\omega^{(0)} = \omega_p^{(0)}$ to emphasize the existence of multiple modes,

$$\frac{\Lambda - \tan\Lambda}{2\tan\Lambda} - \sum_{k=1}^{\infty}\frac{(\omega_p^{(0)})^2}{(1-l_{2k}^2)\left[(\omega_p^{(0)})^2 + l_{2k}(1-l_{2k}^2)\frac{I_0'(l_{2k})}{I_0(l_{2k})}\right]} = 0,$$

$$\frac{1}{2}\Lambda\tan\Lambda + \sum_{k=1}^{\infty}\frac{(\omega_p^{(0)})^2}{(1-l_{2k-1}^2)\left[(\omega_p^{(0)})^2 + l_{2k-1}(1-l_{2k-1}^2)\frac{I_0'(l_{2k-1})}{I_0(l_{2k-1})}\right]} = 0.$$

Numerical solutions for given Λ can be obtained from these dispersion (eigenvalue) equations. The $p = 2$ mode can be obtained from (3.30) and the $p = 3$ mode can be obtained from (3.29), where p is the number of half-waves in the interface deformation. The modes with an odd (even) number of surface deformations are obtained by using equations (3.29) ((3.30), respectively). A root finding technique was used in order to determine the $\omega_p^{(0)}$ values for each Λ. The program is listed in the Appendix. Graphical results are presented in Figure 3.1 and the numerical results are listed in Table 3.1.

The linear solutions of this order ($O(\epsilon)$) recovers A. Sanz's solutions (1985).

ROOT DETERMINATION BY BISECTION METHOD TO FIND OMEGA.

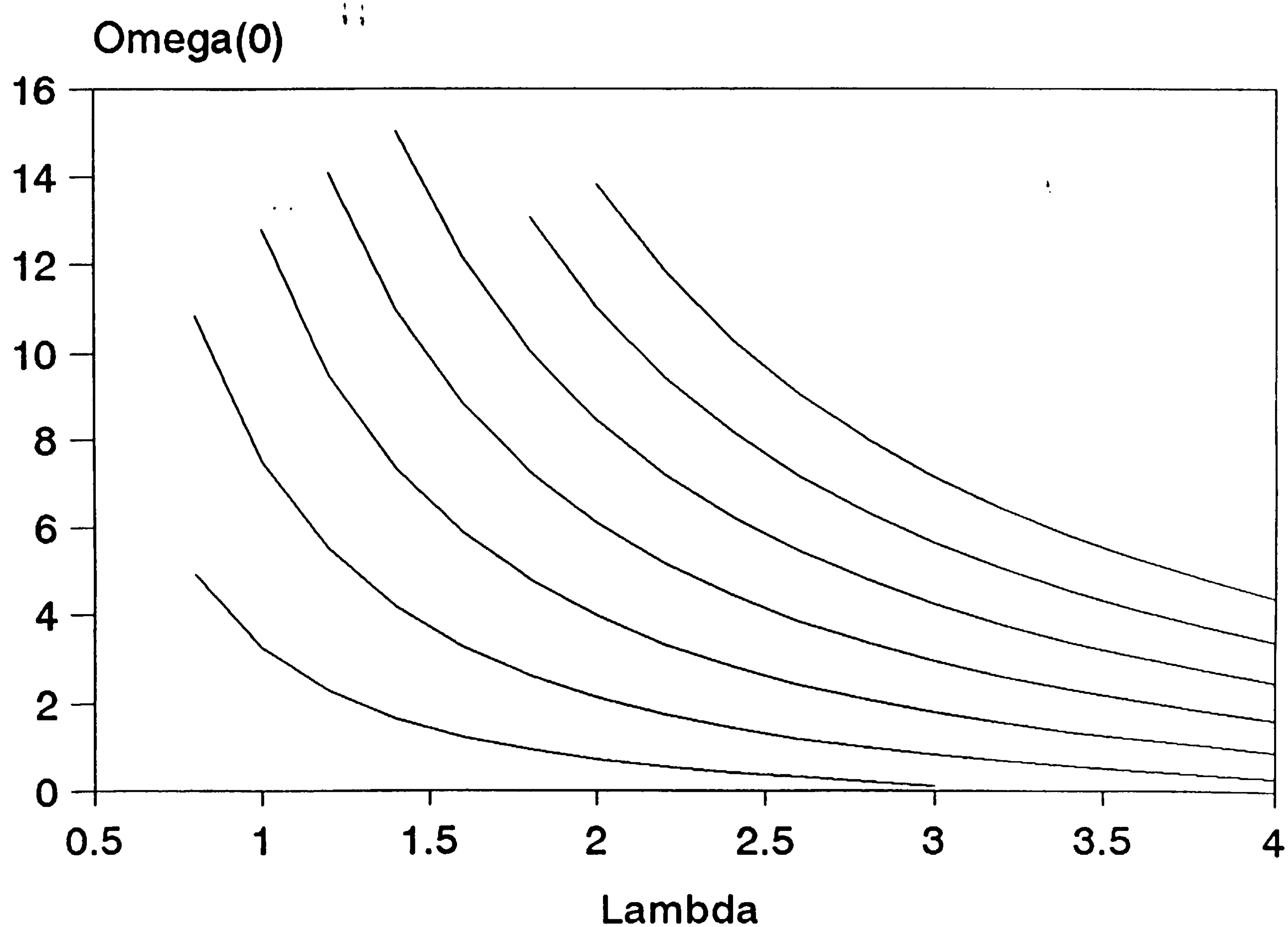
LAMBDA	OMEGA (P=2 MODE)	OMEGA (P=3 MODE)
1.0000	3.264606138984453400	7.523140628393082400
1.2000	2.290288785914530890	5.517359210610919050
1.4000	1.670226411974154330	4.203184144471603600
1.6000	1.251121188390937440	3.290224461262170050
1.8000	0.953872402882467108	2.628156961557847950
2.0000	0.734523546383059167	2.131828795651618870
2.2000	0.566652921197410286	1.749636681888166120
2.4000	0.433454446539947594	1.448627121229342450
2.6000	0.323269135939490443	1.206860255054819530
2.8000	0.226462506734397689	1.009208756750524130
3.0000	0.129816430336827382	0.844915285737982580
3.2000		0.706097678690390085
3.4000		0.586748700766904663
3.6000		0.482135179607233483
3.8000		0.388141417997999019
4.0000		0.300614117118225790

LAMBDA	OMEGA (P=4 MODE)	OMEGA (P=5 MODE)
1.0000	12.797858131049062900	18.777629760152812800
1.2000	9.527490411879419700	14.084428617905016900
1.4000	7.383664680146235560	11.005851738380386000
1.6000	5.889552334017784220	8.858385874800074330
1.8000	4.799506245931250750	7.290301709168400810
2.0000	3.976672194738063660	6.104153871476119210
2.2000	3.338290221303994660	5.181477567644345860
2.4000	2.831998994119034000	4.447332002919017310
2.6000	2.423088801453758820	3.852210648810926230
2.8000	2.087696090144671060	3.362202349416284710
3.0000	1.808906068875178570	2.953358820877600270
3.2000	1.574420289854091330	2.608325526860350460
3.4000	1.375094424256466660	2.314158395188188780
3.6000	1.204001042690194370	2.061181711922818180
3.8000	1.055797902929259900	1.841870305525759920
4.0000	0.926300795696662571	1.650362380063753460

LAMBDA	OMEGA (P=6 MODE)	OMEGA (P=7 MODE)
1.0000	25.467738030123200600	32.754142031693433500
1.2000	19.168957516886681700	24.712642351512379000
1.4000	15.039590677929982800	19.439369813704285400
1.6000	12.161136645430588000	15.762671957877159900
1.8000	10.059471455407461300	13.079191696480551200
2.0000	8.470268297490264330	11.049919726683331300
2.2000	7.233823344526586930	9.471294339905082180
2.4000	6.249413915126632580	8.214513832394541030
2.6000	5.450582931543062730	7.194550792016014110
2.8000	4.791893868648658740	6.353266606388104480
3.0000	4.241331957985241540	5.649717318975978400
3.2000	3.775746660722400170	5.054338388853206740
3.4000	3.378012864456509720	4.545165435275326570
3.6000	3.035212736699548720	4.105857244407967070
3.8000	2.737425280715521710	3.723756342366369540
4.0000	2.476916703999844090	3.389043470572185690

LAMBDA	OMEGA (P=8 MODE)
1.0000	40.629465277129657600
1.2000	30.697886249731954700
1.4000	24.185997431375284300
1.6000	19.646925116789443400
1.8000	16.333418982145715900
2.0000	13.828645817765947300
2.2000	11.880501548046547500
2.4000	10.329839411058587800
2.6000	9.071571819568198690
2.8000	8.033823073607900160
3.0000	7.165957682358648610
3.2000	6.431391677787967830
3.4000	5.803106046519861620
3.6000	5.260761645784036710
3.8000	4.788783435320783880
4.0000	4.375064459219278220

OMEGA(0) VS LAMBDA



Chapter 4 Solutions for the Order of $O(\epsilon^2)$

In this chapter, we shall discuss the solution to the system at the order $O(\epsilon^2)$. Recall that in a previous chapter, the governing equations for the order $O(\epsilon^2)$ were listed, and they are now repeated here:

The Laplace equation:

$$\nabla^2\Phi^{(2)}(\mu, z, t) = 0, \ (0 \le \mu \le 1, \ -\Lambda \le z \le \Lambda). \tag{4.1}$$

Condition on radial velocity required for axisymmetry:

$$\frac{\partial\Phi^{(2)}}{\partial\mu} = 0, \ (\mu = 0, \ -\Lambda \le z \le \Lambda). \tag{4.2a}$$

Kinematic condition:

$$-\omega^{(0)}\frac{\partial F^{(2)}}{\partial t} + \frac{\partial\Phi^{(2)}}{\partial\mu} = 2\omega^{(1)}\frac{\partial F^{(1)}}{\partial t} - 2F^{(1)}\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} + 2\frac{\partial\Phi^{(1)}}{\partial z}\frac{\partial F^{(1)}}{\partial z}, \ (\mu = 1). \tag{4.2b}$$

Normal force balance:

$$\omega^{(0)}\frac{\partial\Phi^{(2)}}{\partial t} - F^{(2)} - \frac{\partial^2 F^{(2)}}{\partial z^2}$$

$$= -2\omega^{(1)}\frac{\partial \Phi^{(1)}}{\partial t} - 2\omega^{(0)}F^{(1)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial t} - \left(\frac{\partial \Phi^{(1)}}{\partial \mu}\right)^2 - \left(\frac{\partial \Phi^{(1)}}{\partial z}\right)^2$$

$$-2(F^{(1)})^2 + \left(\frac{\partial F^{(1)}}{\partial z}\right)^2 + BN^{(2)}, \ (\mu = 1). \tag{4.2c}$$

Periodicity in time:

$$\nabla \Phi^{(2)}(\mu, z, t + 2\pi) = \nabla \Phi^{(2)}(\mu, z, t). \tag{4.2d}$$

Zero normal velocity at end disks:

$$\left.\frac{\partial \Phi^{(2)}}{\partial z}\right|_{z=\pm\Lambda} = 0. \tag{4.2e}$$

Conservation of mass condition:

$$\int_{-\Lambda}^{\Lambda} \left[F^{(2)} + \left(F^{(1)}\right)^2\right] dz = 0. \tag{4.2f}$$

Triple contact line condition:

$$F^{(2)}(\pm\Lambda, t) = 0. \tag{4.2g}$$

In general, a solution of $\Phi^{(2)}(\mu, z, t)$ would be solved by using equations (4.1), (4.2a) and (4.2e). Consider the Laplace equation (4.1). In this order $O(\epsilon^2)$, there are nonlinear forcing terms involved. In the process of solving $\Phi^{(2)}(\mu, z, t)$, it is recognized that an additional potential term without z-dependence, denoted by $\phi_M^{(2)}(\mu, t)$, should be added to balance the system. Specifically, the form of the nonlinear forcing terms in the kinematic condition together with the requirements of the conservation of mass condition requires an additional contribution to the function $\Phi^{(2)}(\mu, z, t)$.

Therefore we may assume that $\Phi^{(2)}(\mu,z,t)$ to have the following form.

$$\Phi^{(2)}(\mu,z,t) = \phi^{(2)}(\mu,z,t) + \phi_M^{(2)}(\mu,t).$$

It is remarked that $\phi_M^{(2)}(\mu,t)$ (as well as $\phi^{(2)}$) must satisfy any conditions required on $\Phi^{(2)}(\mu,z,t)$.

Thus (4.1) becomes

$$\nabla^2\phi^{(2)}(\mu,z,t) = 0 \tag{4.3}$$

$$\nabla^2\phi_M^{(2)}(\mu,t) = 0 \tag{4.4}$$

Equation (4.3) can be solved, in a similar way to the case of $O(\epsilon)$, with the frequency of t-dependence being "2".

$$\phi^{(2)}(\mu,z,t) = \sin(2t)\sum_{m=0}^{\infty}\gamma_m I_0(l_m\mu)\cos l_m(z+\Lambda),\ (0\le\mu\le 1,\ -\Lambda\le z\le\Lambda). \tag{4.5}$$

To solve (4.4), we let $\phi_M^{(2)}(\mu,t) = T(t)\tilde{\phi}_M^{(2)}(\mu)$. Then (4.4) yields

$$\frac{\partial^2\tilde{\phi}_M^{(2)}}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial\tilde{\phi}_M^{(2)}}{\partial\mu} = 0. \tag{4.6}$$

Integrate both sides of (4.6) to obtain

$$\ln\left(\frac{\partial\tilde{\phi}_M^{(2)}}{\partial\mu}\right) = -ln\mu + \text{Const.},$$

which implies that

$$\frac{\partial\tilde{\phi}_M^{(2)}}{\partial\mu} = \frac{1}{\mu}\text{Const.}$$

By integrating both sides of the equality above, we obtain the solution of (4.6):

$$\tilde{\phi}_M^{(2)} = E_1 \ln \mu, \tag{4.7}$$

where E_1 is a constant. Set $T(t) = \sin 2t$. Then the solution of (4.4) is

$$\phi_M^{(2)} = \sin(2t) E_1 \ln \mu, \tag{4.8}$$

Physically. this is a source term correction to $\Phi^{(2)}(\mu, z, t)$. Therefore, the solution form for $\Phi^{(2)}$ is:

$$\Phi^{(2)}(\mu, z, t) = \sin(2t) \left\{ \sum_{m=0}^{\infty} \gamma_m I_0(l_m \mu) \cos l_m(z + \Lambda) + E_1 \ln \mu \right\}. \tag{4.9}$$

Considering the form of the nonlinear terms of $O(\epsilon^2)$, in the kinematic and normal force conditions, it may be assumed that the appropriate form of the solution for $F^{(2)}$ has a time-dependent term and a steady state (time-independent) term. The time-independent term will balance certain the nonlinear terms in the normal force balance at this order. Thus, by using similar techniques in solving this problem as were used in the linear order, the appropriate solution form for $F^{(2)}$ is assumed as follows:

$$F^{(2)}(z, t) = \cos(2t) \sum_{m=0}^{\infty} \delta_m \cos l_m(z + \Lambda) + \sum_{m=0}^{\infty} \hat{\delta}_m \cos l_m(z + \Lambda). \tag{4.10}$$

Let

$$F^{(1)}(z, t) = \cos t \tilde{F}^{(1)}(z) \text{ and } \Phi^{(1)}(\mu, z, t) = \sin t \tilde{\Phi}^{(1)}(\mu, z). \tag{4.11}$$

Substitute these expressions in (4.11) into the kinematic condition (4.2b) to get

$$2\omega^{(0)}\sin 2t\tilde{F}^{(2)} + \sin 2t\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu} = -2\omega^{(1)}\sin t\tilde{F}^{(1)} - \sin 2t\tilde{F}^{(1)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} + \sin 2t\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(1)}}{\partial z}. \tag{4.12}$$

Substitute these expressions in (4.11) into the normal force equation (4.2c) and then partially differentiate both sides with respect to t to get

$$-4\omega^{(0)}\sin 2t\tilde{\Phi}^{(2)} + \sin 2t\tilde{F}^{(2)} + 2\sin 2t\frac{\partial^2\tilde{F}^{(2)}}{\partial z^2}$$

$$= 2\omega^{(1)}\sin t\tilde{\Phi}^{(1)} + 2\omega^{(0)}\sin 2t\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu} - \sin 2t\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\right)^2$$

$$-\sin 2t\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\right)^2 + \sin 2t(\tilde{F}^{(1)})^2 - \sin 2t\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2. \tag{4.13}$$

Combine (4.12) and (4.13) to get

$$-4\omega^{(0)}\sin 2t\tilde{\Phi}^{(2)} - \sin 2t\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu} - \sin 2t\frac{\partial^2}{\partial z^2}\left(\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}\right)$$

$$= 2\omega^{(1)}\sin t\tilde{\Phi}^{(1)} + \sin 2t\tilde{F}^{(1)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \sin 2t\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(1)}}{\partial z}$$

$$+2\omega^{(1)}\sin t\frac{\partial^2\tilde{F}^{(1)}}{\partial z^2} + \sin 2t(\tilde{F}^{(1)})^2 - \sin 2t\frac{\partial^2}{\partial z^2}\left(\tilde{F}^{(1)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}\right)$$

$$+\sin 2t\frac{\partial^2}{\partial z^2}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)$$

$$+2\omega^{(1)}\sin t\tilde{\Phi}^{(1)} + 2\omega^{(0)}\sin 2t\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu} - \sin 2t\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\right)^2$$

$$- \sin 2t \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial z}\right)^2 + \sin 2t (\tilde{F}^{(1)})^2 - \sin 2t \left(\frac{\partial \tilde{F}^{(1)}}{\partial z}\right)^2 . \qquad (4.14)$$

Rewriting (4.14) gives

$$\begin{aligned}
0 \quad = \quad & \sin t \omega^{(1)} \left[4\tilde{\Phi}^{(1)} + 2\frac{\partial^2 \tilde{F}^{(1)}}{\partial z^2} \right] \\
& + \sin 2t \left\{ 4\omega^{(0)}\tilde{\Phi}^{(2)} + \frac{\partial \tilde{\Phi}^{(2)}}{\partial \mu} + \frac{\partial^2}{\partial z^2}\left(\frac{\partial \tilde{\Phi}^{(2)}}{\partial \mu}\right) \right. \\
& \tilde{F}^{(1)}\frac{\partial^2 \tilde{\Phi}^{(1)}}{\partial \mu^2} - \frac{\partial \tilde{\Phi}^{(1)}}{\partial z}\frac{\partial \tilde{F}^{(1)}}{\partial z} \\
& + (\tilde{F}^{(1)})^2 - \frac{\partial^2}{\partial z^2}\left(\tilde{F}^{(1)}\frac{\partial^2 \tilde{\Phi}^{(1)}}{\partial \mu^2}\right) \\
& + \frac{\partial^2}{\partial z^2}\left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial z}\frac{\partial \tilde{F}^{(1)}}{\partial z}\right) + 2\omega^{(0)}\tilde{F}^{(1)}\frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu} \\
& \left. - \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu}\right)^2 - \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial z}\right)^2 + (\tilde{F}^{(1)})^2 - \left(\frac{\partial \tilde{F}^{(1)}}{\partial z}\right)^2 \right\} \quad ; \quad ; \qquad (4.15)
\end{aligned}$$

By (4.15), in order to vanish the secular term, the coeficient of $\sin t$ has to be zero.

Since

$$4\tilde{\Phi}^{(1)} + 2\frac{\partial^2 \tilde{F}^{(1)}}{\partial z^2} \neq 0, \qquad (4.16)$$

we must have

$$\omega^{(1)} = 0. \qquad (4.17)$$

Applying the triple contact line condition (4.2g) for the order of $O(\epsilon^2)$, and by

(4.10), one obtains, when $z = \Lambda$

$$F^{(2)}(\Lambda, t) = \cos(2t) \sum_{m=0}^{\infty} \delta_m \cos m\pi + \sum_{m=0}^{\infty} \hat{\delta}_m \cos m\pi = 0, \tag{4.18}$$

and when $z = -\Lambda$

$$F^{(2)}(-\Lambda, t) = \cos(2t) \sum_{m=0}^{\infty} \delta_m + \sum_{m=0}^{\infty} \hat{\delta}_m = 0. \tag{4.19}$$

Combine (4.18) and (4.19) to get

$$\cos(2t) \sum_{k=0}^{\infty} \delta_{2k} + \sum_{k=0}^{\infty} \hat{\delta}_{2k} = 0. \tag{4.20}$$

Subtract (4.19) from (4.18) to get

$$\cos(2t) \sum_{k=1}^{\infty} \delta_{2k-1} + \sum_{k=1}^{\infty} \hat{\delta}_{2k-1} = 0. \tag{4.21}$$

Since (4.20) and (4.21) are valid for all t, we must have the following systems.

$$\begin{cases} \sum_{k=0}^{\infty} \delta_{2k} = 0 \\ \sum_{k=1}^{\infty} \delta_{2k-1} = 0, \end{cases} \tag{4.22a}$$

and

$$\begin{cases} \sum_{k=0}^{\infty} \hat{\delta}_{2k} = 0 \\ \sum_{k=1}^{\infty} \hat{\delta}_{2k-1} = 0. \end{cases} \tag{4.22b}$$

By the conservation of mass condition (4.2f), and by (4.10)

$$\int_{-\Lambda}^{\Lambda}\left[\cos(2t)\sum_{m=0}^{\infty}\delta_m\cos l_m(z+\Lambda)+\sum_{m=0}^{\infty}\hat{\delta}_m\cos l_m(z+\Lambda)+\frac{\cos 2t}{2}\left(\tilde{F}^{(1)}\right)^2\right.$$

$$\left.+\frac{\left(\tilde{F}^{(1)}\right)^2}{2}\right]dz=0,$$

which is valid for all t. It follows that both

$$\int_{-\Lambda}^{\Lambda}\left[\sum_{m=0}^{\infty}\delta_m\cos l_m(z+\Lambda)+\frac{1}{2}\left(\tilde{F}^{(1)}\right)^2\right]dz=0,$$

and

$$\int_{-\Lambda}^{\Lambda}\left[\sum_{m=0}^{\infty}\tilde{\delta}_m\cos l_m(z+\Lambda)+\frac{1}{2}\left(\tilde{F}^{(1)}\right)^2\right]dz=0.$$

Hence, we have the following conclusion.

$$\delta_0=\hat{\delta}_0=-\frac{1}{4\Lambda}\int_{-\Lambda}^{\Lambda}(\tilde{F}^{(1)})^2dz. \tag{4.23}$$

Apply the kinematic condition to obtain

$$2\omega^{(0)}\sum_{m=0}^{\infty}\delta_m\cos l_m(z+\Lambda)+\sum_{m=0}^{\infty}\gamma_m l_m I_0'(l_m)\cos l_m(z+\Lambda)+E_1$$

$$=2\omega^{(1)}\frac{\partial\tilde{F}^{(1)}}{\partial t}-\tilde{F}^{(1)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}+\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(1)}}{\partial z}, \tag{4.24}$$

where $\omega^{(1)}=0$.

Using the orthogonal properties to eliminate the z-dependence, one obtains

$$E_1=-2\omega^{(0)}\delta_0-\frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda}\left[\tilde{F}^{(1)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}-\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(1)}}{\partial z}\right]dz, \tag{4.25}$$

and for $m = 1, 2, \cdots$,

$$\Lambda l_m I_0'(l_m)\gamma_m + 2\omega^{(0)}\Lambda\delta_m$$

$$= -\int_{-\Lambda}^{\Lambda} \tilde{F}^{(1)} \frac{\partial^2 \tilde{\Phi}^{(1)}}{\partial \mu^2} \cos l_m(z+\Lambda)dz + \int_{-\Lambda}^{\Lambda} \frac{\partial \tilde{\Phi}^{(1)}}{\partial z} \frac{\partial \tilde{F}^{(1)}}{\partial z} \cos l_m(z+\Lambda)dz. \quad (4.26)$$

It is noted that the use of the orthogonal properties involves multiplication through by $\cos l_q(z+\Lambda)$ and integration over the range $(-\Lambda, \Lambda)$, with the appropriate (integer) q value.

By the normal force balance condition, we have

$$\omega^{(0)} \frac{\partial \Phi^{(2)}}{\partial t} - F^{(2)} - \frac{\partial^2 F^{(2)}}{\partial z^2} = -2\omega^{(1)} \frac{\partial \Phi^{(1)}}{\partial t} - 2\omega^{(0)} F^{(1)} \frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial t} - \left(\frac{\partial \Phi^{(1)}}{\partial \mu}\right)^2$$

$$- \left(\frac{\partial \Phi^{(1)}}{\partial z}\right)^2 - 2[F^{(1)}]^2 + \left(\frac{\partial F^{(1)}}{\partial z}\right)^2 + BN^{(2)}. \quad (4.27)$$

For the normal force condition without t-dependence, the orthogonal properties are used to obtain

$$BN^{(2)} = \hat{\delta}_0 - \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} \left[\omega^{(0)} \tilde{F}^{(1)} \frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu} + \frac{1}{2} \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu}\right)^2 \right.$$

$$\left. + \frac{1}{2} \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial z}\right)^2 + (\tilde{F}^{(1)})^2 - \frac{1}{2} \left(\frac{\partial \tilde{F}^{(1)}}{\partial \mu}\right)^2 \right] dz \quad (4.28)$$

and for $m = 1, 2, \cdots$,

$$\hat{\delta}_m = \frac{1}{\Lambda(1 - l_m^2)} \int_{-\Lambda}^{\Lambda} \left[\omega^{(0)} \tilde{F}^{(1)} \frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu} + \frac{1}{2} \left(\frac{\partial \tilde{\Phi}^{(1)}}{\partial \mu}\right)^2 \right.$$

$$+\frac{1}{2}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\right)^2 + (\tilde{F}^{(1)})^2 - \frac{1}{2}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\Bigg] \cos l_m(z+\Lambda)dz. \tag{4.29}$$

For the normal force condition with t-dependence, the orthogonal properties are used to obtain

$$\gamma_0 = \frac{1}{4\omega^{(0)}\Lambda}\int_{-\Lambda}^{\Lambda}\Bigg[-\omega^{(0)}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu} + \frac{1}{2}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\right)^2$$

$$+\frac{1}{2}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\right)^2 - \frac{3}{2}(\tilde{F}^{(1)})^2 + \frac{1}{2}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\Bigg] dz \tag{4.30}$$

and for $m = 1, 2, \cdots$,

$$2\Lambda\omega^{(0)}I_0(l_m)\gamma_m - \Lambda(1-l_m^2)\delta_m$$

$$= -\omega^{(0)}\int_{-\Lambda}^{\Lambda}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\cos l_m(z+\Lambda)dz + \frac{1}{2}\int_{-\Lambda}^{\Lambda}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\right)^2\cos l_m(z+\Lambda)dz$$

$$+\frac{1}{2}\int_{-\Lambda}^{\Lambda}\left(\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\right)^2\cos l_m(z+\Lambda)dz$$

$$-\int_{-\Lambda}^{\Lambda}(\tilde{F}^{(1)})^2\cos l_m(z+\Lambda)dz + \frac{1}{2}\int_{-\Lambda}^{\Lambda}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\cos l_m(z+\Lambda)dz. \tag{4.31}$$

By the Lanzcos Tau method and using (4.31), (4.26) and (4.21a), a set of linear algebraic nonhomogeneous equations in δ_m and γ_m are developed. Using techniques of numerical linear algebra, the truncated system can be solved for γ_m and δ_m for each $m \in \{1, 2, \cdots, M\}$. Using (4.29) and (4.21b), $\hat{\delta}_m$ can be solved also.

Knowing of the γ_m and δ_m can then be used to construct $\Phi^{(2)}(\mu, z, t)$ and $F^{(2)}(z, t)$. It is the shape function $F^{(2)}$ which is of most interest at second order, for it represents the first nonlinear correction to the deformed interface shape of the finite length liquid column in natural harmonic oscillation. Also, the steady state correction to $F^{(2)}$ indicates a modification of the mean form.

Chapter 5 Results at Second Order of $O(\epsilon^2)$: The Shape Function and Velocity Potential Corrections

In this chapter, some numerical results at the second order of $O(\epsilon^2)$ are displayed. Results for the first six modes ($p = 2,3,4,5,6,7$) are presented. For each of these modes, the shape function $F^{(2)}$ in $O(\epsilon^2)$ is computed The initial parameters $\alpha^{(1)}$ and $\beta^{(1)}$ are so chosen that either $\alpha^{(1)} = 1$ and $\beta^{(1)} = 0$, or $\alpha^{(1)} = 0$ and $\beta^{(1)} = 1$. The value of ϵ is set to be 0.4. With these values of the parameters, the corrected deformation up to $O(\epsilon^2)$ is plotted.

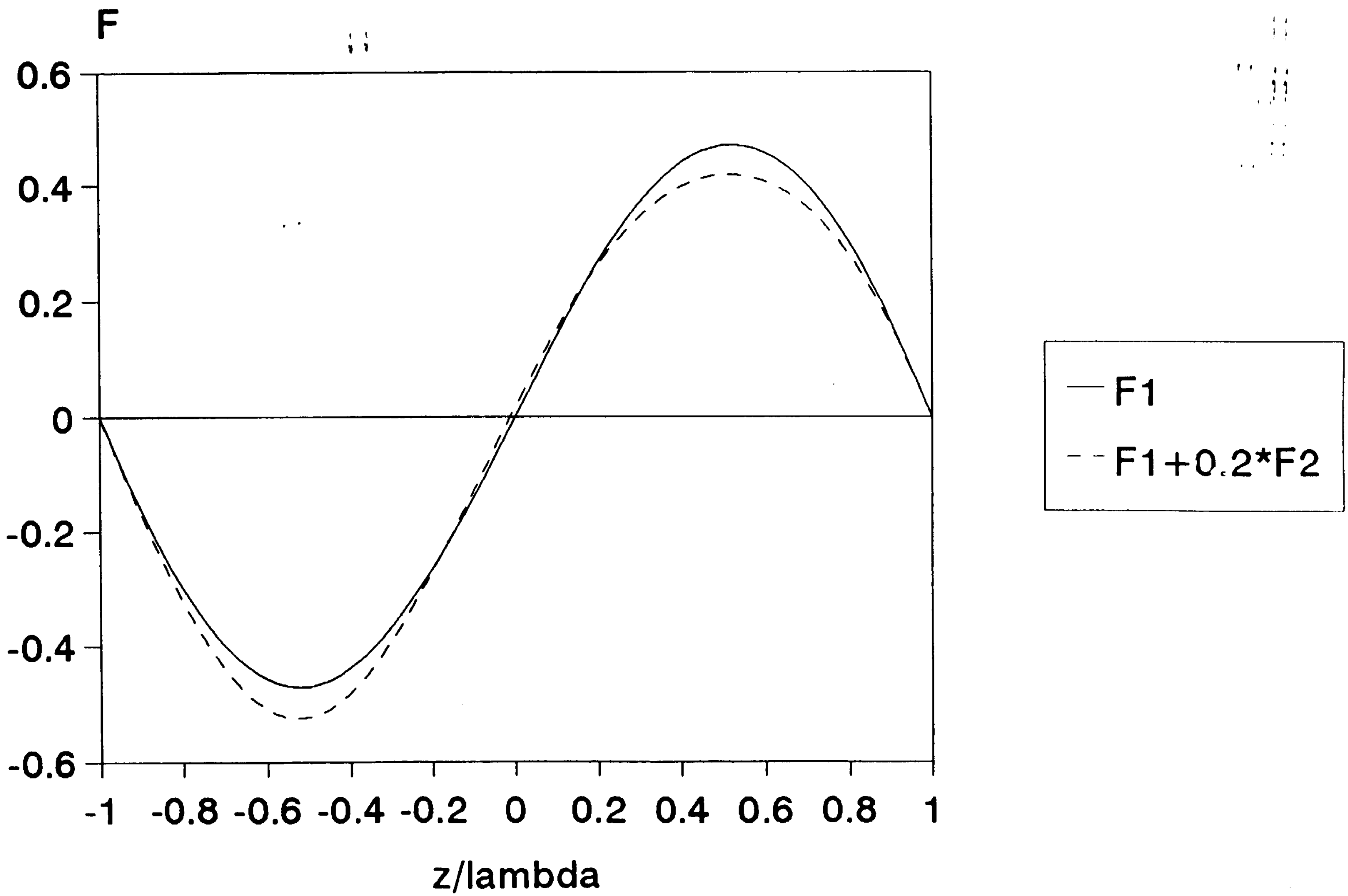
Note that

$$\begin{aligned} F(z,t,;\epsilon) &= F^{(0)}(z,t) + \epsilon F^{(1)}(z,t) + \frac{\epsilon^2}{2} F^{(2)}(z,t) \\ &= 1 + \epsilon F^{(1)}(z,t) + \frac{\epsilon^2}{2} F^{(2)}(z,t) \qquad (5.1) \end{aligned}$$

The perturbation contribution to F from both order $O(\epsilon)$ and $O(\epsilon^2)$ is graphed for various modes and Λ values.

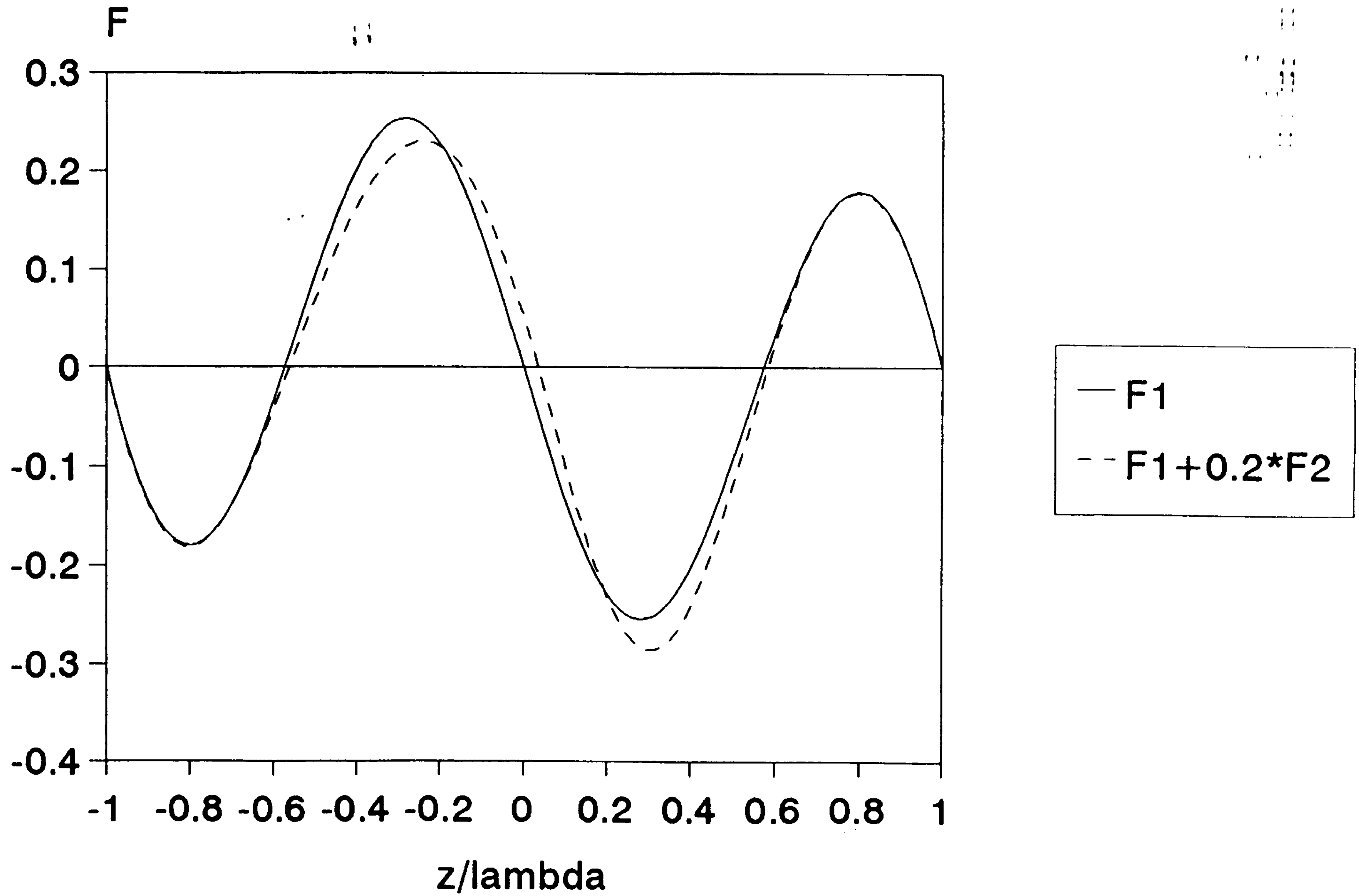
FREE SURFACE PROFILES: LINEAR AND SECOND ORDER CORRECTED

lambda=2.4, alpha1=0, beta1=1.0, p=2(mode), time=2*pi



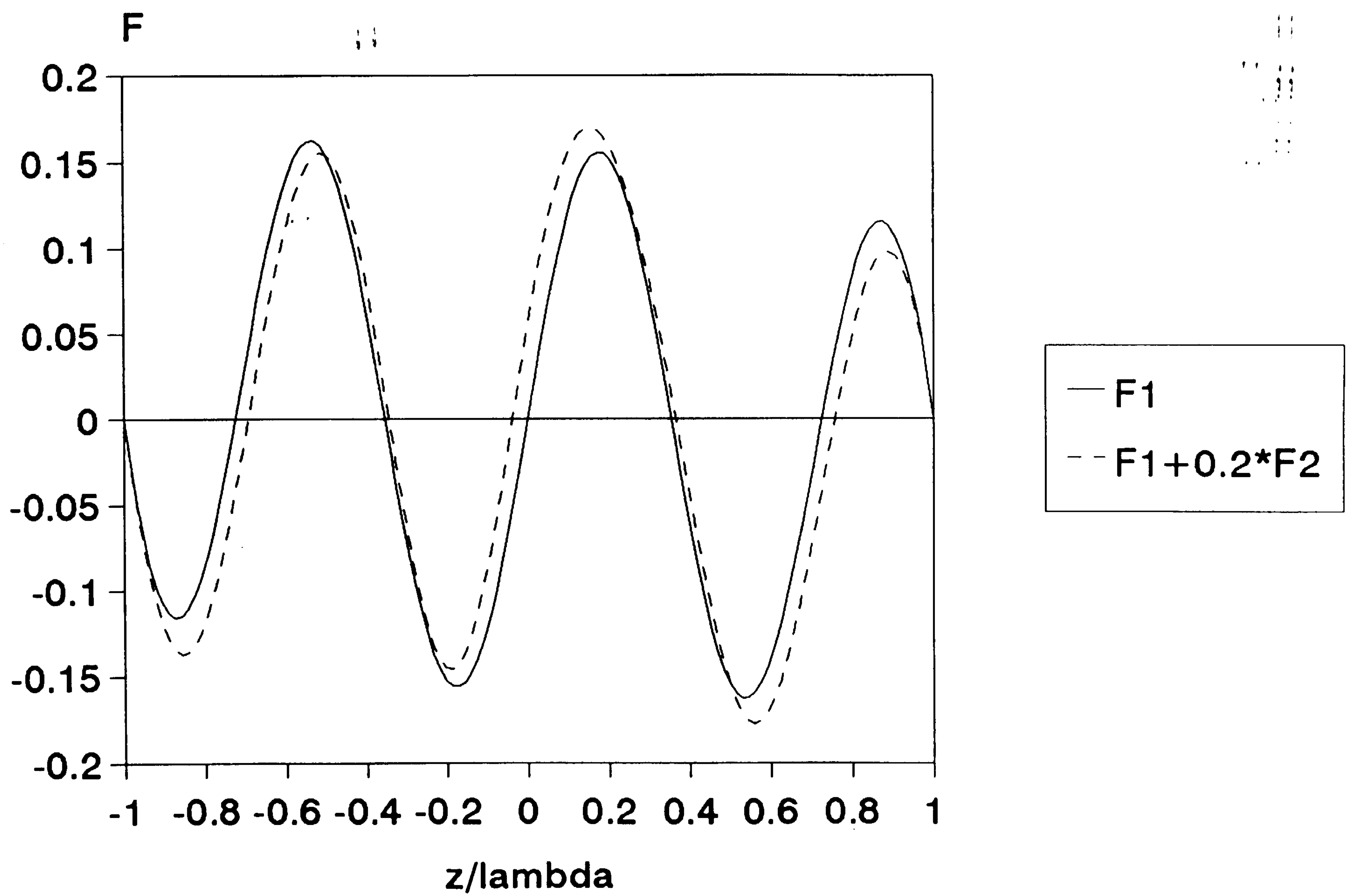
FREE SURFACE PROFILES: LINEAR AND SECOND ORDER CORRECTED

lambda=2.4, alpha1=0, beta1=1.0, p=4(mode), time=2*pi



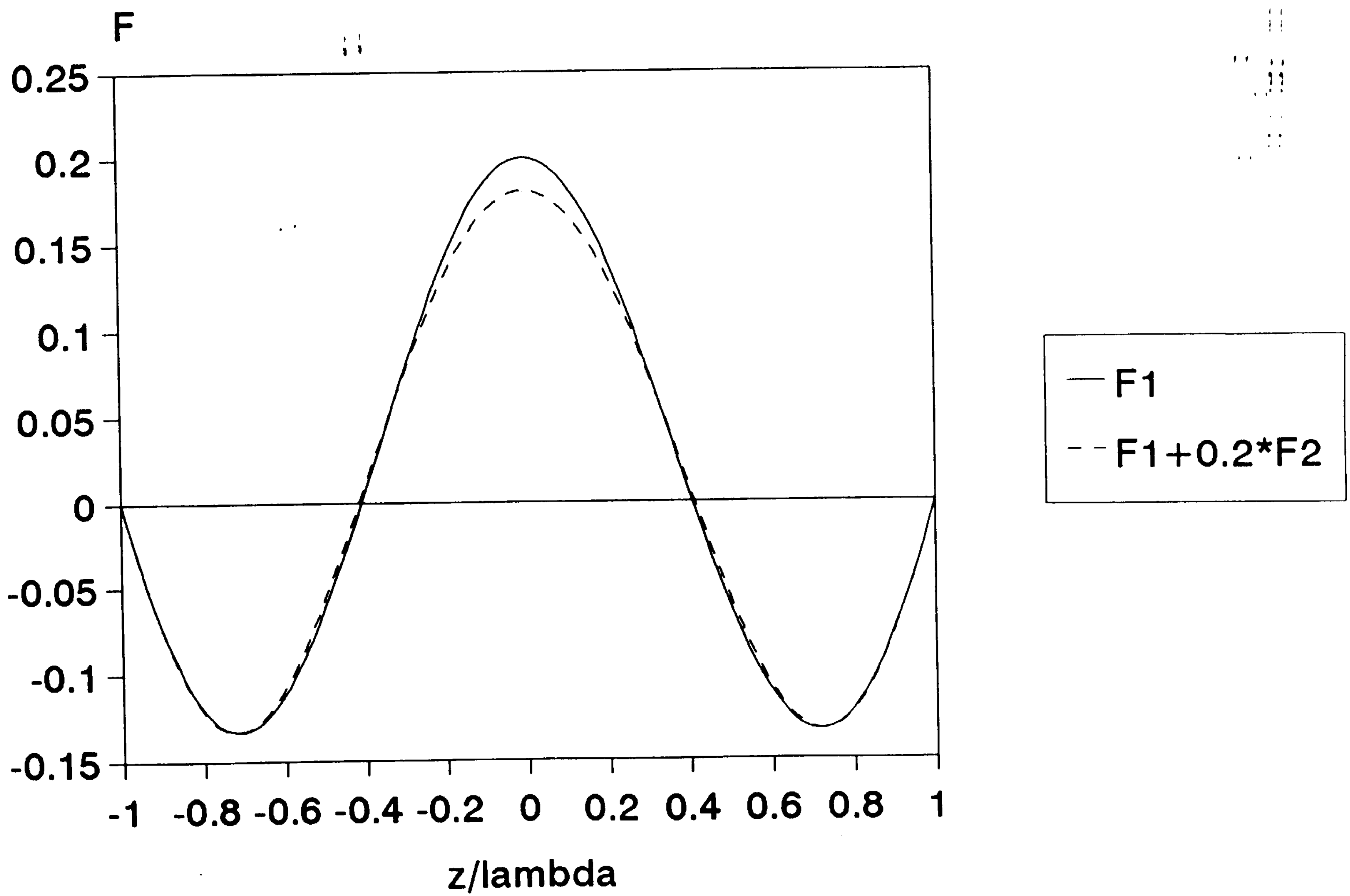
FREE SURFACE PROFILES: LINEAR AND SECOND ORDER CORRECTED

lambda=2.4, alpha1=0, beta1=1.0, p=6(mode), time=2*pi



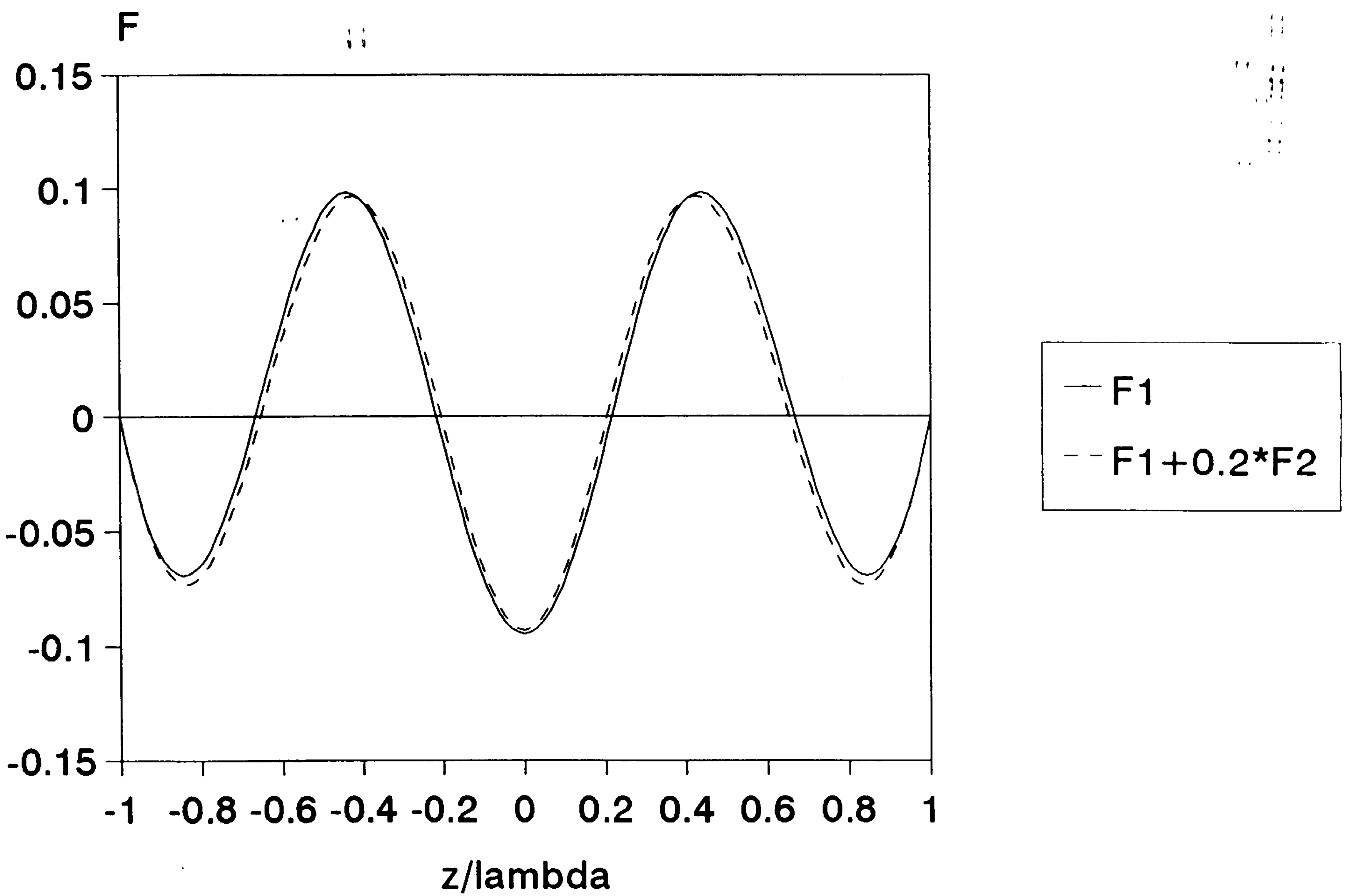
FREE SURFACE PROFILES: LINEAR AND SECOND ORDER CORRECTED

lambda=3.4, alpha1=1.0, beta1=0, p=3(mode), time=2*pi



FREE SURFACE PROFILES: LINEAR AND SECOND ORDER CORRECTED

lambda=3.4, alpha1=1.0, beta1=0, p=5(mode), time=2*pi



Chapter 6 Derivation of the Solvability Condition at Order of $O(\epsilon^3)$

The nonlinear correction to the interface shape has been determined at order ϵ^2, and the forms of the theoretical solutions have been presented.

However, it remains to investigate the nonlinear corrections to the interface for various families of parameters.

Of interest is how the shape is modified by nonlinear corrections for various values of the slenderness parameter, for an even or odd number of half-wave deformations (at the linear order), for higher order modes in general, and for various forms of the initial disturbance ($\alpha^{(1)}$ and $\beta^{(1)}$ values).

The third order (in ϵ) system has been listed in Chapter 2. It is repeated below. In this order, the corrections to the time frequency, $\omega^{(2)}$, will be solved.

The system equations are:

The Laplace equation:

$$\nabla^2 \Phi^{(3)}(\mu, z, t) = 0, \quad (0 \le \mu \le 1, \ -\Lambda \le z \le \Lambda). \tag{6.1}$$

Radial velocity condition:

$$\frac{\partial \Phi^{(3)}}{\partial \mu} = 0, \quad (\mu = 0, \ -\Lambda \le z \le \Lambda). \tag{6.2a}$$

Kinematic condition:

$$\begin{aligned}
-\omega^{(0)}\frac{\partial F^{(3)}}{\partial t} + \frac{\partial \Phi^{(3)}}{\partial \mu} \quad = \quad & 3\omega^{(2)}\frac{\partial F^{(1)}}{\partial t} + 3\omega^{(1)}\frac{\partial F^{(2)}}{\partial t} - 3F^{(1)}\frac{\partial^2 \Phi^{(2)}}{\partial \mu^2} - 3F^{(2)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu^2} \\
& -3(F^{(1)})^2\frac{\partial^3 \Phi^{(1)}}{\partial \mu^3} + 3\frac{\partial \Phi^{(2)}}{\partial z}\frac{\partial F^{(1)}}{\partial z} \\
& +6F^{(1)}\frac{\partial F^{(1)}}{\partial z}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial z} + 3\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial F^{(2)}}{\partial z}, \quad (\mu = 1).
\end{aligned} \tag{6.2b}$$

Normal force equation:

$$\begin{aligned}
\omega^{(0)}\frac{\partial \Phi^{(3)}}{\partial t} - F^{(3)} - \frac{\partial^2 F^{(3)}}{\partial z^2} \quad = \quad & -3\omega^{(2)}\frac{\partial \Phi^{(1)}}{\partial t} - 3\omega^{(1)}\frac{\partial \Phi^{(2)}}{\partial t} - 6\omega^{(1)}F^{(1)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial t} \\
& -3\omega^{(0)}F^{(1)}\frac{\partial^2 \Phi^{(2)}}{\partial \mu \partial t} - 3\omega^{(0)}F^{(2)}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial t} \\
& -3\omega^{(0)}(F^{(1)})^2\frac{\partial^3 \Phi^{(1)}}{\partial \mu^2 \partial t} - 3\frac{\partial \Phi^{(1)}}{\partial \mu}\frac{\partial \Phi^{(2)}}{\partial \mu} \\
& -6F^{(1)}\frac{\partial \Phi^{(1)}}{\partial \mu}\frac{\partial^2 \Phi^{(1)}}{\partial \mu^2} - 3\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial \Phi^{(2)}}{\partial z} \\
& -6F^{(1)}\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial^2 \Phi^{(1)}}{\partial \mu \partial z} - 6F^{(1)}F^{(2)}
\end{aligned}$$

$$+6(F^{(1)})^3 - 3F^{(1)}\left(\frac{\partial F^{(1)}}{\partial z}\right)^2 + 3\frac{\partial F^{(1)}}{\partial z}\frac{\partial F^{(2)}}{\partial z}$$

$$-9\left(\frac{\partial F^{(1)}}{\partial z}\right)^2 \frac{\partial^2 F^{(1)}}{\partial z^2} + BN^{(3)}, \quad (\mu = 1). \tag{6.2c}$$

Periodicity in time:

$$\nabla\Phi^{(3)}(\mu, z, t + 2\pi) = \nabla\Phi(\mu, z, t). \tag{6.2d}$$

Normal velocity:

$$\left.\frac{\partial \Phi^{(3)}}{\partial z}\right|_{z=\pm\Lambda} = 0. \tag{6.2e}$$

Conservation of mass:

$$\int_{-\Lambda}^{\Lambda}\left[\frac{1}{3}F^{(3)} + F^{(1)}F^{(2)}\right]dz = 0. \tag{6.2f}$$

Triple contact line condition:

$$F^{(3)}(\pm\Lambda, t) = 0. \tag{6.2g}$$

Assume the appropriate solution forms of $\Phi^{(3)}$ and $F^{(3)}$ as

$$\Phi^{(3)}(\mu, z, t) = \sum_{m=0}^{\infty} \gamma_m^{(3)}(t) I_0(l_m\mu)\cos l_m(z + \Lambda), \tag{6.3}$$

$$F^{(3)}(z, t) = \sum_{m=0}^{\infty} \delta_m^{(3)}(t)\cos l_m(z + \Lambda). \tag{6.4}$$

This is consisted with all boundary conditions.

Let

$$\Phi^{(1)}(\mu, z, t) = \sin t\ \tilde{\Phi}^{(1)}(\mu, z),$$

$$\Phi^{(2)}(\mu, z, t) = \sin 2t\ \left[\tilde{\Phi}^{(2)}(\mu, z) + E_1 \ln \mu\right],$$

$$F^{(1)}(z, t) = \cos t\ \tilde{F}^{(1)}(z),$$

$$F^{(2)}(z, t) = \cos 2t\ \tilde{F}^{(2)}(z) + \tilde{\tilde{F}}^{(2)}(z).$$

Then substitute (6.3) and (6.4) into the Kinematic condition (6.2b) to get

$$-\omega^{(0)} \sum_{m=0}^{\infty} \frac{d\delta_m^{(3)}(t)}{dt} \cos l_m(z + \Lambda) + \sum_{m=0}^{\infty} \gamma_m^{(3)}(t) l_m I_0'(l_m) \cos l_m(z + \Lambda)$$

$$= \sin t[KH_1] + \sin 3t[KH_3], \tag{6.5}$$

where

$$[KH_1] = -3\omega^{(2)}\tilde{F}^{(1)} - \frac{3}{2}\tilde{F}^{(1)}\left[\frac{\partial^2\tilde{\Phi}^{(2)}}{\partial\mu^2} - E_1\right] + \frac{3}{2}\tilde{F}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}$$

$$-3\tilde{\tilde{F}}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \frac{3}{4}(\tilde{F}^{(1)})^2\frac{\partial^3\tilde{\Phi}^{(1)}}{\partial\mu^3} + \frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}$$

$$+\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z} - \frac{3}{2}\frac{\partial\tilde{F}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z} + 3\frac{\partial\tilde{\tilde{F}}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}, \tag{6.6}$$

and where

$$[KH_3] = -\frac{3}{2}\tilde{F}^{(1)}\left[\frac{\partial^2\tilde{\Phi}^{(2)}}{\partial\mu^2} - E_1\right] - \frac{3}{2}\tilde{F}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \frac{3}{4}(\tilde{F}^{(1)})^2\frac{\partial^3\tilde{\Phi}^{(1)}}{\partial\mu^3}$$

$$+\frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z} + \frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z} + \frac{3}{2}\frac{\partial\tilde{F}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}. \tag{6.7}$$

Substitute (6.3) and (6.4) into the normal force balance equation (6.2c) to get

$$\omega^{(0)}\sum_{m=0}^{\infty}\frac{d\gamma_m^{(3)}(t)}{dt}I_0(l_m)\cos l_m(z+\Lambda)-\sum_{m=0}^{\infty}(1-l_m^2)\delta_m^{(3)}(t)\cos l_m(z+\Lambda)$$

$$= \cos t[NH_1]+\cos 3t[KH_3]+BN^{(3)} \tag{6.8}$$

where

$$\begin{aligned}[NH_1] \;=\; & -3\omega^{(2)}\tilde{\Phi}^{(1)}-3\omega^{(0)}\tilde{F}^{(1)}\left[\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right]-\frac{3}{2}\omega^{(0)}\tilde{F}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\\
& -3\omega^{(0)}\tilde{\tilde{F}}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}-\frac{9}{4}\omega^{(0)}(\tilde{F}^{(1)})^2\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}-\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\left[\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right]\\
& -\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}-\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}-\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z}\\
& -3\tilde{F}^{(1)}\tilde{F}^{(2)}-6\tilde{F}^{(1)}\tilde{\tilde{F}}^{(2)}+\frac{9}{2}(\tilde{F}^{(1)})^3-\frac{9}{4}\tilde{F}^{(1)}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\\
& +\frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(2)}}{\partial z}+3\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\tilde{F}}^{(2)}}{\partial z}-\frac{27}{4}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\frac{\partial^2\tilde{F}^{(1)}}{\partial z^2},\end{aligned} \tag{6.9}$$

and where

$$\begin{aligned}[NH_3] \;=\; & -3\omega^{(0)}\tilde{F}^{(1)}\left[\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right]-\frac{3}{2}\omega^{(0)}\tilde{F}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}-\frac{3}{4}\omega^{(0)}(\tilde{F}^{(1)})^2\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}\\
& +\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\left[\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right]+\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}+\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}\\
& +\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z}-3\tilde{F}^{(1)}\tilde{F}^{(2)}+\frac{3}{2}(\tilde{F}^{(1)})^3-\frac{3}{4}\tilde{F}^{(1)}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\\
& +\frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(2)}}{\partial z}-\frac{9}{4}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\frac{\partial^2\tilde{F}^{(1)}}{\partial z^2}.\end{aligned} \tag{6.10}$$

Taking derivative with respect to t in the normal force balance equation, we obtain

$$\omega^{(0)} \sum_{m=0}^{\infty} \frac{d^2\gamma_m^{(3)}(t)}{dt^2} I_0(l_m) \cos l_m(z+\Lambda) - \sum_{m=0}^{\infty} (1-l_m^2) \frac{d\delta_m^{(3)}(t)}{dt} \cos l_m(z+\Lambda)$$

$$= \; -\sin t[NH_1] - 3\sin 3t[KH_3] + \frac{\partial BN^{(3)}}{\partial t}. \qquad (6.11)$$

Apply the orthogonal properties to the kinematic condition and to the normal force balance equation to eliminate the z-dependence.

Thus the kinematic condition becames the following: for $m = 0$,

$$-2\Lambda\omega^{(0)} \frac{d\delta_0^{(3)}(t)}{dt} + 0 = \sin t \int_{-\Lambda}^{\Lambda} [KH_1] dz + \sin 3t \int_{-\Lambda}^{\Lambda} [KH_3] dz, \qquad (6.12)$$

and for $m \geq 1$,

$$-\Lambda\omega^{(0)} \frac{d\delta_m^{(3)}(t)}{dt} + \Lambda l_m I_0'(l_m) \gamma_m^{(3)}(t)$$

$$= \; \sin t \int_{-\Lambda}^{\Lambda} [KH_1] \cos l_m(z+\Lambda) dz + \sin 3t \int_{-\Lambda}^{\Lambda} [KH_3] \cos l_m(z+\Lambda) dz. \qquad (6.13)$$

And the normal force balance equation becomes the following: for $m = 0$,

$$2\Lambda\omega^{(0)} \frac{d^2\gamma_0^{(3)}(t)}{dt^2} - 2\Lambda \frac{d\delta_0^{(3)}(t)}{dt}$$

$$= \; -\sin t \int_{-\Lambda}^{\Lambda} [NH_1] dz - 3\sin 3t \int_{-\Lambda}^{\Lambda} [NH_3] dz + \int_{-\Lambda}^{\Lambda} \frac{\partial BN^{(3)}}{\partial t} dz, \qquad (6.14)$$

and for $m \geq 1$,

$$\Lambda\omega^{(0)} I_0(l_m) \frac{d^2\gamma_m^{(3)}(t)}{dt^2} - \Lambda(1-l_m^2) \frac{d\delta_m^{(3)}(t)}{dt}$$

$$= \; -\sin t \int_{-\Lambda}^{\Lambda} [NH_1] \cos l_m(z+\Lambda) dz - 3\sin 3t \int_{-\Lambda}^{\Lambda} [NH_3] \cos l_m(z+\Lambda) dz. \qquad (6.15)$$

Note that equation (6.15) can be rewritten as

$$\frac{d\delta_m^{(3)}(t)}{dt} = \sin t \int_{-\Lambda}^{\Lambda} \frac{[NH_1]}{\Lambda(1-l_m^2)} \cos l_m(z+\Lambda)dz$$

$$+3\sin 3t \int_{-\Lambda}^{\Lambda} \frac{[NH_3]}{\Lambda(1-l_m^2)} \cos l_m(z+\Lambda)dz + \omega^{(0)} \frac{I_0(l_m)}{1-l_m^2} \frac{d^2\gamma_m^{(3)}(t)}{dt^2} \quad (6.16)$$

Substituting normal force banlance equation into kinematic condition, we obtain the following: for $m = 0$,

$$-2\Lambda\omega^{(0)} \left[\omega^{(0)} \frac{d^2\gamma_0^{(3)}(t)}{dt^2} + \frac{1}{2\Lambda} \sin t \int_{-\Lambda}^{\Lambda} [NH_1]dz \right.$$

$$\left. + \frac{3}{2\Lambda} \sin 3t \int_{-\Lambda}^{\Lambda} [NH_3]dz - \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} \frac{\partial BN^{(3)}}{\partial t} dz \right]$$

$$= \sin t \int_{-\Lambda}^{\Lambda} [KH_1]dz + \sin 3t \int_{-\Lambda}^{\Lambda} [KH_3]dz.$$

This can be rewritten as

$$-2\Lambda(\omega^{(0)})^2 \frac{d^2\gamma_0^{(3)}(t)}{dt^2}$$

$$= \sin t \int_{-\Lambda}^{\Lambda} \left\{ [KH_1] + \omega^{(0)}[NH_1] \right\} dz + \sin 3t \int_{-\Lambda}^{\Lambda} \left\{ [KH_3] + 3\omega^{(0)}[NH_3] \right\} dz. \quad (6.17)$$

For $m \geq 1$,

$$-\Lambda\omega^{(0)} \left[\omega^{(0)} \frac{I_0(l_m)}{1-l_m^2} \frac{d^2\gamma_m^{(3)}(t)}{dt^2} + \sin t \int_{-\Lambda}^{\Lambda} \frac{[NH_1]}{\Lambda(1-l_m^2)} \cos l_m(z+\Lambda)dz \right.$$

$$\left. +3\sin 3t \int_{-\Lambda}^{\Lambda} \frac{[NH_3]}{\Lambda(1-l_m^2)} \cos l_m(z+\Lambda)dz \right] + \Lambda l_m I_0'(l_m)\gamma_m^{(3)}(t)$$

$$= \sin t \int_{-\Lambda}^{\Lambda} [KH_1] \cos l_m(z+\Lambda)dz + \sin 3t \int_{-\Lambda}^{\Lambda} [KH_3] \cos l_m(z+\Lambda)dz.$$

This can be rewritten as

$$
\begin{aligned}
& -\Lambda(\omega^{(0)})^2 \frac{I_0(l_m)}{1-l_m^2} \frac{d^2\gamma_m^{(3)}(t)}{dt^2} + \Lambda l_m I_0'(l_m)\gamma_m^{(3)}(t) \\
= & \sin t \int_{-\Lambda}^{\Lambda} \left[[KH_1] + \frac{\omega^{(0)}[NH_1]}{1-l_m^2} \right] \cos l_m(z+\Lambda)dz \\
+ & \sin 3t \int_{-\Lambda}^{\Lambda} \left[[KH_3] + \frac{3\omega^{(0)}[NH_3]}{1-l_m^2} \right] \cos l_m(z+\Lambda)dz. \qquad (6.18)
\end{aligned}
$$

Equation (6.18) is an ordinary differential equation for $\gamma_m^{(3)}$.

In order to get the solvability condition, set the coefficients of $\sin t$ to be zero. Therefore, for $m = 0$,

$$
\int_{-\Lambda}^{\Lambda} \left\{ [KH_1] + \omega^{(0)}[NH_1] \right\} \cos(0)dz = 0, \qquad (6.19)
$$

and for $m \geq 1$,

$$
\int_{-\Lambda}^{\Lambda} \left\{ [KH_1] + \frac{\omega^{(0)}}{1-l_m^2}[NH_1] \right\} \cos l_m(z+\Lambda)dz = 0, \quad m = 1, 2, 3, \cdots \qquad (6.20)
$$

Note that $l_0 = 0$ and $l_m = \frac{m\pi}{2\Lambda}$. Hence

$$
\int_{-\Lambda}^{\Lambda} \left\{ [KH_1] + \frac{\omega^{(0)}}{1-l_m^2}[NH_1] \right\} \cos l_m(z+\Lambda)dz = 0, \quad m = 1, 2, 3, \cdots \qquad (6.21)
$$

By (6.6) and (6.9), the definitions of $[KH_1]$ and $[NH_1]$, the solvability condition (6.21) can be rewritten as:

$$
0 = \sum_{m=0}^{\infty} \int_{-\Lambda}^{\Lambda} \left\{ \left[-3\omega^{(2)}\tilde{F}^{(1)} - \frac{3}{2}\tilde{F}^{(1)} \left(\frac{\partial^2 \tilde{\Phi}^{(2)}}{\partial \mu^2} - E_1 \right) + \frac{3}{2}\tilde{F}^{(2)} \frac{\partial^2 \tilde{\Phi}^{(1)}}{\partial \mu^2} \right. \right.
$$

$$-3\tilde{\bar{F}}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \frac{3}{4}(\tilde{F}^{(1)})^2\frac{\partial^3\tilde{\Phi}^{(1)}}{\partial\mu^3} + \frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}$$

$$+\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z} - \frac{3}{2}\frac{\partial\tilde{F}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z} + 3\frac{\partial\tilde{\bar{F}}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\Bigg]$$

$$+\frac{\omega^{(0)}}{1-l_m^2}\Bigg[-3\omega^{(2)}\tilde{\Phi}^{(1)} - 3\omega^{(0)}\tilde{F}^{(1)}\left(\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu} + E_1\right)$$

$$-\frac{3}{2}\omega^{(0)}\tilde{F}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu} - 3\omega^{(0)}\tilde{\bar{F}}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu} - \frac{9}{4}\omega^{(0)}(\tilde{F}^{(1)})^2\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}$$

$$-\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\left(\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu} + E_1\right) - \frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}$$

$$-\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z} - 3\tilde{F}^{(1)}\tilde{F}^{(2)} - 6\tilde{F}^{(1)}\tilde{\bar{F}}^{(2)} + \frac{9}{2}(\tilde{F}^{(1)})^3$$

$$-\frac{9}{4}\tilde{F}^{(1)}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2 + \frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(2)}}{\partial z}$$

$$+3\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\bar{F}}^{(2)}}{\partial z} - \frac{27}{4}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\frac{\partial^2\tilde{F}^{(1)}}{\partial z^2}\Bigg]\Bigg\}\cos l_m(z+\Lambda)dz. \qquad (6.22)$$

It follows that

$$\omega^{(2)} = \frac{1}{\sum_{m=0}^{\infty}\int_{-\Lambda}^{\Lambda}\left(3\tilde{F}^{(1)} + \frac{3\omega^{(0)}\tilde{\Phi}^{(1)}}{1-l_m^2}\right)\cos l_m(z+\Lambda)dz}\sum_{m=0}^{\infty}\int_{-\Lambda}^{\Lambda}\Bigg\{\Bigg[-\frac{3}{2}\tilde{F}^{(1)}\left(\frac{\partial^2\tilde{\Phi}^{(2)}}{\partial\mu^2}\right.$$

$$\left.-E_1\right) + \frac{3}{2}\tilde{F}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - 3\tilde{\bar{F}}^{(2)}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2} - \frac{3}{4}(\tilde{F}^{(1)})^2\frac{\partial^3\tilde{\Phi}^{(1)}}{\partial\mu^3} + \frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}$$

$$+\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z} - \frac{3}{2}\frac{\partial\tilde{F}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z} + 3\frac{\partial\tilde{\bar{F}}^{(2)}}{\partial z}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\Bigg]$$

$$
+\frac{\omega^{(0)}}{1-l_m^2}\left[-3\omega^{(0)}\tilde{F}^{(1)}\left(\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right)\right.
$$

$$
-\frac{3}{2}\omega^{(0)}\tilde{F}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}-3\omega^{(0)}\tilde{\tilde{F}}^{(2)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}-\frac{9}{4}\omega^{(0)}(\tilde{F}^{(1)})^2\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}
$$

$$
-\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\left(\frac{\partial\tilde{\Phi}^{(2)}}{\partial\mu}+E_1\right)-\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial\mu}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu^2}-\frac{3}{2}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial\tilde{\Phi}^{(2)}}{\partial z}
$$

$$
-\frac{3}{2}\tilde{F}^{(1)}\frac{\partial\tilde{\Phi}^{(1)}}{\partial z}\frac{\partial^2\tilde{\Phi}^{(1)}}{\partial\mu\partial z}-3\tilde{F}^{(1)}\tilde{F}^{(2)}-6\tilde{F}^{(1)}\tilde{\tilde{F}}^{(2)}+\frac{9}{2}(\tilde{F}^{(1)})^3
$$

$$
-\frac{9}{4}\tilde{F}^{(1)}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2+\frac{3}{2}\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{F}^{(2)}}{\partial z}
$$

$$
\left.\left.+3\frac{\partial\tilde{F}^{(1)}}{\partial z}\frac{\partial\tilde{\tilde{F}}^{(2)}}{\partial z}-\frac{27}{4}\left(\frac{\partial\tilde{F}^{(1)}}{\partial z}\right)^2\frac{\partial^2\tilde{F}^{(1)}}{\partial z^2}\right]\right\}\cos l_m(z+\Lambda)dz. \tag{6.23}
$$

Therefore, $\omega^{(2)}$ can be determined numerically.

Chapter 7 Results of Third Order of ϵ, $O(\epsilon^3)$: Corrections to the Frequency, ω

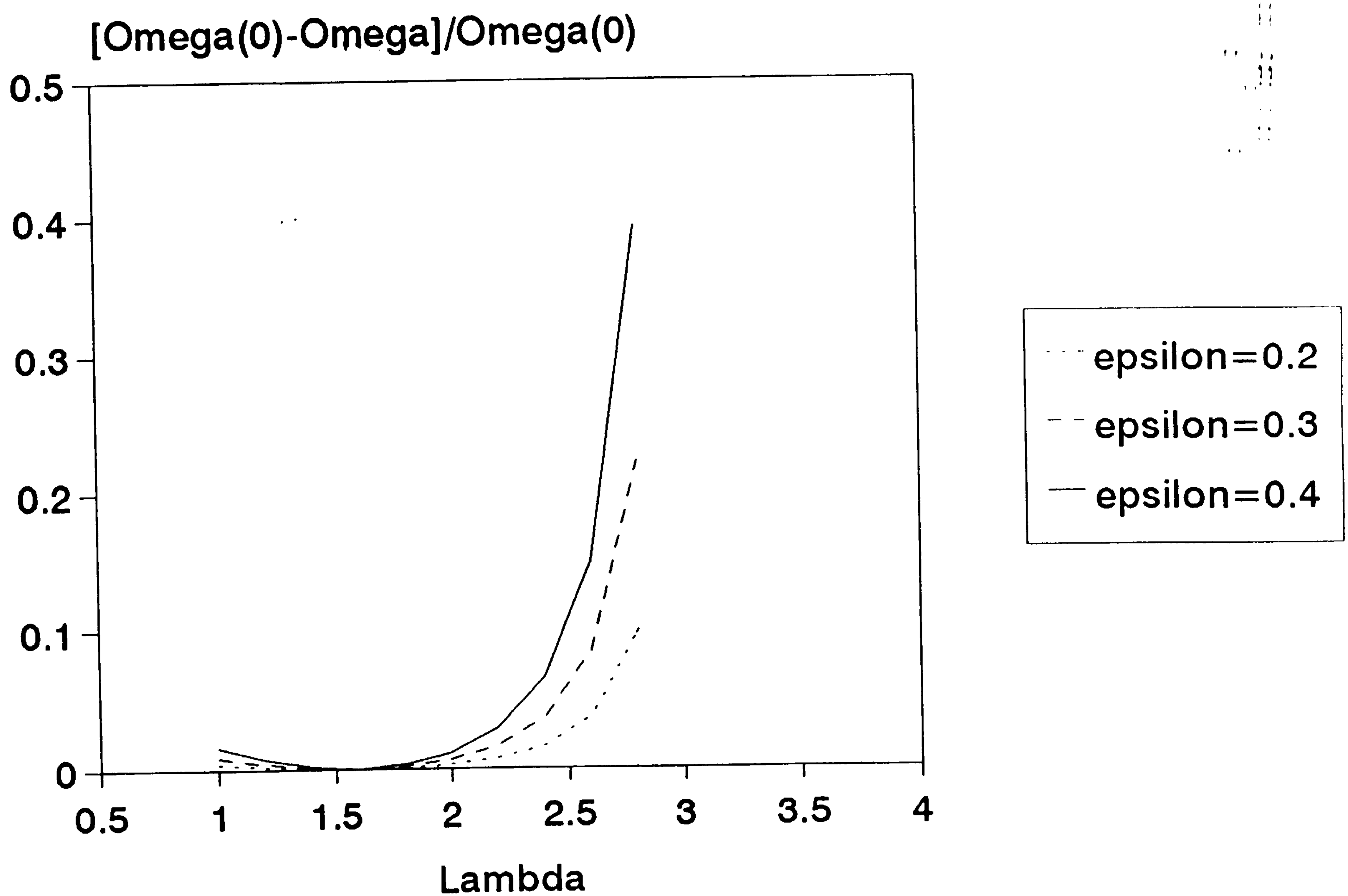
Preliminary calculations have yielded the corrections to ω. Note that

$$\omega = \omega^{(0)} + \frac{\epsilon^2}{2}\omega^{(2)}. \tag{7.1}$$

Results are plotted for $\dfrac{\omega^{(0)} - \omega}{\omega^{(0)}}$ versus the slenderness parameter Λ. This is done for modes $p = 2, 3$ and 6.

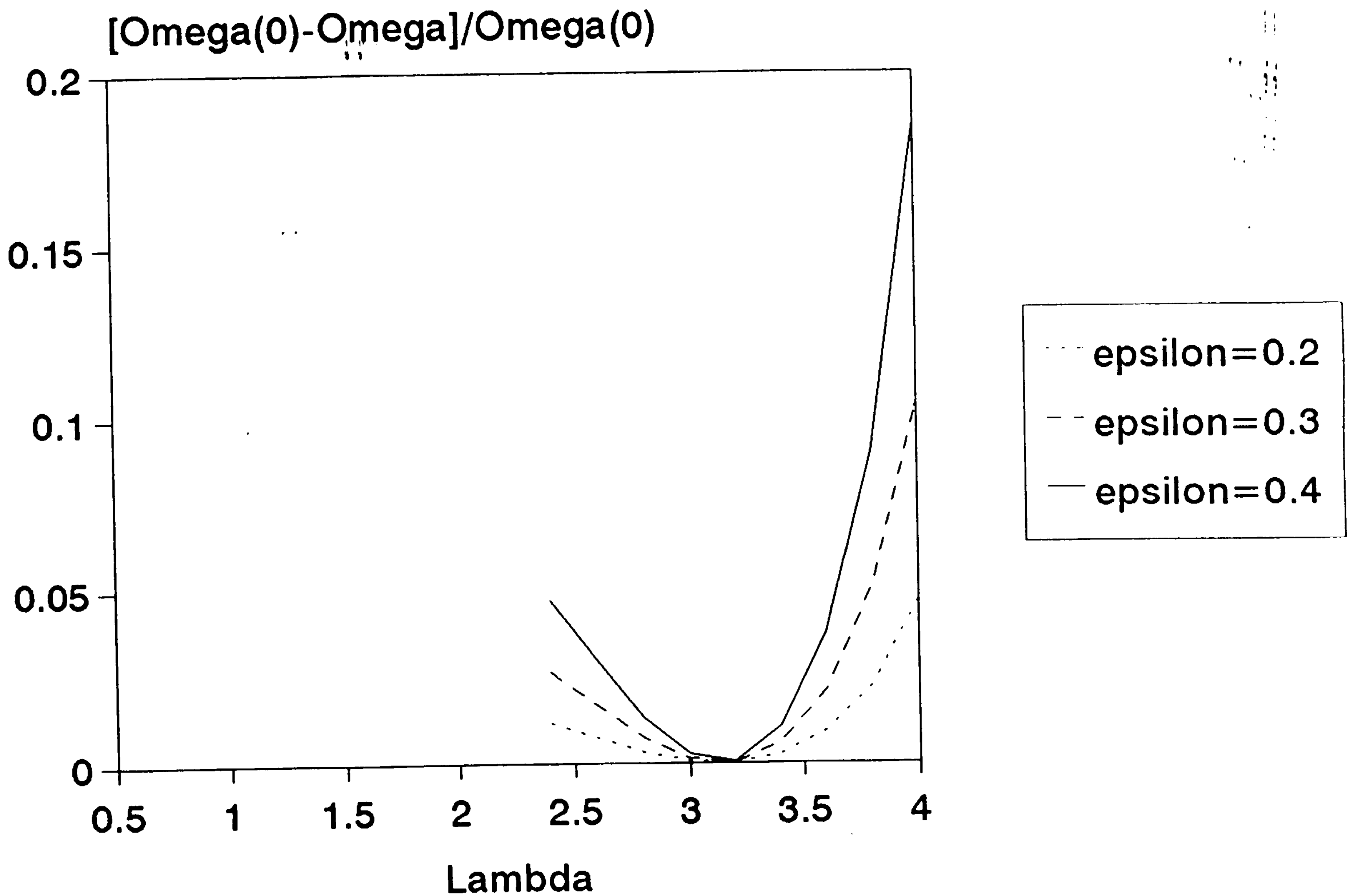
RATIO OF NONLINEAR FREQUENCY CORRECTION TO LINEAR FREQUENCY VS LAMBDA

P=2 (mode), beta1=1.0



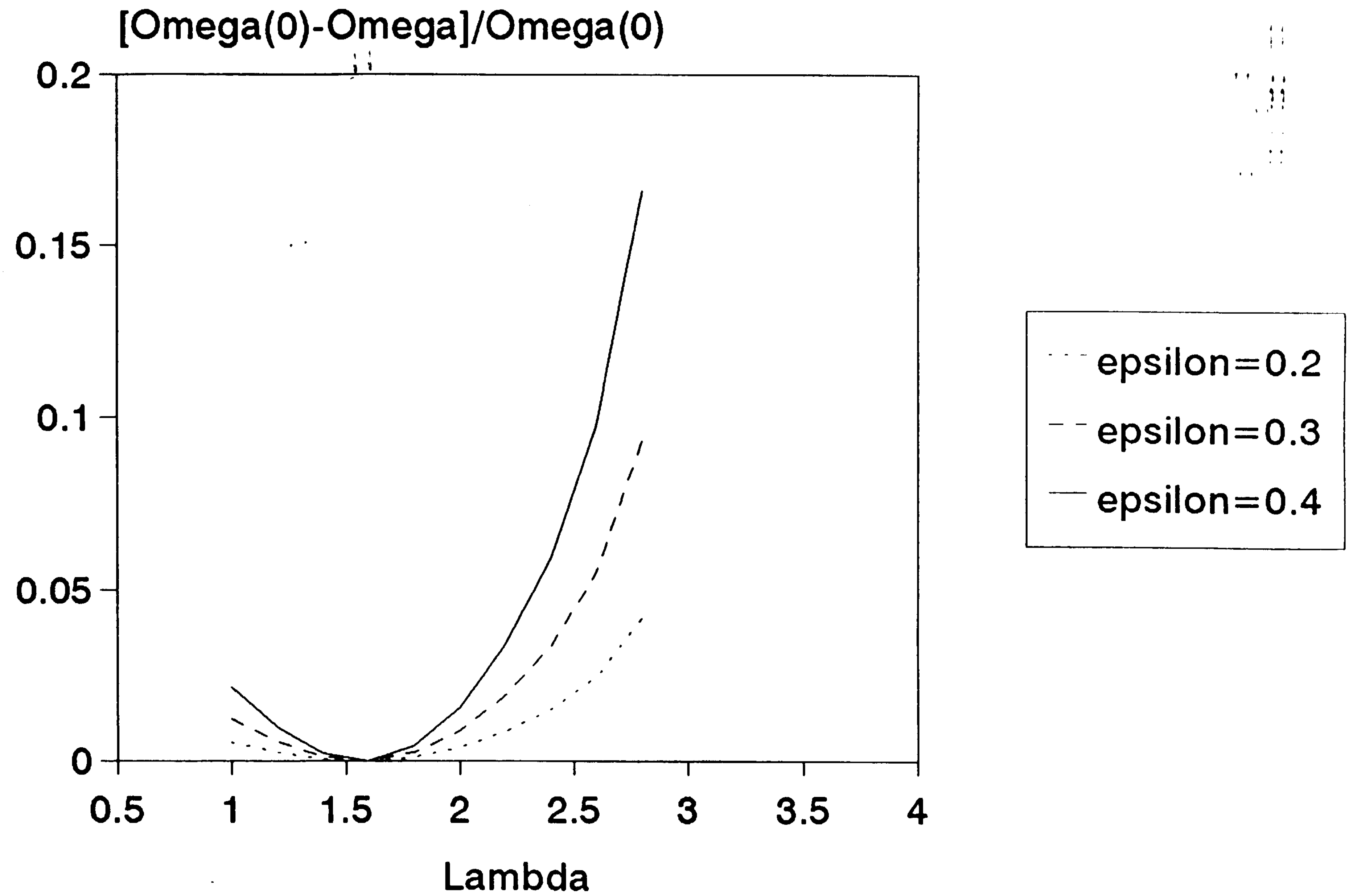
RATIO OF NONLINEAR FREQUENCY CORRECTION TO LINEAR FREQUENCY VS LAMBDA

P=3 (mode), alpha1=1.0



RATIO OF NONLINEAR FREQUENCY CORRECTION TO LINEAR FREQUENCY VS LAMBDA

P=6 (mode), beta1=1.0



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Appendix A

Code for Linear Root Finding

```
C    *   *   *   *   *   *   *   *   *   *   *   *
C
C     FROM THE FIRST ORDER OF EPSILON WE HAVE TWO EQUATIONS WHICH
C     GIVE THE RELATIONSHIP BETWEEN THE SLENDERNESS LAMBDA AND THE
C     ANGULAR FREQUENCY OMEGA(0) AT THE ZERO ORDER OF EPSILON.
C     ONE OF THE EQUATION IS FOR EVEN MODE. THE OTHER ONE IS FOR
C     ODD MODE.
C     HERE WE CALCULATE THE ODD MODE EQAUTION.
C    *   *   *   *   *   *   *   *   *   *   *   *
C
      DIMENSION BL(80),B1(80),B0(80)
      REAL*8 BL,B1,B0,A,PI,LAMBDA,S,S1,SAVG,
     * X,XMAX,DELTX,FMAX,XAVG,FAVG,X1,FX1,FX
      INTEGER N,I,K
C
C     INITIAL THE VALUE OF LAMBDA
C
      LAMBDA=1.80D0
      WRITE(6,1)LAMBDA
 1    FORMAT(' ','LAMBDA=',F7.4)
      PI=3.1415926D0
      A=(0.5D0)*((LAMBDA-DTAN(LAMBDA))/DTAN(LAMBDA))
C
C    *   *   *   *   *   *   *   *   *   *   *   *
C
C     CALCULATE THE WAVELENGTH BL(N), THE MODIFIED BESSEL FUNCTIONS
C     OF ZERO AND FIRST ORDER OF FIRST KIND I0(X) AND I1(X).
C    *   *   *   *   *   *   *   *   *   *   *   *
C
      DO 20 K=1,80
      BL(K)=(K*PI)/(2.D0*LAMBDA)
      B0(K)=BESSI0(BL(K))
      B1(K)=BESSI1(BL(K))
 20   CONTINUE
C
C    *   *   *   *   *   *   *   *   *   *   *   *
C
C     USE BISECTION METHOD TO SOLVE THE ODD MODES EQUATION.
C     DEFINE FX TO BE THE LEFT-HAND SIDE OF THE EQUATION.
C     OMEGA IS DENOTIED BY X WHICH WILL BE THE ROOTS OF THE EQUATION.
C
C     THE INITIAL INTERVAL WILL BE CHOSEN AS [X, XMAX].
C     DELTX IS CHOSEN AS 0.005.
C     LET N=80 BE THE NUMBER OF BISECTIONS DESIRED.
C     CONSIDERING THE FX GOES TO INFINIT WHEN FX > FMAX(=10000).
C
C    *   *   *   *   *   *   *   *   *   *   *   *
      READ(5,200) X,XMAX
      DELTX=0.005
      FMAX=10000
      N=80
 200  FORMAT(4F10.4,I2)
C
C    *   *   *   *   *   *   *   *   *   *   *   *
C     FROM THE LEFT END VALUE OF THE INITIAL INTERVAL, X, WE ARE
C     GOING TO HAVE THE FUNCTION FX.
C     THE LOOP IS TO ADD 40 TERMS FOR THE SUMATION WHICH IS
C     DEFINED AS S.
C    *   *   *   *   *   *   *   *   *   *   *   *
C
```

```
      S=0.D0
      DO 201 K=1,20
      S=S+(1.D0/(1.D0-BL(2*K)**2))*(1.D0/(X**2+BL(2*K)*
     * (1.D0-BL(2*K)**2)*(B1(2*K)/B0(2*K))))
 201  CONTINUE
 300  FX=A-S*(X**2)
C
C   *   *   *   *   *   *   *   *   *   *   *   *
C     NOW WE DEFINE A NEW VALUE AS X1=X+DELTX.
C     REPLACE X BY X1 FOR FX. THEN WE HAVE FX1.
C     THE LOOP IS TO ADD 40 TERMS OF THE SUMATION, WHICH IS
C     REDEFINED AS S1.
C   *   *   *   *   *   *   *   *   *   *   *   *
C
 400  X1=X+DELTX
      S1=0.D0
      DO 401 K=1,20
      S1=S1+(1.D0/(1.D0-BL(2*K)**2))*(1.D0/(X1**2+BL(2*K)*
     * (1.D0-BL(2*K)**2)*(B1(2*K)/B0(2*K))))
 401  CONTINUE
      FX1=A-S1*(X1**2)
C
C   *   *   *   *   *   *   *   *   *   *   *   *
C     NOW WE NEED TO CONSIDER THE TWO RESULTS FX AND FX1.
C     IF FX*FX1 IS LESS THAN ZERO, THE ROOT MUST BE IN THE INTERVAL
C     [X, X1]. (WHICH MEANS FX AND FX1 HAVE DIFFERENT SIGNS.)
C     IF FX*FX1=0, THEN X1 IS THE ROOT OF THE FUNCTION.
C     IF FX*FX1 IS GREATER THAN AERO, THE ROOT MUST BE IN THE
C     INTERVAL [X1, XMAX].
C
C     FURTHER, IF FX*FX1<0, LET XAVG=(X+X1)/2.
C              IF FX*FX1>0, LET XAVG=(X1+XMAX)/2.
C              BY SUBSTITUTING XAVG WE WILL GET FAVG. (DEFINE AS SAVG
C              FOR THE SUM OF THE FIRST 40 TERMS OF THE SUMATION.)
C              CONTINUE THE SAME PROCEDURE UNTIL THE 'REAL ROOT',X,
C              IS OBTAINED WHICH MAKES THE FUNCTION FX ZERO.
C   *   *   *   *   *   *   *   *   *   *   *   *
C
      IF(FX*FX1) 800,500,700
 500  WRITE(6,600) X1
 600  FORMAT(' ','X=',F24.18,' IS A REAL ROOT')
      X=X1+DELTX
      GO TO 300
 700  IF(X1.GE.XMAX) STOP
      X=X1
      FX=FX1
      GO TO 400
 800  DO 1100 I=1,N
      XAVG=(X+X1)/(2.D0)
      SAVG=0.D0
      DO 801 K=1,20
      SAVG=SAVG+(1.D0/(1.D0-BL(2*K)**2))*(1.D0/(XAVG**2+
     * BL(2*K)*(1.D0-BL(2*K)**2)*(B1(2*K)/B0(2*K))))
 801  CONTINUE
      FAVG=A-SAVG*(XAVG**2)
      IF(ABS(FAVG).GT.FMAX) GO TO 1400
      IF(FX*FAVG) 1000,1200,900
 900  X=XAVG
      FX=FAVG
      GO TO 1100
```

```
 1000 X1=XAVG
      FX1=FAVG
 1100 CONTINUE
 1200 WRITE(6,600) XAVG
 1300 FX=FX1
      X=X1
      GO TO 400
 1400 WRITE(6,1500) XAVG
 1500 FORMAT(' ','FUNCTION APPROACHING INFINITY FOR X=',F7.4)
      GO TO 1300
      END
C
      FUNCTION BESSI0(X)
C     RETRURNS THE MODIFIED BESSEL I0 FOR ANY REAL X.
C
      REAL*8 Y, P1, P2, P3, P4, P5, P6, P7
      REAL*8 AX,X,BESSI0
C     ACCUMULATE POLYNOMIALS IN DOULBLE PRECISION
      REAL*8 Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9
      P1=1.0D0
      P2=3.5156229D0
      P3=3.0899424D0
      P4=1.2067492D0
      P5=0.2659732D0
      P6=0.360768D-1
      P7=0.45813D-2
C
      Q1=0.39894228D0
      Q2=0.1328592D-1
      Q3=0.225319D-2
      Q4=-0.157565D-2
      Q5=0.916281D-2
      Q6=-0.2057706D-1
      Q7=0.2635537D-1
      Q8=-0.1647633D-1
      Q9=0.392377D-2
C
C     POLYNOMIAL FIT
      IF (DABS(X).LT.3.75D0) THEN
      Y=(X/3.75D0)**2
      BESSI0=P1+Y*(P2+Y*(P3+Y*(P4+Y*(P5+Y*(P6+Y*P7)))))
      ELSE
      AX=DABS(X)
      Y=3.75D0/AX
      BESSI0=(DEXP(AX)/DSQRT(AX))*(Q1+Y*(Q2+Y*(Q3+Y*(Q4
     *+Y*(Q5+Y*(Q6+Y*(Q7+Y*(Q8+Y*Q9))))))))
      ENDIF
      RETURN
      END
C
      FUNCTION BESSI1(X)
C     RETURNS THE MODIFIED BESSEL I1 FOR ANY REAL X.
C
      REAL*8 Y, P1, P2, P3, P4, P5, P6, P7
      REAL*8 AX,X,BESSI1
C     ACCUMULATE POLYNOMIALS IN DOUBLE PRECISION
      REAL*8 Q1,Q2,Q3,Q4,Q5,Q6,Q7,Q8,Q9
      P1=0.5D0
      P2=0.87890594D0
      P3=0.51498869D0
```

```
      P4=0.15084934D0
      P5=0.2658733D-1
      P6=0.301532D-2
      P7=0.32411D-3
C
      Q1=0.39894228D0
      Q2=-0.3988024D-1
      Q3=-0.362018D-2
      Q4=0.163801D-2
      Q5=-0.1031555D-1
      Q6=0.2282967D-1
      Q7=-0.2895312D-1
      Q8=0.1787654D-1
      Q9=-0.420059D-2
C
C     POLYNOMIAL FIT
      IF (DABS(X).LT.3.75D0) THEN
      Y=(X/3.75D0)**2
      BESSI1=P1+Y*(P2+Y*(P3+Y*(P4+Y*(P5+Y*(P6+Y*P7)))))
      BESSI1=X*BESSI1
      ELSE
      AX=DABS(X)
      Y=3.75D0/AX
      BESSI1=(DEXP(AX)/DSQRT(AX))*(Q1+Y*(Q2+Y*(Q3+Y*(Q4
     *+Y*(Q5+Y*(Q6+Y*(Q7+Y*(Q8+Y*Q9))))))))
      ENDIF
      RETURN
      END
```

```
C    *    *    *    *    *    *    *    *    *    *    *    *
C
C      FROM THE FIRST ORDER OF EPSILON WE HAVE TWO EQUATIONS WHICH
C      GIVE THE RELATIONSHIP BETWEEN THE SLENDERNESS LAMBDA AND THE
C      ANGULAR FREQUENCY OMEGA(0) AT THE ZERO ORDER OF EPSILON.
C      ONE OF THE EQUATION IS FOR EVEN MODE. ANOTHER ONE IS FOR ODD
C      MODE.
C      HERE WE CALCULATE THE EVEN MODE EQAUTION.
C    *    *    *    *    *    *    *    *    *    *    *    *
C
       DIMENSION BL(80),B1(80),B0(80)
       REAL*8 BL,B1,B0,A,PI,LAMBDA,S,S1,SAVG,
     * X,XMAX,DELTX,FMAX,XAVG,FAVG,X1,FX1,FX
       INTEGER N,I,K
C
C *** INITIAL THE VALUE OF LAMBDA
C
       LAMBDA=1.80D0
       WRITE(6,1)LAMBDA
 1     FORMAT(' ','LAMBDA=',F7.4)
       PI=3.1415926D0
       A=(0.5D0)*(LAMBDA)*(DTAN(LAMBDA))
C
C    *    *    *    *    *    *    *    *    *    *    *    *
C
C      CALCULATE THE WAVELENGTH BL(N), THE MODIFIED BESSEL FUNCTIONS
C      OF ZERO AND FIRTS ORDER OF FIRST KIND I0(X) AND I1(X).
C    *    *    *    *    *    *    *    *    *    *    *    *
C
       DO 20 K=1,80
       BL(K)=(K*PI)/(2.D0*LAMBDA)
       B0(K)=BESSI0(BL(K))
       B1(K)=BESSI1(BL(K))
 20    CONTINUE
C
       READ(5,200) X,XMAX
       DELTX=0.005
       FMAX=10000
       N=80
 200   FORMAT(4F10.4,I2)
       S=0.D0
       DO 201 K=1,20
       S=S+(1.D0/(1.D0-BL(2*K-1)**2))*(1.D0/(X**2+BL(2*K-1)*
     * (1.D0-BL(2*K-1)**2)*(B1(2*K-1)/B0(2*K-1))))
 201   CONTINUE
 300   FX=A+S*(X**2)
 400   X1=X+DELTX
       S1=0.D0
       DO 401 K=1,20
       S1=S1+(1.D0/(1.D0-BL(2*K-1)**2))*(1.D0/(X1**2+BL(2*K-1)*
     * (1.D0-BL(2*K-1)**2)*(B1(2*K-1)/B0(2*K-1))))
 401   CONTINUE
       FX1=A+S1*(X1**2)
       IF(FX*FX1) 800,500,700
 500   WRITE(6,600) X1
 600   FORMAT(' ','X=',F24.18,' IS A REAL ROOT')
       X=X1+DELTX
       GO TO 300
 700   IF(X1.GE.XMAX) STOP
       X=X1
```

```
      FX=FX1
      GO TO 400
 800  DO 1100 I=1,N
      XAVG=(X+X1)/(2.D0)
      SAVG=0.D0
      DO 801 K=1,20
      SAVG=SAVG+(1.D0/(1.D0-BL(2*K-1)**2))*(1.D0/(XAVG**2+
     * BL(2*K-1)*(1.D0-BL(2*K-1)**2)*(B1(2*K-1)/B0(2*K-1))))
 801  CONTINUE
      FAVG=A+SAVG*(XAVG**2)
      IF(ABS(FAVG).GT.FMAX) GO TO 1400
      IF(FX*FAVG) 1000,1200,900
 900  X=XAVG
      FX=FAVG
      GO TO 1100
 1000 X1=XAVG
      FX1=FAVG
 1100 CONTINUE
 1200 WRITE(6,600) XAVG
 1300 FX=FX1
      X=X1
      GO TO 400
 1400 WRITE(6,1500) XAVG
 1500 FORMAT(' ','FUNCTION APPROACHING INFINITY FOR X=',F7.4)
      GO TO 1300
      END
C
      FUNCTION BESSI0(X)
C     RETRURNS THE MODIFIED BESSEL I0 FOR ANY REAL X.
C
      REAL*8 Y, P1, P2, P3, P4, P5, P6, P7
      REAL*8 AX,X,BESSI0
C     ACCUMULATE POLYNOMIALS IN DOULBLE PRECISION
      REAL*8 Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9
      P1=1.0D0
      P2=3.5156229D0
      P3=3.0899424D0
      P4=1.2067492D0
      P5=0.2659732D0
      P6=0.360768D-1
      P7=0.45813D-2
C
      Q1=0.39894228D0
      Q2=0.1328592D-1
      Q3=0.225319D-2
      Q4=-0.157565D-2
      Q5=0.916281D-2
      Q6=-0.2057706D-1
      Q7=0.2635537D-1
      Q8=-0.1647633D-1
      Q9=0.392377D-2
C
C     POLYNOMIAL FIT
      IF (DABS(X).LT.3.75D0) THEN
      Y=(X/3.75D0)**2
      BESSI0=P1+Y*(P2+Y*(P3+Y*(P4+Y*(P5+Y*(P6+Y*P7)))))
      ELSE
      AX=DABS(X)
      Y=3.75D0/AX
      BESSI0=(DEXP(AX)/DSQRT(AX))*(Q1+Y*(Q2+Y*(Q3+Y*(Q4
```

```
     *+Y*(Q5+Y*(Q6+Y*(Q7+Y*(Q8+Y*Q9)))))))
      ENDIF
      RETURN
      END
C
      FUNCTION BESSI1(X)
C     RETURNS THE MODIFIED BESSEL I1 FOR ANY REAL X.
C
      REAL*8 Y, P1, P2, P3, P4, P5, P6, P7
      REAL*8 AX,X,BESSI1
C     ACCUMULATE POLYNOMIALS IN DOUBLE PRECISION
      REAL*8 Q1,Q2,Q3,Q4,Q5,Q6,Q7,Q8,Q9
      P1=0.5D0
      P2=0.87890594D0
      P3=0.51498869D0
      P4=0.15084934D0
      P5=0.2658733D-1
      P6=0.301532D-2
      P7=0.32411D-3
C
      Q1=0.39894228D0
      Q2=-0.3988024D-1
      Q3=-0.362018D-2
      Q4=0.163801D-2
      Q5=-0.1031555D-1
      Q6=0.2282967D-1
      Q7=-0.2895312D-1
      Q8=0.1787654D-1
      Q9=-0.420059D-2
C
C     POLYNOMIAL FIT
      IF (DABS(X).LT.3.75D0) THEN
      Y=(X/3.75D0)**2
      BESSI1=P1+Y*(P2+Y*(P3+Y*(P4+Y*(P5+Y*(P6+Y*P7)))))
      BESSI1=X*BESSI1
      ELSE
      AX=DABS(X)
      Y=3.75D0/AX
      BESSI1=(DEXP(AX)/DSQRT(AX))*(Q1+Y*(Q2+Y*(Q3+Y*(Q4
     *+Y*(Q5+Y*(Q6+Y*(Q7+Y*(Q8+Y*Q9)))))))
      ENDIF
      RETURN
      END
```

Appendix B

Solvability Condition:

Alternative Formulation

Determining the Solvability Condition Formally

Start with the basic governing equation at $O(\epsilon^3)$ which is

$$\nabla^2\Phi^{(3)} = 0, \quad \text{on } 0 \leq \mu \leq 1,\ 0 \leq \theta \leq 2\pi, \text{ and } -\Lambda \leq z \leq \Lambda.$$

Multiply by $\Phi^{(1)}$ to integrate over the volume

$$\int_{\text{volume}} \Phi^{(1)}\nabla^2\Phi^{(3)} dV = 0.$$

Use cylindrical coordinates to write $dV = d\mu(\mu d\theta)dz$, and integrate in μ

$$\int\int\int \Phi^{(1)} \left\{ \frac{\partial^2\Phi^{(3)}}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial\Phi^{(3)}}{\partial\mu} + \frac{1}{\mu^2}\frac{\partial^2\Phi^{(3)}}{\partial\theta^2} + \frac{\partial^2\Phi^{(3)}}{\partial z^2} \right\} d\mu(\mu d\theta)dz = 0$$

Since

$$\frac{1}{\mu^2}\frac{\partial^2\Phi^{(3)}}{\partial\theta^2} = 0,$$

the basic governing equation can be rewritten as follows.

$$\int\int\int \mu\Phi^{(1)}\frac{\partial^2\Phi^{(3)}}{\partial\mu^2} d\mu d\theta dz + \int\int\int \Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu} d\mu d\theta dz$$
$$+ \int\int\int \mu\Phi^{(1)}\frac{\partial^2\Phi^{(3)}}{\partial z^2} d\mu d\theta dz = 0.$$

Use integration by parts to get the adjoint system.

Denote

$$\mu\Phi^{(1)} = u, \text{ and } V = \frac{\partial\Phi^{(3)}}{\partial\mu}.$$

Then

$$du = \Phi^{(1)} + \frac{d\Phi^{(1)}\mu}{d\mu} \text{ and } dV = \frac{\partial^2\Phi^{(3)}}{\partial\mu^2}.$$

It follows from integration by parts that

$$\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \mu\Phi^{(1)}\frac{\partial^2\Phi^{(3)}}{\partial\mu^2}d\mu d\theta dz$$

$$= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\mu\Phi^{(1)}\Phi^{(3)}_{\mu}\right]_0^1 d\theta dz - \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}_{\mu}(\Phi^{(1)} + \mu\Phi^{(1)}_{\mu})d\mu d\theta dz$$

$$= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\mu\Phi^{(1)}\Phi^{(3)}_{\mu}\right]_0^1 d\theta dz - \int_{-\Lambda}\Lambda\int_0^{2\pi}\int_0^1 \Phi^{(3)}_{\mu}\Phi^{(1)}d\mu d\theta dz$$

$$-\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \mu\Phi^{(1)}_{\mu}\Phi^{(3)}_{\mu}d\mu d\theta dz.$$

$$= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\Phi^{(1)}\Phi^{(3)}_{\mu}\right]_{\mu=1} d\theta dz$$

$$-\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\Phi^{(1)}\Phi^{(3)}\right]_0^1 d\theta dz + \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}\Phi^{(1)}_{\mu}d\mu d\theta dz$$

$$-\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\Phi^{(1)}_{\mu}\Phi^{(3)}\right]_{\mu=1} d\theta dz + \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1(\Phi^{(1)}_{\mu} + \mu\Phi^{(3)}_{\mu\mu})\Phi^{(3)}d\mu d\theta dz.$$

For the second integral, we have

$$\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu}d\mu d\theta dz$$

$$= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\left[\mu\Phi^{(1)}\Phi^{(3)}\right]_0^1 d\theta dz - \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}\frac{\partial\Phi^{(1)}}{\partial\mu}d\mu d\theta dz$$

For the third integral, we have

$$\int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \mu\Phi^{(1)}\frac{\partial^2\Phi^{(3)}}{\partial z^2}d\mu d\theta dz$$

$$= \int_0^{2\pi}\int_0^1\int_{-\Lambda}^{\Lambda} \mu\Phi^{(1)}\frac{\partial^2\Phi^{(3)}}{\partial z^2}d\mu d\theta dz$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^1 \left[\mu\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial z}\right]_{-\Lambda}^{\Lambda} d\mu d\theta - \int_0^{2\pi}\int_0^1\int_{-\Lambda}^{\Lambda} \mu\Phi_z^{(1)}\Phi_z^{(3)} dz d\mu d\theta \\
&= \int_0^{2\pi}\int_0^1 \left[\mu\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial z}\right]_{-\Lambda}^{\Lambda} d\mu d\theta \\
&\quad - \left[\int_0^{2\pi}\int_0^1 \left[\mu\Phi_\mu^{(1)}\Phi^{(3)}\right]_{-\Lambda}^{\Lambda} - \int_0^{2\pi}\int_0^1\int_{-\Lambda}^{\Lambda} \Phi^{(3)}\mu\frac{\partial^2\Phi^{(1)}}{\partial z^2} d\mu d\theta dz\right]
\end{aligned}
$$

Let $d^3\tau = d\mu(\mu d\theta)dz$. Then by

$$
\left.\frac{\partial\Phi^{(1)}}{\partial z}\right|_{z=\pm\Lambda} = 0,
$$

and by setting

$$
\left.\frac{\partial\Phi^{(3)}}{\partial z}\right|_{z=\pm\Lambda} = 0,
$$

one obtains

$$
\begin{aligned}
0 &= \iiint_{\text{volume}} \Phi^{(1)}\nabla^2\Phi^{(3)} d^3\tau \\
&= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi} \left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu} - \frac{\partial\Phi^{(1)}}{\partial\mu}\Phi^{(3)}\right]_{\mu=1} d\theta dz - \int_{-\Lambda}^{\Lambda}\int_0^{2\pi} [\Phi^{(1)}\Phi^{(3)}]_0^1 d\theta dz \\
&\quad + \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}\left[\mu\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} + 2\Phi_\mu^{(1)}\right] d\mu d\theta dz + \int_{-\Lambda}^{\Lambda}\int_0^{2\pi} [\Phi^{(1)}\Phi^{(3)}]_0^1 d\theta dz \\
&\quad - \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}\Phi_\mu^{(1)} d\mu d\theta dz + \int_0^{2\pi}\int_0^1 \left[\mu\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial z}\right]_{-\Lambda}^{\Lambda} d\mu d\theta \\
&\quad - \int_0^{2\pi}\int_0^1 \left[\mu\Phi^{(3)}\frac{\partial\Phi^{(1)}}{\partial z}\right]_{-\Lambda}^{\Lambda} d\mu d\theta + \int_0^{2\pi}\int_0^1\int_{-\Lambda}^{\Lambda} \Phi^{(3)}\mu\frac{\partial^2\Phi^{(1)}}{\partial z^2} dz d\mu d\theta \\
&= \int_{-\Lambda}^{\Lambda}\int_0^{2\pi} \left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu} - \frac{\partial\Phi^{(1)}}{\partial\mu}\Phi^{(3)}\right]_{\mu=1} d\theta dz \\
&\quad + \int_{-\Lambda}^{\Lambda}\int_0^{2\pi}\int_0^1 \Phi^{(3)}\left[\mu\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} + 2\Phi_\mu^{(1)} - \Phi_\mu^{(1)} + \mu\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right] d\mu d\theta dz \\
&= \iiint_{\text{volume}} \Phi^{(3)}\left[\frac{\partial^2\Phi^{(1)}}{\partial\mu^2} + \frac{1}{\mu}\Phi_\mu^{(1)}\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right] d^3\tau
\end{aligned}
$$

$$+\int_{-\Lambda}^{\Lambda}\int_{0}^{2\pi}\left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu}-\frac{\partial\Phi^{(1)}}{\partial\mu}\Phi^{(3)}\right]_{\mu=1}d\theta dz$$

By Laplace equation on $O(\epsilon)$ and on $O(\epsilon^3)$,

$$\nabla^2\Phi^{(1)}=0 \text{ and } \nabla^2\Phi^{(3)}=0.$$

Thus the solvability on $\mu=1$ becomes

$$\int_{-\Lambda}^{\Lambda}\int_{0}^{2\pi}\left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu}-\Phi^{(3)}\frac{\partial\Phi^{(1)}}{\partial\mu}\right]d\theta dz=0$$

or

$$2\pi\int_{-\Lambda}^{\Lambda}\left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu}-\Phi^{(3)}\frac{\partial\Phi^{(1)}}{\partial\mu}\right]dz=0.$$

Solvability on free surface is then

$$\int_{-\Lambda}^{\Lambda}\left[\Phi^{(1)}\frac{\partial\Phi^{(3)}}{\partial\mu}-\Phi^{(3)}\frac{\partial\Phi^{(1)}}{\partial\mu}\right]dz=0.$$

Kinematic

$$-\omega^{(0)}\frac{\partial F^{(3)}}{\partial t}+\frac{\partial\Phi^{(3)}}{\partial\mu}=\sin t\,[KH_1]+\sin 3t\,[KH_3]$$

Normal force

$$\omega^{(0)}\frac{\partial F^{(3)}}{\partial t}-F^{(3)}-\frac{\partial^2F^{(3)}}{\partial z^2}=\cos t\,[NH_1]+\cos 3t\,[NH_3]+BN^{(3)}.$$

$$\Phi^{(3)}=\sum_{m=0}^{\infty}\gamma_m^{(3)}(t)I_0(l_m\mu)\cos(l_m(z+\Lambda))+G(\mu,t).$$

$$F^{(3)}=\sum_{m=0}^{\infty}\delta_m^{(3)}(t)\cos(l_m(z+\Lambda)).$$

Solvability condition

$$\int_{-\Lambda}^{\Lambda}\left[\Phi^{(1)}\Phi_\mu^{(3)}-\Phi_\mu^{(1)}\Phi^{(3)}\right]dz=0.$$

$$\Phi^{(1)} = \sum_{m=0}^{\infty} \sin t A_m(t) I_0(l_m\mu)\cos(l_m(z+\Lambda)).$$

$$\Phi^{(3)} = \sum_{m=0}^{\infty} \gamma_m^{(3)}(t) I_0(l_m\mu)\cos(l_m(z+\Lambda)). \qquad (C)$$

$$F^{(3)} = \sum_{m=0}^{\infty} \delta_m^{(3)}(t) I_0(l_m\mu)\cos(l_m(z+\Lambda)). \qquad (A)$$

$$\sum_{m=0}^{\infty} A_m I_0(l_m\mu)|_{\mu=1} \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda))\Phi_\mu^{(3)} dz$$
$$- \sum_{m=0}^{\infty} l_m A_m I_0'(l_m\mu)|_{\mu=1} \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda))\Phi^{(3)} dz = 0$$

$$\sum_{m=0}^{\infty} A_m I_0(l_m) \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda)) \left[-\omega^{(0)}\frac{\partial F^{(3)}}{\partial t} + \sin t\,[KH_1] + \sin 3t\,[KH_3]\right]$$
$$- \sum_{m=0}^{\infty} l_m A_m I_0'(l_m) \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda))\Phi^{(3)} dz = 0$$

$$\sum_{m=0}^{\infty} A_m I_0(l_m) \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda))\left[\sin t\,[KH_1] + \sin 3t\,[KH_3]\right] dz$$
$$- \sum_{m=0}^{\infty} \omega^{(0)} A_m I_0(l_m) \int_{-\Lambda}^{\Lambda} \frac{\partial F^{(3)}}{\partial t} \cos(l_m(z+\Lambda)) dz$$
$$- \sum_{m=0}^{\infty} l_m A_m I_0'(l_m) \int_{-\Lambda}^{\Lambda} \cos(l_m(z+\Lambda))\Phi^{(3)} dz = 0$$

Apply normal force equation to get

$$\omega^{(0)}\frac{\partial \Phi^{(3)}}{\partial t} - F^{(3)} - \frac{\partial^2 F^{(3)}}{\partial z^2} = \cos t[NH_1] + \cos 3t[NH_3].$$