

Development of Homotopy Algorithms for Fixed-Order Mixed H_2/H_∞ Controller Synthesis

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TECHNICAL PAPER

DEVELOPMENT OF HOMOTOPY ALGORITHMS FOR FIXED-ORDER MIXED H_2/H_∞ CONTROLLER SYNTHESIS

I. INTRODUCTION

Modern control theory has revolutionized control system design over the past decade. H_2 and H_∞ methods have gained widespread recognition and are used in controller synthesis for *SISO* and *MIMO* problems. A significant disadvantage of modern control techniques is that the resulting compensator is the same order as the generalized plant, which is often larger than the original plant due to the inclusion of frequency-dependent weights to achieve the desired performance and robustness characteristics. Real-time implementation of these controllers for large-order systems is prohibitive due to the computational burden of fast throughput times.

One indirect approach to alleviating this is to reduce the order of the plant and to synthesize a controller based on this reduced-order design plant. An alternate indirect approach is to design a full-order controller and then to apply model reduction to the controller. In either case, indirect methods cannot guarantee closed-loop stability and are suboptimal in performance. Direct methods impose constraints on controller order or architecture. In an optimization based synthesis procedure, necessary conditions are formulated for the constrained closed-loop system that ensure internal stability. The optimal projection approach of reference 1 is an example whereby order constraints are imposed on the controller, and the necessary conditions for minimizing a quadratic cost functional with respect to the fixed-order controller are derived. The resulting necessary conditions consist of two modified Riccati equations and two modified Lyapunov equations coupled by an oblique projection matrix. However, solution of the necessary conditions for realistic large order systems is a difficult task. Reference 2 employs homotopy methods to solve the optimal projection equations. Newton methods have also been applied.³

Optimal projection has also been extended to *LQG* control with an H_∞ norm over-bound.⁴ This mixed H_2/H_∞ optimization problem seeks to minimize the H_2 norm of one transfer function for performance while satisfying a bound on the H_∞ norm of another transfer function for robustness. The true mixed problem has two inputs and two outputs, indicating different classes of disturbances and performance variables. Much of the research into the mixed problem considers variations of the true problem with only one input or only one output. Reference 4 considered the case of two outputs and one input, with both full-order and fixed-order control. The difficulty here is the size of the gap between the H_∞ over-bound and the true H_∞ norm.⁵ Reference 6 addresses the two input, two output problem with output feedback including the fixed-order problem, but does not attempt to solve it. Recently, reference 7 used a differential game formulation to obtain fixed-order controllers for the true mixed problem. A conjugate gradient technique was applied to solve these resulting necessary conditions.

The objective of this paper is to build on the results of references 4 and 7 by presenting homotopy algorithms for solving the H_2 , H_∞ , and true mixed H_2/H_∞ fixed-order compensator synthesis problems. The paper is organized as follows. Section II presents a formulation of the problem with the compensator in controller canonical form. The necessary conditions for the fixed-order H_2 controller are developed. These results are then extended to the fixed-order H_∞ and mixed H_2/H_∞

compensator design using the differential game results of reference 7. Section III introduces homotopy methods and develops homotopy algorithms for solution of the H_2 , H_∞ , and the mixed H_2/H_∞ problem formulations of section II. Section IV presents numerical evaluations of these homotopy algorithms, and section V concludes the paper.

II. PROBLEM FORMULATION

The generalized plant of a standard control problem is given by:

$$\dot{x} = Ax + B_1 w + B_2 u , \quad (1)$$

$$z = C_1 x + D_{12} u , \quad (2)$$

$$y = C_2 x + D_{21} w + D_{22} u , \quad (3)$$

where $x \in R^n$ is the state vector, $w \in R^{n_w}$ is the disturbance vector, $u \in R^{n_u}$ is the control vector, $z \in R^{n_z}$ is the performance vector, and $y \in R^{n_y}$ is the observation vector. It is assumed that:

- (A, B_1, C_1) is stabilizable and detectable
- (A, B_2, C_2) is stabilizable and detectable
- D_{12} has full column rank
- D_{21} has full row rank.

A general compensator for this system is

$$\dot{x}_c = A_c x_c + B_c y , \quad (4)$$

$$u = C_c x_c , \quad (5)$$

where $x_c \in R^{n_c}$ is the state vector of the controller, the dimension of which can be specified. Figure 1 illustrates this design framework. Closing the loop using negative feedback yields the closed-loop system dynamics:

$$\dot{\bar{x}} = \tilde{A} \bar{x} + \tilde{B} w , \quad (6)$$

$$z = \tilde{C} \bar{x} , \quad (7)$$

where

$$\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} , \quad (8)$$

$$\tilde{A} = \begin{bmatrix} A & -B_2 C_c \\ B_c C_2 & A_c - B_c D_{22} C_c \end{bmatrix}, \quad (9)$$

$$\tilde{B} = \begin{bmatrix} B_1 \\ B_c D_{21} \end{bmatrix}, \quad (10)$$

$$\tilde{C} = [C_1 \quad -D_{12} C_c]. \quad (11)$$

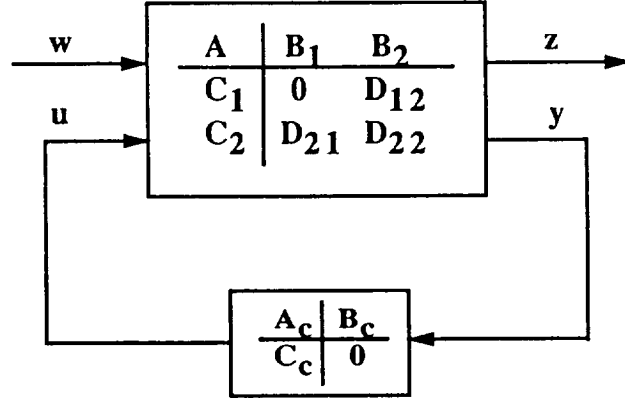


Figure 1. Generalized plant with general compensator.

The set of all internally stabilizing compensators is defined as:

$$S_c = \{ (A_c, B_c, C_c) : \tilde{A} \text{ is asymptotically stable} \}. \quad (12)$$

For an H_2 problem, the objective is to minimize the H_2 -norm on the closed-loop transfer function from disturbance inputs to performance outputs:

$$T_{zw} = \tilde{C}(sI - \tilde{A})^{-1} \tilde{B}, \quad (13)$$

where the disturbances are confined to the set of signals with bounded power and fixed spectra. This leads to three equivalent H_2 optimization problems. For impulsive inputs $w_i = \delta(t)$, the objective is

$$\min_{S_c} \left\{ J(A_c, B_c, C_c) = \|T_{zw}\|_2 = \left(\sum_i \|z_i\|_2^2 \right)^{1/2}, \quad i = 1, \dots, ny \right\}, \quad (14)$$

where z_i is the response to w_i and $\|z_i\|_2$ denotes the L_2 norm. For $w_i = a_i \delta(t)$ with $E\{a_i\} = 0$, $E\{a_i a_j\} = \delta(i-j)$ the objective becomes:

$$\min_{S_c} \left\{ J(A_c, B_c, C_c) = E_{x_0} \left\{ \int_0^\infty z(t)^T z(t) dt \right\} \right\}, \quad (15)$$

where $z(t)$ is the response to an initial condition $x(0)$, and $E_{x_0}\{\cdot\}$ denotes the expectation over a distribution of initial conditions defined by $E\{\bar{x}(0)\} = 0$, $E\{\bar{x}(0)\bar{x}(0)^T\} = \tilde{B}\tilde{B}^T$. If the disturbance is modeled as white noise, the objective is:

$$\min_{S_c} \left\{ J(A_c, B_c, C_c) = \lim_{t \rightarrow \infty} E \{ z(t)^T z(t) \} \right\} . \quad (16)$$

It can be shown that the cost in the three formulations given above can be expressed as:

$$J(A_c, B_c, C_c) = \text{tr} \{ Q \tilde{B} \tilde{B}^T \} = \text{tr} \{ P \tilde{C}^T \tilde{C} \} , \quad (17)$$

where

$$\tilde{A} P + P \tilde{A}^T + \tilde{B} \tilde{B}^T = 0 , \quad (18)$$

$$\tilde{A}^T Q + Q \tilde{A} + \tilde{C}^T \tilde{C} = 0 . \quad (19)$$

P is the controllability grammian of (\tilde{A}, \tilde{B}) and Q is the observability grammian of (\tilde{C}, \tilde{A}) .

In order to obtain the H_2 -optimal compensator, the Lagrangian is defined as:

$$\mathcal{L}(Q, L, A_c, B_c, C_c) = \text{tr} \left\{ Q \tilde{B} \tilde{B}^T + (\tilde{A}^T Q + Q \tilde{A} + \tilde{C}^T \tilde{C}) L \right\} , \quad (20)$$

where L is a symmetric matrix of multipliers. Matrix gradients are taken to determine the first-order necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial Q} = \frac{\partial \mathcal{L}}{\partial L} = \frac{\partial \mathcal{L}}{\partial A_c} = \frac{\partial \mathcal{L}}{\partial B_c} = \frac{\partial \mathcal{L}}{\partial C_c} = 0 . \quad (21)$$

Hence, computing an H_2 -optimal controller of fixed-order $nc < n$ for the general controller structure given in equations (4) and (5) requires the simultaneous solution of five coupled equations. This is not only computationally expensive, but is also further complicated by the fact that the problem is overparametrized with such a compensator.

To avoid the problem of overparametrization, a canonical form description for the controller can be used. It was shown in reference 8 that if either a controller or observer canonical form is imposed on the compensator structure, the number of parameters is reduced to its minimal number. The internal structure of the compensator is prespecified by assigning a set of feedback invariant indices v_i . In controller canonical form the compensator is defined as:

$$\dot{x}_c = P^0 x_c + N^0 u_c - N^0 y , \quad (22)$$

$$u_c = -P x_c , \quad (23)$$

$$u = -H x_c , \quad (24)$$

where $x_c \in R^{nc}$ and $u_c \in R^{ny}$. P and H are free-parameter matrices, and P^0 and N^0 are fixed matrices of zeros and ones determined by the choice of controllability indices v_i as follows:

$$P^0 = \text{block diag} \{ P_1^0, \dots, P_{ny}^0 \} , \quad (25)$$

$$P_i^0 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}_{v_i \times v_i}, \quad i = 1, \dots, n_y, \quad (26)$$

$$N^0 = \text{block diag}\{[0 \dots 01]_{1 \times v_i}^T\}, \quad (27)$$

where $i=1, \dots, n_y$. The controllability indices must satisfy the following condition:

$$\sum_i v_i = n_c, \quad i = 1, \dots, n_y. \quad (28)$$

Figure 2 shows the structure of the controller. Similarly, a compensator in observer canonical form can be constructed. In this paper, only the controller canonical form is employed, which imposes the lower bound $n_c \geq n_y$ on the order of the compensator.

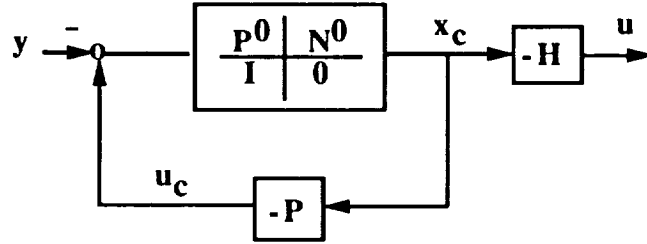


Figure 2. Compensator in controller canonical form.

Let

$$\bar{u} = \begin{bmatrix} u \\ u_c \end{bmatrix}. \quad (29)$$

The augmented system may be expressed as:

$$\dot{\bar{x}} = \begin{bmatrix} A & 0 \\ -N^0 C_2 & P^0 \end{bmatrix} \bar{x} + \begin{bmatrix} B_1 \\ -N^0 D_{21} \end{bmatrix} w + \begin{bmatrix} B_2 & 0 \\ -N^0 D_{22} & N^0 \end{bmatrix} \bar{u} = \bar{A}\bar{x} + \bar{B}_1 w + \bar{B}_2 \bar{u}, \quad (30)$$

$$z = [C_1 \ 0] \bar{x} + [D_{12} \ 0] \bar{u} = \bar{C}_1 \bar{x} + \bar{D}_{12} \bar{u}, \quad (31)$$

$$\bar{y} = [0 \ I] \bar{x} = \bar{C}_2 \bar{x}, \quad (32)$$

$$\bar{u} = - \begin{bmatrix} H \\ P \end{bmatrix} \bar{y} = -G \bar{y}. \quad (33)$$

Equations (30) through (33) define a static gain output feedback problem where the compensator is represented by a minimal number of free parameters in the design matrix, G . The augmented system is shown in figure 3. The closed-loop system is given by:

$$\dot{x} = (\bar{A} - \bar{B}_2 G \bar{C}_2)x + \bar{B}_1 w = \tilde{A}x + \tilde{B}w , \quad (34)$$

$$z = (\bar{C}_1 - \bar{D}_{12} G \bar{C}_2)x = \tilde{C}x . \quad (35)$$

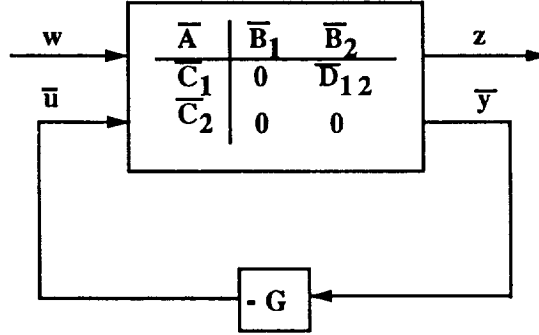


Figure 3. Augmented system with compensator in controller canonical form.

Minimizing the H_2 -norm of $T_{zw} = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ utilizes the same Lagrangian as given in equation (20), but now \mathcal{L} is only a function of three parameter matrices, i.e., $\mathcal{L}(Q, L, G)$. Thus, only three first-order necessary conditions result:

$$\frac{\partial \mathcal{L}}{\partial Q} = \tilde{A}L + L\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 , \quad (36)$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T \tilde{C} = 0 , \quad (37)$$

$$\frac{\partial \mathcal{L}}{\partial G} = 2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q)L \bar{C}_2^T = 0 . \quad (38)$$

This demonstrates that using the controller canonical form yields simpler expressions for the necessary conditions, with the additional benefit of minimizing the number of compensator parameters.

Controller canonical forms can also be used to solve the H_∞ problem. The objective is now to minimize the ∞ -norm of the transfer function from disturbance inputs w to performance outputs z given in equation (13). In this case, the necessary conditions for an H_∞ -optimal fixed-order compensator gain G are:⁷

$$\frac{\partial \mathcal{L}}{\partial Q_\infty} = (\tilde{A} + \gamma^{-2} \tilde{B}\tilde{B}^T Q_\infty)L + L(\tilde{A} + \gamma^{-2} \tilde{B}\tilde{B}^T Q_\infty)^T + \tilde{B}\tilde{B}^T = 0 , \quad (39)$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q_\infty + Q_\infty \tilde{A} + \tilde{C}^T \tilde{C} + \gamma^{-2} Q_\infty \tilde{B}\tilde{B}^T Q_\infty = 0 , \quad (40)$$

$$\frac{\partial \mathcal{L}}{\partial G} = 2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty)L \bar{C}_2^T = 0 , \quad (41)$$

where

$$\mathcal{L}(Q_\infty, L, G) = \text{tr}\{Q_\infty \bar{B} \bar{B}^T + (\bar{A}^T Q_\infty + Q_\infty \bar{A} + \bar{C}^T \bar{C} + \gamma^{-2} Q_\infty \bar{B} \bar{B}^T Q_\infty) L\} . \quad (42)$$

As in the H_2 problem, three coupled equations have to be solved to obtain a fixed-order compensator which satisfies the constraint $\|T_{zw}\|_\infty < \gamma$.

Using this approach, fixed-order H_∞ -design can also be extended to fixed-order μ -synthesis. Since H_∞ controller design is a subproblem when designing for robust performance with structured uncertainty, the fixed-order technique introduced above has the potential to constrain the order of the controller which is normally subject to significant increases in the μ -synthesis procedure.

The mixed H_2/H_∞ problem can be approached in a similar fashion. In this case, the generalized plant has additional inputs and outputs w_p and z_p , respectively, which define the H_2 performance criterion (fig. 4). The inputs w and outputs z are used to define the H_∞ performance criterion. Using the controller canonical form for the compensator, the augmented system for the mixed problem is:

$$\dot{\bar{x}} = \bar{A} \bar{x} + \bar{B}_1 w + \bar{B}_p w_p + \bar{B}_2 \bar{u} , \quad (43)$$

$$z_p = \bar{C}_p \bar{x} + \bar{D}_{1p} \bar{u} , \quad (44)$$

$$z = \bar{C}_1 \bar{x} + \bar{D}_{12} \bar{u} , \quad (45)$$

$$y = \bar{C}_2 \bar{x} , \quad (46)$$

$$\bar{u} = -G \bar{y} , \quad (47)$$

where

$$\bar{B}_p = \begin{bmatrix} B_p \\ -N^0 D_{2p} \end{bmatrix} , \quad (48)$$

$$\bar{C}_p = [C_p \ 0] , \quad (49)$$

$$\bar{D}_{1p} = [D_{1p} \ 0] . \quad (50)$$

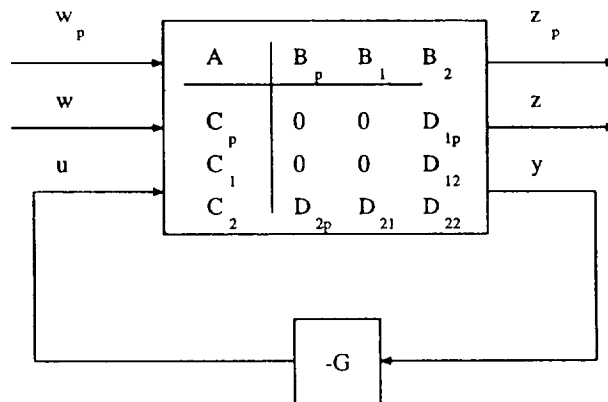


Figure 4. Mixed H_2/H_∞ problem.

The other expressions are the same as in equations (30) through (33). Consequently, the closed-loop system is given by:

$$\dot{\bar{x}} = (\bar{A} - \bar{B}_2 G \bar{C}_2) \bar{x} + \bar{B}_p w_p + \bar{B}_1 w = \tilde{A} \bar{x} + \tilde{B}_p w_p + \tilde{B} w , \quad (51)$$

$$z_p = (\bar{C}_p - \bar{D}_{1p} G \bar{C}_2) \bar{x} = \tilde{C}_p \bar{x} , \quad (52)$$

$$z = (\bar{C}_1 - \bar{D}_{12} G \bar{C}_2) \bar{x} = \tilde{C} \bar{x} . \quad (53)$$

In order to formulate the performance index of the mixed problem, the Lagrangian for the H_2 problem in equation (20) is adjoined to the Lagrangian for the H_∞ problem in equation (42) by a scalar weight λ :

$$\mathcal{L} = \text{tr} \{ Q_\infty \tilde{B} \tilde{B}^T + (\tilde{A}^T Q_\infty + Q_\infty \tilde{A} + \tilde{C}^T \tilde{C} + \gamma^{-2} Q_\infty \tilde{B} \tilde{B}^T Q_\infty) L + \lambda X \tilde{C}_p^T \tilde{C}_p + (\tilde{A} X + X \tilde{A}^T + \tilde{B}_p \tilde{B}_p^T) L_p \} . \quad (54)$$

The weight λ on the H_2 -norm allows a tradeoff between (H_2) performance and the H_∞ norm. The first order necessary conditions are:

$$\frac{\partial \mathcal{L}}{\partial Q_\infty} = (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty) L + L (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty)^T + \tilde{B} \tilde{B}^T = 0 , \quad (55)$$

$$\frac{\partial \mathcal{L}}{\partial L} = \tilde{A}^T Q_\infty + Q_\infty \tilde{A} + \tilde{C}^T \tilde{C} + \gamma^{-2} Q_\infty \tilde{B} \tilde{B}^T Q_\infty = 0 , \quad (56)$$

$$\frac{\partial \mathcal{L}}{\partial X} = \tilde{A}^T L_p + L_p \tilde{A} + \lambda \tilde{C}_p^T \tilde{C}_p = 0 , \quad (57)$$

$$\frac{\partial \mathcal{L}}{\partial L_p} = \tilde{A} X + X \tilde{A}^T + \tilde{B}_p \tilde{B}_p^T = 0 , \quad (58)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial G} = & 2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 L \bar{C}_2^T - \bar{D}_{12}^T \bar{C}_1 L \bar{C}_2^T - \bar{B}_2^T Q_\infty L \bar{C}_2^T + \lambda \bar{D}_{1p}^T \bar{D}_{1p} G \bar{C}_2 X \bar{C}_2^T \\ & - \lambda \bar{D}_{1p}^T \bar{C}_p X \bar{C}_2^T - \bar{B}_2^T L_p X \bar{C}_2^T) = 0 . \end{aligned} \quad (59)$$

As demonstrated above, imposing a controller canonical form on the compensator structure provides a powerful tool for the design of fixed-order controllers. Promising results have been obtained for the H_∞ and the mixed problem in reference 7 where a conjugate gradient method was used in the computation. A disadvantage of this method is that convergence slows down near the optimum. Also, an initial starting guess for the compensator gain G has to be provided that stabilizes the closed-loop system. In this paper, a homotopy method is used to continuously deform the solution of a simple problem formulation to the solution of the desired problem formulation.

III. HOMOTOPY METHODS

Homotopy methods offer an attractive alternative to more standard approaches of optimal controller synthesis such as sequential and conjugate gradient methods. The basic philosophy of homotopy methods is to deform a problem which is relatively easily solved into the problem for which a solution is desired.

Homotopy (or continuation) methods, arising from algebraic and differential topology, embed a given problem in a parameterized family of problems. More specifically, consider sets U and $Y \in \mathfrak{R}^n$ and a mapping $F : U \rightarrow Y$, where solutions of the problem

$$F(u) = 0 \quad (60)$$

are desired with $u \in U$ and $F(u) \in Y$. The homotopy function is defined by the mapping $H: U \times [0, 1] \rightarrow \mathfrak{R}^n$ such that:

$$H(u_1, 1) = F(u) \quad , \quad (61)$$

and there exists a known (or easily calculated) solution, u_0 , such that:

$$H(u_0, 0) = 0 \quad . \quad (62)$$

The homotopy function is a continuously differentiable function given by:

$$H(u(\alpha), \alpha) = 0, \quad \forall \alpha \in [0, 1] \quad . \quad (63)$$

Thus, the homotopy begins with a simple problem with a known solution (equation (62)) which is deformed by continuously varying the parameter until the solution of the original problem (equation (60)) is obtained.¹⁰ The power of homotopy methods is that minimization is not strongly dependent on starting solution, but depends on local, small variations in the solution. Theoretically, these methods are globally convergent for a wide range of complex optimization problems, but in actuality, finite wordlength computation often introduces numerical ill-conditioning resulting in difficulties with convergence. In light of these numerical limitations, a judicious choice of the initial problem is necessary for convergence and efficient computation. However, the ability to select an initial problem with a simple solution renders homotopy methods more widely applicable than sequential- or gradient-based methods, which have a stringent requirement for an initial stabilizing solution.

Both discrete and continuous methods are used to solve the homotopy. Discrete methods simply partition the interval $[0, 1]$ to obtain a finite chain of problems:

$$H(u, \alpha_n) = 0, \quad 0 = \alpha_0 < \alpha_1 < \dots < \alpha_N = 1 \quad . \quad (64)$$

Starting with a known solution at α_n , the solution for $H(u, \alpha_{n+1})$ is computed by a local iteration scheme. Continuous methods involve integration of Davidenko's differential equation, which is obtained by differentiating equation (63) with respect to α , yielding:

$$\frac{du}{d\alpha} = -\left(\frac{\partial H}{\partial u}\right)^{-1} \frac{\partial H}{\partial \alpha} . \quad (65)$$

Given $u(0) = u_0$, this initial value problem may be numerically integrated to obtain the solution at $\alpha = 1$ if the solution exists and is uniquely defined.

Research remains to be done in the application of homotopy algorithms. Efficient application of homotopy methods depends on the initial problem, the final problem, and the deformation undertaken. Given a good initial solution, the key to convergence is the ability to accurately track the solution curve, which is determined by the deformation undertaken. The ability to predict the solution along the homotopy path via Davidenko's differential equation makes continuous homotopy methods superior to discrete methods. Efficient computation of the Hessian is the primary issue for practical implementation of continuous homotopy algorithms. In the following sections, continuous homotopy algorithms are presented for reduced-order H_2 , H_∞ , and mixed H_2/H_∞ compensator designs.

A. Homotopy Algorithm

This section describes the algorithm used for implementing the continuous homotopy. In essence, a mixed discrete and continuous approach is employed where Davidenko's differential equation (65) is integrated along the homotopy path, and at discrete points along the trajectory, a Newton's correction is used for local optimization to remove integration error. Newton's method, which has quadratic convergence properties in a neighborhood of the local minimum, allows a crude integration procedure with large step sizes to be employed for efficiently tracking the solution curve. This approach follows closely that of references 11 and 12 and is employed in the following algorithm.

1. Find initial solution ($\alpha = 0$).
2. Advance α .

$$\alpha_{1,k} = \alpha_0 + \Delta\alpha_{0,k} .$$

3. Predict θ .

$$\theta(\alpha_{1,k}) = \theta(\alpha_0) + \Delta\alpha_{0,k} \theta'(\alpha_0) ,$$

where

$$\theta'(\alpha) = \frac{d\theta}{d\alpha} = -\left(\frac{\partial H}{\partial \theta}\right)^{-1} \frac{\partial H}{\partial \alpha} .$$

4. Check prediction error.

$$e_k(\theta, \alpha) = \|J_\theta(\theta(\alpha_{1,k}))\| < \varepsilon .$$

- a. If error less than tolerance, continue.
- b. If not, $0.5\Delta\alpha_{0,k} \rightarrow \Delta\alpha_{0,k+1}$.

- c. Increment k and repeat steps 2 to 4.
5. Correct with Newton's method to compute local minimum.
6. If $\alpha = 1$, stop. Otherwise, go to step 2.

Various approaches may be taken when selecting the deformation, but the general procedure applied in this effort is outlined as follows:

- Synthesize a low-authority H_2 (full-order) compensator
- Reduce the compensator to desired order and transform to canonical form⁹
- Set γ large enough so that the H_2 and H_∞ compensators are approximately equivalent
- Use homotopy to deform the initial low-authority, reduced-order H_2 compensator into a full-authority reduced-order H_2 compensator (H_2 homotopy)
- Deform the full-authority H_2 compensator into a nearly optimal H_∞ compensator with γ approaching its infimum (H_∞ homotopy)
- At discrete values of λ , fix γ and deform the compensator into the mixed H_2/H_∞ compensator by varying λ (mixed H_2/H_∞ homotopy).

This procedure was chosen because it has been observed numerically that order reduction techniques tend to work best for low-authority LQG controllers.¹¹ A canonical form is imposed on the compensator structure to minimize the number of free parameters, which in some cases can also lead to numerical ill-conditioning. A balancing transformation which does not affect the controller characteristics relaxes the strict structure in the P^0 and N^0 matrices in equations (25) through (27) and improves the conditioning of the problem.

The procedure outlined above separates the compensator synthesis into distinct phases. The initial reduced-order, full-authority compensator is synthesized using the H_2 homotopy, which is then deformed into the reduced-order H_∞ compensator. During the H_∞ phase, the scalar H_2 norm weight λ is fixed (as are the plant matrices) and only the H_∞ norm overbound γ is varied. At discrete values, γ is fixed and λ is varied to perform the H_2 norm minimization. Thus, the procedure alternates between the H_∞ and H_2 norm minimization.

During the homotopy, both the predicted and corrected gains are checked to assure closed-loop stability. After each correction step, the cost gradient is checked to verify descent. During the H_∞ homotopy, the solvability of the Riccati equation using predicted or corrected gains must also be checked. If any of these conditions are violated during correction, the correction step size is scaled and the condition is checked again. If scaling the correction step size is ineffective, the prediction step size is decreased and the prediction phase is repeated. This process continues until the homotopy is completed or until the prediction step size is decreased below a prespecified tolerance. The following sections detail the derivations employed in the homotopy algorithm.

B. The H_2 Case

In reference 11, a continuous homotopy algorithm is presented for H_2 compensator synthesis when the compensator has a general architecture, which requires the solution of the five coupled equations given by equation (21). The following development parallels that of reference 11 except in this formulation, a controller canonical form is employed for the compensator dynamics which results in the three necessary conditions (equations (36) through (38)).

Consider equations (34) and (35). The necessary conditions for an H_2 optimal compensator are given by equations (36) through (38). Define θ to be a vector comprised of the free compensator parameters

$$\theta = \text{vec}(G) , \quad (66)$$

where G is the output feedback gain matrix defined in equation (33). The gradient of the cost is:

$$f(\theta) = \frac{\partial \mathcal{L}}{\partial \theta} = \text{vec} \left(\frac{\partial \mathcal{L}}{\partial G} \right) = 0 , \quad (67)$$

where $\partial \mathcal{L} / \partial G$ is given by equation (38).

The homotopy function is defined as

$$H(\theta, \alpha) = \frac{\partial \mathcal{L}(\theta, \alpha)}{\partial \theta} = \text{vec} \left(\frac{\partial \mathcal{L}(\theta, \alpha)}{\partial G} \right) = 0 . \quad (68)$$

Note that \mathcal{L} is now a function of the homotopy parameter α since the system matrices are now functions of α . The gradient of the homotopy function is

$$\nabla[H^T(\theta, \alpha)] = [H_\theta \ H_\alpha] . \quad (69)$$

1. *Computation of Hessian.* The hessian of the cost, H_θ , is given by:

$$H_\theta = \text{vec} \left(\frac{\partial H(\theta, \alpha)}{\partial \theta_j} \right)^T , \quad j = 1, N , \quad (70)$$

where N is the number of free parameters. Using equation (38),

$$\frac{\partial H(\theta, \alpha)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} [2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q) L \bar{C}_2^T] , \quad (71)$$

$$= 2(\bar{D}_{12}^T \bar{D}_{12} G^{(j)} \bar{C}_2 - \bar{B}_2^T Q^{(j)}) L \bar{C}_2^T + 2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q) L^{(j)} \bar{C}_2^T , \quad (72)$$

where

$$(*)^{(j)} = \frac{\partial (*)}{\partial \theta_j} . \quad (73)$$

For $\theta_j = g_{ik}$,

$$G^{(j)} = \frac{\partial G}{\partial \theta_j} = E_{ik} , \quad (74)$$

which is a matrix of zeros except for a one in the ik element. From equations (36) and (37), derivatives of L and Q are obtained by solving the Lyapunov equations:

$$\tilde{A}L^{(j)} + L^{(j)}\tilde{A}^T + [\tilde{A}^{(j)}L + L\tilde{A}^{(j)T} + (\tilde{B}\tilde{B}^T)^{(j)}] = 0 , \quad (75)$$

$$\tilde{A}^T Q^{(j)} + Q^{(j)}\tilde{A} + [\tilde{A}^{(j)T}Q + Q\tilde{A}^{(j)} + (\tilde{C}^T\tilde{C})^{(j)}] = 0 , \quad (76)$$

where from equations (34) and (35),

$$\tilde{A}^{(j)} = -\bar{B}_2 G^{(j)} \bar{C}_2 , \quad (77)$$

$$(\tilde{B}\tilde{B}^T)^{(j)} = 0 , \quad (78)$$

$$\begin{aligned} (\tilde{C}^T\tilde{C})^{(j)} &= (\bar{C}_2^T G^{(j)T} \bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2) + (\bar{C}_2^T G^{(j)T} \bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2)^T \\ &\quad - (\bar{C}_1^T \bar{D}_{12} G^{(j)} \bar{C}_2) - (\bar{C}_1^T \bar{D}_{12} G^{(j)} \bar{C}_2)^T . \end{aligned} \quad (79)$$

2. *Computation of H_α .* The derivative of the homotopy function with respect to the homotopy parameter, α , is:

$$H_\alpha = \text{vec} \left(\frac{\partial H(\theta, \alpha)}{\partial \alpha} \right) , \quad (80)$$

where

$$\begin{aligned} \frac{\partial H(\theta, \alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha} [2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q) L \bar{C}_2^T] , \quad (81) \\ &= 2[(\dot{\bar{D}}_{12}^T \bar{D}_{12} G \bar{C}_2 + \bar{D}_{12}^T \dot{\bar{D}}_{12} G \bar{C}_2 + \bar{D}_{12}^T \bar{D}_{12} G \dot{\bar{C}}_2 - \dot{\bar{D}}_{12}^T \bar{C}_1 - \bar{D}_{12}^T \dot{\bar{C}}_1 - \dot{\bar{B}}_2^T Q \\ &\quad - \bar{B}_2^T \dot{Q}) L \bar{C}_2^T + (\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q) (\dot{L} \bar{C}_2^T + L \dot{\bar{C}}_2^T)] , \end{aligned} \quad (82)$$

and

$$(\dot{*}) = \frac{\partial (*)}{\partial \alpha} . \quad (83)$$

The derivative expressions (equations (80) through (82)) depend on the deformation undertaken in the specified problem, i.e., the initial and final problem. In general, suppose that the deformation of the matrix \bar{A} is prescribed to be:

$$\bar{A}(\alpha) = \bar{A}_0(\alpha) + \alpha(\bar{A}_f(\alpha) - \bar{A}_0(\alpha)) , \quad (84)$$

where the 0 and f subscripts indicate the initial and final system matrices, respectively. It follows that

$$\dot{\bar{A}} = \bar{A}_f - \bar{A}_0 . \quad (85)$$

The derivative of other plant matrices are determined accordingly. The derivatives of L and Q with respect to α are obtained from equations (36) and (37) by solving the Lyapunov equations:

$$0 = \bar{A}L + L\bar{A}^T + (\dot{\bar{A}}L + L\dot{\bar{A}}^T + \dot{\bar{B}}\bar{B}^T + \bar{B}\dot{\bar{B}}^T) , \quad (86)$$

$$0 = \bar{A}^T Q + Q\bar{A} + (\dot{\bar{A}}^T Q + Q\dot{\bar{A}} + \dot{\bar{C}}^T \bar{C} + \bar{C}^T \dot{\bar{C}}) , \quad (87)$$

where from equations (34) and (35)

$$\dot{\bar{A}} = \dot{\bar{A}} - \dot{\bar{B}}_2 G \bar{C}_2 - \bar{B}_2 G \dot{\bar{C}}_2 , \quad (88)$$

$$\dot{\bar{B}} = \dot{\bar{B}}_1 , \quad (89)$$

$$\dot{\bar{C}} = \dot{\bar{C}}_1 - \dot{\bar{D}}_{12} G \bar{C}_2 - \bar{D}_{12} G \dot{\bar{C}}_2 . \quad (90)$$

By employing canonical compensator formulations of equations (30) through (33), the expressions for the derivatives of the augmented system matrices reduce to:

$$\dot{\bar{A}} = \begin{bmatrix} A_f - A_0 & 0 \\ -N^0(C_{2,f} - C_{2,0}) & 0 \end{bmatrix} , \quad (91)$$

$$\dot{\bar{B}}_1 = \begin{bmatrix} B_{1,f} - B_{1,0} \\ -N^0(D_{21,f} - D_{21,0}) \end{bmatrix} , \quad (92)$$

$$\dot{\bar{B}}_2 = \begin{bmatrix} B_{2,f} - B_{2,0} & 0 \\ -N^0(D_{22,f} - D_{22,0}) & 0 \end{bmatrix} , \quad (93)$$

$$\dot{\bar{C}}_1 = [C_{1,f} - C_{1,0} \ 0] , \quad (94)$$

$$\dot{\bar{C}}_2 = [0 \ 0] , \quad (95)$$

$$\dot{\bar{D}}_{12} = [D_{12,f} - D_{12,0} \ 0] . \quad (96)$$

Thus, the use of canonical forms not only simplifies the necessary conditions by grouping all the free compensator parameters into one feedback gain matrix, but also simplifies the derivative expressions. The presence of the zero subblocks significantly enhances the computational efficiency of this approach.

When implementing the procedure described at the beginning of this section, the above equations may be further specialized. In this procedure, the initial and final plant matrices are the same, and the homotopy is performed only on the measurement and process noise intensities, D_{12} and D_{21} . Hence, $\dot{\tilde{A}}$, $\dot{\tilde{B}}_2$, $\dot{\tilde{C}}_1$, and $\dot{\tilde{C}}_2$ are identically zero.

C. The H_∞ Case

A continuous homotopy algorithm has also been developed to solve the fixed-order H_∞ control problem. The development for the H_∞ homotopy algorithm is identical to that of the previous section for the H_2 homotopy algorithm with the exception of additional terms in the necessary conditions (equations (39) through (41)) resulting from the γ weighted disturbance term in the cost function (equation (42)). Only the portions that differ from the previous section will be presented here.

As with the H_2 case, when implementing the procedure described at the beginning of this section, the previous equations may be further specialized for the H_∞ case. In this procedure, the H_∞ homotopy begins with a full-authority H_2 compensator (obtained by the H_2 homotopy) which is then deformed into a full-authority H_∞ compensator by decreasing γ . Note that for large values of γ , the H_∞ necessary conditions are equivalent to the H_2 case. The initial and final plant matrices and weights are identical, and the homotopy is performed on γ only, resulting in considerable simplifications in the computation of H_α . The value of γ is linearly varied from an initial high value toward a lower bound (determined from a full-order design) according to:

$$\gamma = \gamma_{\max} - \alpha(\gamma_{\max} - \gamma_{\min}). \quad (97)$$

Thus, the homotopy function defined by equations (68) and (41) is implicitly a function of α .

1. *Computation of Hessian.* The hessian for the H_∞ homotopy is identical to the hessian for the H_2 case, except that the observability grammian, Q , is replaced by Q_∞ , which is the solution of equation (39). The hessian is given by:

$$\frac{\partial H(\theta, \alpha)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} \left[2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L \bar{C}_2^T \right], \quad (98)$$

$$= 2(\bar{D}_{12}^T \bar{D}_{12} G^{(j)} \bar{C}_2 - \bar{B}_2^T Q_\infty^{(j)}) L \bar{C}_2^T + 2(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L^{(j)} \bar{C}_2^T. \quad (99)$$

To obtain expressions for $L^{(j)}$ and $Q_\infty^{(j)}$, differentiate equations (39) and (40) to obtain:

$$0 = (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty) L^{(j)} + L^{(j)} (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty)^T + [(\tilde{A}^{(j)} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty^{(j)}) L + L(\tilde{A}^{(j)} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty^{(j)})^T], \quad (100)$$

$$0 = (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty)^T Q_\infty^{(j)} + Q_\infty^{(j)} (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty) + (\tilde{A}^{(j)T} Q_\infty + Q_\infty \tilde{A}^{(j)} + (\tilde{C}^T \tilde{C})^{(j)}), \quad (101)$$

where $\tilde{A}^{(j)}$ and $(\tilde{C}^T \tilde{C})^{(j)}$ are given by equations (77) and (79), respectively.

2. *Computation of H_α .* The derivative of the homotopy function with respect to the homotopy parameter, α , is given by equations (80) and (81) where Q is replaced by Q_∞ , and equation (82) reduces to:

$$\frac{\partial H(\theta, \alpha)}{\partial \alpha} = 2[(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L \bar{C}_2^T - \bar{B}_2^T \dot{Q}_\infty L \bar{C}_2^T] . \quad (102)$$

L and \dot{Q}_∞ are obtained by differentiating equations (39) and (40) and are given by:

$$0 = (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty) L + L (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty)^T + (\Gamma L + L \Gamma^T) , \quad (103)$$

$$0 = (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty)^T \dot{Q}_\infty + \dot{Q}_\infty (\tilde{A} + \gamma^{-2} \tilde{B} \tilde{B}^T Q_\infty) - (2\gamma^{-3} \dot{\gamma} Q_\infty \tilde{B} \tilde{B}^T Q_\infty) , \quad (104)$$

where

$$\Gamma = -2\gamma^{-3} \dot{\gamma} \tilde{B} \tilde{B}^T Q_\infty + \gamma^{-2} \tilde{B} \tilde{B}^T \dot{Q}_\infty . \quad (105)$$

The derivative of γ with respect to α is obtained from equation (97):

$$\dot{\gamma} = -(\gamma_{\max} - \gamma_{\min}) . \quad (106)$$

Note that a general homotopic deformation of the plant matrices and weights is allowable by introducing the variation of the plant matrices with respect to the homotopy. The two-step procedure presented here can, in principle, be reduced to a one-step homotopy while simultaneously deforming the system matrices (the H_2 case) and γ (the H_∞ case). However, the minimum achievable value of γ when designing a reduced-order H_∞ compensator is usually not known a priori, whereas the system matrices must be fully deformed (i.e., α must attain the value of 1) to obtain the desired system representation. Thus, the two-step procedure described previously was used in computing the fixed-order H_∞ compensators. For the H_∞ homotopy, only the overbound γ is deformed, thereby greatly simplifying the required computation.

D. The Mixed H_2/H_∞ Case

1. *Computation of Hessian.* A homotopy algorithm that solves the necessary conditions for the mixed H_2/H_∞ case equations (55) through (59) is obtained in a straightforward extension of the H_2 and H_∞ homotopy algorithms. The hessian for the mixed case is given by:

$$\begin{aligned} \frac{\partial H(\theta, \alpha)}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j} \left\{ 2[(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L \bar{C}_2^T + (\lambda \bar{D}_{1p}^T \bar{D}_{1p} G \bar{C}_2 - \lambda \bar{D}_{1p}^T \bar{C}_p - \bar{B}_2^T L_p) X \bar{C}_2^T] \right\} , \quad (107) \\ &= 2[(\bar{D}_{12}^T \bar{D}_{12} G^{(j)} \bar{C}_2 - \bar{B}_2^T Q_\infty^{(j)}) L \bar{C}_2^T + (\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L^{(j)} \bar{C}_2^T + (\lambda \bar{D}_{1p}^T \bar{D}_{1p} G^{(j)} \bar{C}_2 \\ &\quad - \bar{B}_2^T L_p^{(j)}) X \bar{C}_2^T + (\lambda \bar{D}_{1p}^T \bar{D}_{1p} G \bar{C}_2 - \lambda \bar{D}_{1p}^T \bar{C}_p - \bar{B}_2^T L_p) X^{(j)} \bar{C}_2^T] . \quad (108) \end{aligned}$$

Expressions for $L_p^{(j)}$ and $X^{(j)}$, obtained by differentiating equations (57) and (58), are given by:

$$0 = \tilde{A}^T L_p^{(j)} + L_p^{(j)} \tilde{A} + [\tilde{A}^{(j)T} L_p + L_p \tilde{A}^{(j)} + \lambda (\tilde{C}_p^T \tilde{C}_p)^{(j)}] , \quad (109)$$

$$0 = \tilde{A} X^{(j)} + X^{(j)} \tilde{A}^T + [\tilde{A}^{(j)} X + X \tilde{A}^{(j)T} + (\tilde{B}_p \tilde{B}_p^T)^{(j)}] , \quad (110)$$

while $L^{(j)}$, $Q_\infty^{(j)}$, and $\tilde{A}^{(j)}$ are the same as the H_∞ case, equations (100), (101), and (77), respectively. From equation (52),

$$(\tilde{C}_p^T \tilde{C}_p)^{(j)} = (-\bar{D}_{1p} G^{(j)} \bar{C}_2)^T (\bar{C}_p - \bar{D}_{1p} G \bar{C}_2) + (\bar{C}_p - \bar{D}_{1p} G \bar{C}_2)^T (-\bar{D}_{1p} G^{(j)} \bar{C}_2) . \quad (111)$$

2. *Computation of H_α .* As with the computation of the hessian, the derivative of the homotopy function with respect to α is given by:

$$\begin{aligned} \frac{\partial H(\theta, \alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha} \left\{ 2 \left[(\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L \bar{C}_2^T + (\lambda \bar{D}_{1p}^T \bar{D}_{1p} G \bar{C}_2 - \lambda \bar{D}_{1p}^T \bar{C}_p - \bar{B}_2^T L_p) X \bar{C}_2^T \right] \right\} , \quad (112) \\ &= 2 \left[(\dot{\bar{D}}_{12}^T \bar{D}_{12} G \bar{C}_2 + \bar{D}_{12}^T \dot{\bar{D}}_{12} G \bar{C}_2 - \dot{\bar{D}}_{12}^T \bar{C}_1 - \bar{B}_2^T \dot{Q}_\infty) L \bar{C}_2^T + (\bar{D}_{12}^T \bar{D}_{12} G \bar{C}_2 \right. \\ &\quad \left. - \bar{D}_{12}^T \bar{C}_1 - \bar{B}_2^T Q_\infty) L \bar{C}_2^T + (\lambda \dot{\bar{D}}_{1p}^T \bar{D}_{1p} G \bar{C}_2 - \lambda \bar{D}_{1p}^T \dot{\bar{C}}_p - \bar{B}_2^T \dot{L}_p) X \bar{C}_2^T \right] . \quad (113) \end{aligned}$$

Expressions for L_p and X are obtained by differentiating equations (57) and (58) to obtain

$$0 = \tilde{A}^T \dot{L}_p + \dot{L}_p \tilde{A} + (\dot{\tilde{A}}^T L_p + L_p \dot{\tilde{A}} + \lambda \dot{\tilde{C}}_p^T \tilde{C}_p + \lambda \tilde{C}_p^T \dot{\tilde{C}}_p + \dot{\lambda} \tilde{C}_p^T \tilde{C}_p) , \quad (114)$$

$$0 = \tilde{A} \dot{X} + \dot{X} \tilde{A}^T + (\dot{\tilde{A}} X + X \dot{\tilde{A}}^T + \dot{\tilde{B}}_p \tilde{B}_p^T + \tilde{B}_p \dot{\tilde{B}}_p^T) , \quad (115)$$

and

$$\dot{\lambda} = \lambda_{\max} - \lambda_{\min} . \quad (116)$$

Note that the procedure described in section III.A, where the H_∞ and H_2 homotopies are performed distinctly, simplifies the computations significantly in that the plant matrices remain fixed and only γ or λ are varied at one time.

IV. DESIGN EXAMPLE

To demonstrate the homotopy algorithm applied to optimal controller synthesis, the four-disk example originally described in reference 13 and more recently by numerous others (see reference 11) will be used. The four-disk model used in the example problem was derived from a laboratory experiment and represents an apparatus developed for testing of pointing control systems for flexible space structures with noncolocated sensors and actuators. As illustrated in figure 5, four disks are

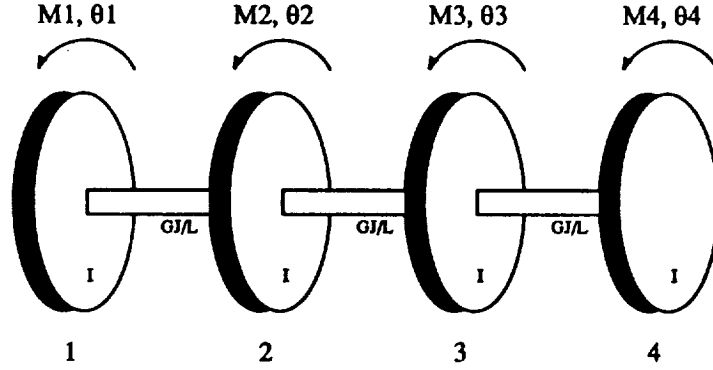


Figure 5. Four-disk system.

rigidly attached to a flexible axial shaft with control torque applied to selected disks and the angular displacement of selected disks measured. The equations of motion may be written as:

$$I_1 \ddot{\theta}_1 + K_1(\theta_1 - \theta_2) = 0 \quad , \quad (117)$$

$$I_2 \ddot{\theta}_2 + K_2(\theta_2 - \theta_3) - K_1(\theta_1 - \theta_2) = 0 \quad , \quad (118)$$

$$I_3 \ddot{\theta}_3 + K_3(\theta_3 - \theta_4) - K_2(\theta_2 - \theta_3) = 0 \quad , \quad (119)$$

$$I_4 \ddot{\theta}_4 - K_3(\theta_3 - \theta_4) = 0 \quad , \quad (120)$$

or

$$\bar{M}\ddot{q} + \bar{K}q = \bar{B}u \quad , \quad (121)$$

where the generalized displacements are the angular displacements of the disks, $q^T = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]$, and the input vector consists of the moments applied to each disk, $u^T = [M_1 \ M_2 \ M_3 \ M_4]$. Defining the state vector as $x^T = [q^T \ \dot{q}^T]$ results in the state space formulation:

$$\dot{x} = Ax + Bu \quad , \quad (122)$$

where

$$A = \begin{bmatrix} 0 & I \\ \bar{M}^{-1}\bar{K} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \bar{M}^{-1}\bar{B} \end{bmatrix} \quad . \quad (123)$$

For simplicity, the stiffness and inertia terms are set to unity ($(GJ/L)_i = K_i = I_i = 1$, $i=1:4$). In this case, the mass matrix is a 4 by 4 identity matrix and the stiffness matrix is:

$$\bar{K} = K \cdot \hat{K} = K \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad . \quad (124)$$

The plant is modeled with parametric uncertainty corresponding to uncertainty in the stiffness of each shaft. Let uncertainty in the shaft stiffness be modeled as:

$$K = K_0 + \Delta K \Rightarrow \bar{K} = (K_0 + \Delta K)\hat{K} . \quad (125)$$

Hence, the A matrix becomes:

$$A = \begin{bmatrix} 0 & I \\ -\bar{M}^{-1}K_0\hat{K} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\bar{M}^{-1}\hat{K} \end{bmatrix} \Delta K [I \ 0] , \quad (126)$$

$$= A_0 + \Delta A . \quad (127)$$

A block diagram representation of the plant with uncertain stiffness is shown in figure 6.

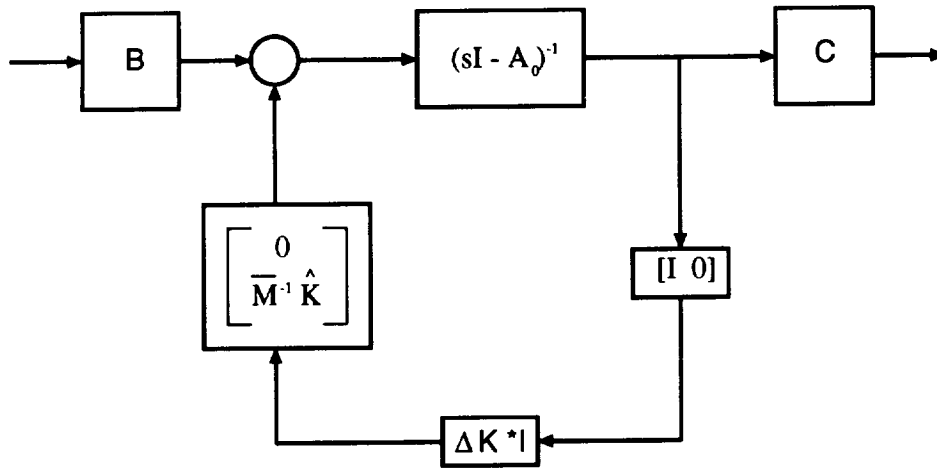


Figure 6. Plant with uncertain stiffness.

A. The H_2 Case

To demonstrate the homotopy algorithm applied to H_2 controller synthesis, the same four-disk model used in reference 11 will be used here. These results provide a direct comparison between the homotopy algorithm of reference 11 (HAO) and the algorithm presented in section III (H2HOM). The main distinction between the two homotopy algorithms for the H_2 case is the compensator architecture. HAO employs a general architecture that may be restricted to various parameterizations including the controllability canonical form (the controller canonical form is used in H2HOM). When the controllability canonical form is implemented in HAO, as in this example, the compensator is still represented in a general architecture, resulting in the evaluation of five necessary conditions (equation (21)). The HAO code has been highly optimized for efficient computation with the result that superfluous computations are not evaluated. The homotopy algorithm of this paper, H2HOM, is patterned after the general approach of HAO and utilizes some of the more efficient computational aspects of the HAO code.

The control design philosophy for this example is to scale the nominal control weight and the nominal sensor noise intensity by the parameter q . As q is reduced, the control authority is

increased. For comparison with the result published in reference 11, a full-order (eighth-order) compensator was synthesized. Although the results can be directly obtained from the LQG Riccati equations, the full-order compensator was chosen to tax the H2HOM algorithm, which must optimize over a greater number of parameters with increasing compensator order.

Table 1 shows a comparison of the results from the H2HOM and HAO algorithms for the full-order compensator along with results from the H2HOM algorithm for sixth- and second-order compensators. All pertinent parameters as well as logic for step size scaling and the computation of the prediction and correction errors are identical in both algorithms, which are implemented in MATLAB™ on a 486 66MHz computer. Whereas the HAO code required a minimum step size of 1.907e-7, the H2HOM code was much better conditioned and required a minimum step size of 0.025. As a consequence of the smaller step sizes with HAO, 2,504 hessian computations were required as opposed to only 63 hessian computations with H2HOM. The HAO code has been tuned extensively for efficient computation as is reflected in the small number of flops required. In spite of the significantly smaller number of flops required for HAO, the H2HOM code required significantly less clock time for convergence to the same final compensator. (The results generated by the author using the HAO code differ slightly from those reported in reference 11, although the parameters in the HAO algorithm are the same. It is likely that the published results were generated with an earlier version of the HAO code. The qualitative trends remain the same.)

Table 1. Comparison of H2HOM and HAO algorithms.

Algorithm	HAO	H2HOM	H2HOM	H2HOM
Compensator Order	8	8	6	2
Number Hessian Comp.	2,504	63	60	30
Minimum Step Size	1.907e-7	0.025	0.05	0.1
Maximum Step Size	0.1	0.1	0.1	0.2
Max. Number Correction Iterations	9	7	8	5
Mflops	287	936	455	37
Time (s)	5,104	883	488	73

In reference 11, the controllability canonical form is assessed as poorly conditioned because of the small minimum step size. However, table 1 indicates that the static gain formulation in H2HOM yields a substantial improvement in conditioning along the homotopy path over the HAO implementation of the canonical compensator. However, this may not be the case in general due to the tendency toward ill-conditioning characteristic of canonical forms. An even more significant benefit of this formulation is the straightforward extension to the H_∞ and mixed H_2/H_∞ problems.

B. The Mixed H_2/H_∞ Case

The homotopy algorithm was also used to synthesize reduced-order mixed H_2/H_∞ compensators for the four-disk problem. The problem was formulated in terms of a robustness design and a performance design. Similar to the methodology of Luke,¹⁴ the H_∞ portion uses weighted sensitivity for the tracking problem and minimizes the control energy due to stochastic disturbances using the H_2 norm. The block diagram for this problem is shown in figure 7. The output is angular position of disk 3, and the control input is torque applied to disk 3.

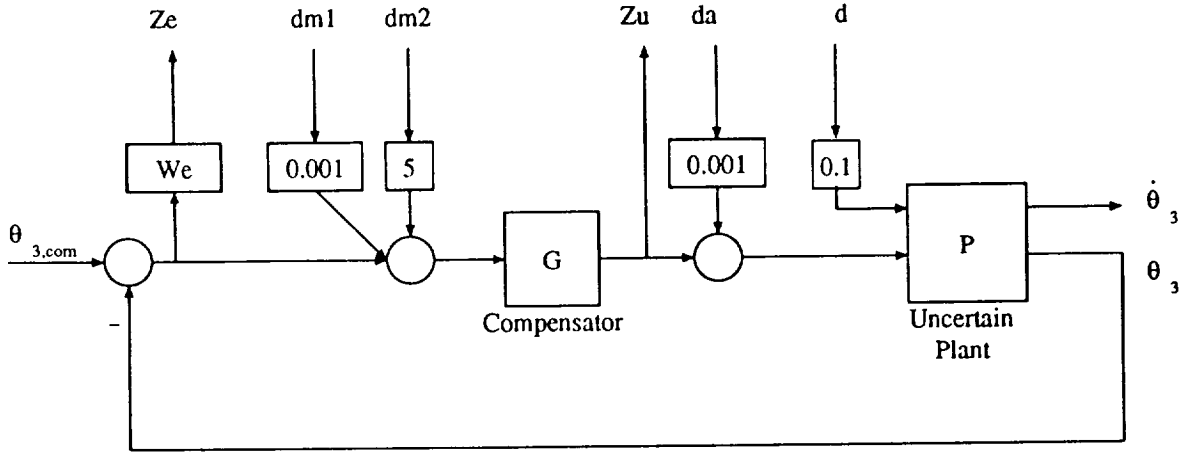


Figure 7. Closed-loop four-disk block diagram.

First, a nominal performance design was performed by varying the sensitivity weight, We , to obtain desirable step responses. Then as described in section IV, the additive uncertainty corresponding to uncertain shaft stiffness was included. This uncertainty representation is somewhat conservative due to the complex uncertainty representation, but still serves to allow robustness to uncertainty in the modes, which in turn adds to the stability margins. The uncertainty weights were then scaled to allow the infinity norm of the closed-loop system to be less than 1. For the H_∞ minimization problem, the input vector w consisted of four inputs corresponding to the uncertainty model (fig. 6); measurement noise, $dm1$; disk 3 position command, $\theta_{3,com}$; and actuator noise, da . The output vector z consisted of four outputs corresponding to the uncertainty model; control energy, Zu ; and weighted error, Ze . For the H_2 minimization problem, the input vector w_p consisted of a stochastic disturbance torque, d , and measurement noise, $dm2$. The output p was the control energy Z_u . The sensitivity weight was given by

$$We = \frac{5}{\frac{1}{0.05}s+1}$$

A full-order compensator was synthesized for the H_∞ portion using the standard 2-Riccati (DGKF) solution¹⁵ for comparison with the full- and reduced-order compensators synthesized by the homotopy algorithm. The homotopy algorithm was able to reproduce the 2-Riccati solution with the exception that the code was stopped prior to obtaining the minimum gamma since convergence slows considerably in the neighborhood of optimum. For the sixth-order H_∞ compensator, the homotopy was terminated at $\gamma = 1.08$. Figure 8 shows the maximum singular value plots of the closed loop for each H_∞ compensator. It should be emphasized that using schur-balanced model reduction and optimal hankel norm model reduction on the full-order H_∞ compensator, it was not possible to obtain a stabilizing reduced-order compensator of any order. Thus, the homotopy algorithm was able to synthesize fixed-order H_∞ controllers where standard model reduction techniques failed.

H_∞ compensators are generated as a special case of the mixed H_2/H_∞ homotopy with the H_2 norm weight λ set to zero. Then for fixed γ , a homotopy is performed on λ , which is increased and the minimum H_2 norm is obtained. Simultaneously, the H_∞ norm increases and the actual infinity norm approaches the overbound. This is the key benefit of the mixed H_2/H_∞ formulation: the

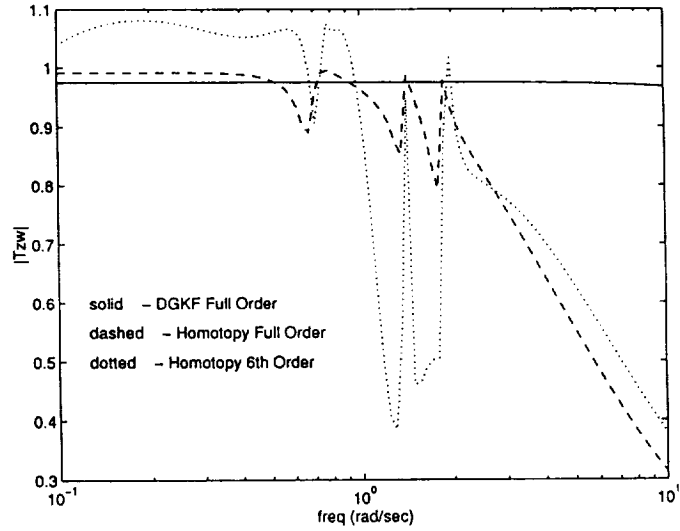


Figure 8. Singular values for H_∞ compensators.

conservatism of the H_∞ design is reduced by reducing the gap between the overbound γ and the actual H_∞ norm, while the performance increases by minimizing the H_2 norm.

The improvement in performance as indicated by the lower H_2 norm is shown in figure 9, where the minimum H_2 norm is plotted versus the H_∞ norm as the overbound γ is decreased. For the DGKF full-order solution, note that the largest H_∞ norm attainable is 1.7, which limits the attainable H_2 norm to 1.088. However, using the mixed H_2/H_∞ formulation, the homotopy algorithm generates a full-order compensator for a wider range of H_∞ norms over which smaller H_2 norms are attainable for a given γ overbound. In this case, the largest H_∞ norm was 3.795 with an H_2 norm of 0.3317, which is a 70-percent reduction of the H_∞ solution. Similarly with the sixth-order mixed H_2/H_∞ compensator, the minimum H_2 norm attained was 0.52 with an H_∞ norm of 4.93. Thus, the presence of the gap between the overbound and the actual H_∞ norm limits the performance range attainable with the H_∞ solution, but is removed when the mixed H_2/H_∞ formulation is employed.

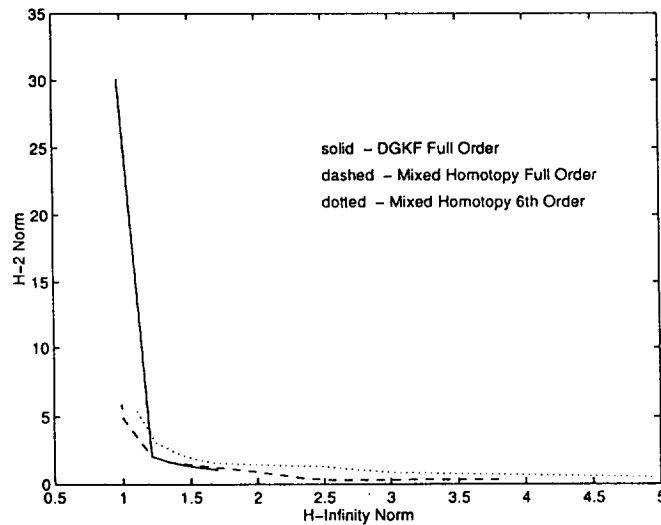


Figure 9. H_∞ versus H_2 norm trades.

V. CONCLUSIONS

A novel homotopy algorithm is developed to synthesize fixed-order H_2 and H_∞ compensators employing a controller canonical form, and a representative flexible structure is used to demonstrate the numerical results. These results indicate that the static gain optimization formulation may be a more efficient means of synthesizing dynamic compensators than employing a general compensator architecture. An even more significant benefit of this approach is the straightforward extension to the fixed-order H_∞ and mixed H_2/H_∞ control problems. The synthesized reduced-order compensators perform well when compared to full-order controllers, which is highlighted by the fact that standard controller order reduction techniques do not yield a stabilizing compensator. The fixed-order mixed H_2/H_∞ formulation is shown to offer improved performance over standard H_∞ compensators by minimizing the H_2 norm while removing (or reducing) the gap between the actual H_∞ norm and the gamma overbound.

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