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Q-DERIVATIVES, COHERENT STATES AND SQUEEZING

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Abstract

We show that the q-deformation of the Weyl-Heisenberg (q-WH) algebra naturally arises in discretized systems, coherent states, squeezed states and systems with periodic potential on the lattice. We incorporate the q-WH algebra into the theory of (entire) analytical functions, with applications to theta and Bloch functions.

1 Introduction

The general properties of q-algebras [1] [2] have been widely studied, in particular in connection with specific physical models. In this paper we will show [3] that the q-deformation of the Weyl-Heisenberg (q-WH) algebra naturally arises in discretized quantum systems, coherent states, squeezed coherent states and systems with periodic potential on the lattice.

q-algebras are deformations of enveloping algebras of Lie algebras and, like the latter, they have Hopf algebras properties. The q-deformation of the Weyl-Heisenberg algebra (q-WH), as well as the WH algebra, is not even a Hopf algebra; it has only the properties of a Hopf superalgebra [4].

In our study of q-deformations we want to preserve the analytic structure of the corresponding Lie algebras and therefore we need to operate in a frame where analyticity is ensured: this is guaranteed by working in the Fock-Bargmann representation (FBR). In this representation it is immediate to show that finite difference operators possess the algebraic structure of q-WH algebra: As a result we recognize that a q-deformation of the algebra occurs whenever a finite length is involved in a physical system, the q-parameter being related with the finite spacing. The q-deformation is also expected in the presence of periodic conditions, since periodicity is a special form of invariance under finite difference operators.

We use the well known mapping of the q-algebra into the universal enveloping algebra of a corresponding Lie structure; to be specific, the mapping of finite difference operators into functions of differential operators, which can be indeed achieved only by operating on C^∞ functions, namely by working in the FBR.

We would like to stress that we succeed into incorporating q-deformation of the WH algebra into the theory of (entire) analytical functions, with specific applications to theta functions and to Bloch functions, a result which may deserve by itself much attention.

In this paper we will use dimensionless units for all physical quantities.

2 Finite difference operators

The FBR operators, solution of the WH commutation relations $[a, a^\dagger] = 1, [N, a] = -a, [N, a^\dagger] = a^\dagger$, are [5]:

$$N \rightarrow z\frac{d}{dz} \ , \qquad a^\dagger \rightarrow z \ , \qquad a \rightarrow \frac{d}{dz} \ . \tag{2.1}$$

The Hilbert space $\mathcal{F}$ is identified with the space of the entire analytical functions. Wave-functions are expressed as $\psi(z) = \sum_{n=0}^{\infty} c_n u_n(z), \ \sum_{n=0}^{\infty} |c_n|^2 = 1, \ u_n(z) = \frac{z^n}{\sqrt{n!}}, \ (n \in Z_+)$. The set $\{u_n(z)\}$ provides an orthonormal basis in $\mathcal{F}$. The finite difference operator $\mathcal{D}_q$

$$\mathcal{D}_q \ f(z) = \frac{f(qz) - f(z)}{(q-1) \ z} \ , \quad f \in \mathcal{F} \ , \tag{2.2}$$

with $q = e^{\varsigma}$, $\varsigma \in \mathcal{C}$, may be written on $\mathcal{F}$ as $\mathcal{D}_q = ((q-1)z)^{-1}(q^{z\frac{d}{dz}} - 1)$. $\mathcal{D}_q$ is the well known [6] [7] [8] [9] q-derivative operator and, for $q \rightarrow 1$ (i.e. $\varsigma \rightarrow 0$), it reduces to the standard derivative. We have the algebra

$$[\mathcal{D}_q \ , z] = q^{z\frac{d}{dz}} \ , \quad [z\frac{d}{dz}, \mathcal{D}_q] = -\mathcal{D}_q \ , \quad [z\frac{d}{dz}, z] = z \ , \tag{2.3}$$

and observe that it is nothing but the q-deformation of the WH algebra. In fact, this can be seen by introducing the following operators in the space $\mathcal{F}$

$$N \rightarrow z\frac{d}{dz} \ , \quad \hat{a}_q \rightarrow z \ , \quad a_q \rightarrow \mathcal{D}_q \ , \tag{2.4}$$

where clearly $\hat{a}_q = \hat{a}_{q=1} = a^\dagger$ and $\lim_{q\rightarrow 1} a_q = a$. The quantum version of the Weyl-Heisenberg algebra is thus realized in terms of these operators $\{a_q, \hat{a}_q, N; \ q \in C\}$ with relations [1] [2]:

$$[N, a_q] = -a_q \ , \quad [N, \hat{a}_q] = \hat{a}_q \ , \quad [a_q, \hat{a}_q] \equiv a_q\hat{a}_q - \hat{a}_q a_q = q^N . \tag{2.5}$$

Equivalently, by introducing $\bar{a}_q \equiv \hat{a}_q q^{-N/2}$, the q-WH algebra eq. (2.5) is rewritten in the more familiar form as $[N, a_q] = -a_q \ , \ [N, \bar{a}_q] = \bar{a}_q \ , \ a_q\bar{a}_q - q^{-\frac{1}{2}}\bar{a}_q a_q = q^{\frac{1}{2}N}$.

The finite difference operator algebra (2.3) in the FBR thus provides a realization of the q-WH algebra (2.5).

The notion of hermiticity associated with (2.5) has been studied in ref. [10] in connection with the discussion of the squeezing of the generalized coherent states $(GCS)_q$, defined in the usual Fock space $\mathcal{F}$.

We note that the commutator $[a_q, \hat{a}_q]$ acts in $\mathcal{F}$ as follows

$$[a_q, \hat{a}_q]f(z) = q^{z\frac{d}{dz}} f(z) = f(qz) \ . \tag{2.6}$$

In conclusion, the strict relation of the q-WH algebra with the finite difference operator $\mathcal{D}_q$ $(q \neq 1)$ suggests that whenever one deals with some lattice or discrete structure, then a deformation of the operator algebra acting on $\mathcal{F}$ should arise.

3 Coherent states, theta functions and squeezing

We summarize now the relation of q-WH algebra with the customary coherent states (CS) $|z>$ [5], with theta functions and with squeezing. Eq.(2.6) is the key relation to establish our results.

For sake of shortness we only report the relevant relations [3]:

$$< n|q^N|z> = \exp\Big(-\frac{|z|^2}{2}\Big)u_n(qz) , \tag{3.1}$$

$$< n|[a_q, \hat{a}_q]|z> = \exp\Big(-(1-\bar{q})(1+q)\frac{|z|^2}{2}\Big) < n|\mathcal{D}((q-1)z)|z> , \tag{3.2}$$

$$\exp\Big((1-|q|^2)\frac{|z|^2}{2}\Big)[a_q, \hat{a}_q]|z> = |qz> , \tag{3.3}$$

$$[a_q, \hat{a}_q]\ f(z) = \exp\Big(-(1-\bar{q})(1+q)\frac{|z|^2}{2}\Big)\ \mathcal{D}((1-\bar{q})\bar{z})\ f(z) , \tag{3.4}$$

where $\mathcal{D}(z)$ denotes the usual CS generator.

We observe that $[a_q, \hat{a}_q]$ acts as mapping operator from $|z>$ to $|qz>$ up to a phase factor. On the other hand, it acts the z-dilatation operator $(z \rightarrow qz)$ in the space of entire analytic functions. When $q = e^{\varsigma}$, with ς pure imaginary, $\varsigma = i\theta$, then $[a_q, \hat{a}_q] : z \rightarrow e^{i\theta}z$, generates the $U(1)$ group of phase transformations in the z-plane. We also observe that eqs. (3.2) and (3.3) provide a non linear realization of the quantum algebra (2.5) in terms of a and $a^{\dagger}$. Vice-versa, the nonlinear operator $\mathcal{D}(z)$ is represented by the linear form $[a_q, \hat{a}_q]$.

In the framework of the formalism of CS on the von Neumann lattice L the defining functional equation for the theta function is [5]

$$\theta_{\epsilon}(z+z_m) = \exp(i\pi F_{\epsilon}(-m))\ \exp\Big(\frac{|z|^2}{2}\Big)\ \exp(\bar{z}_m z)\ \theta_{\epsilon}(z) , \tag{3.5}$$

with $z_m = m_1\omega_1 + m_2\omega_2$ an arbitrary lattice vector and $F_{\epsilon}(m) = m_1 m_2 + m_1\epsilon_1 + m_2\epsilon_2$. A solution of (3.5) can be expressed as

$$\theta_{\epsilon}(z) = \sum_m e^{-i\pi F_{\epsilon}(m)} \exp\Big(-\frac{|z_m|^2}{2}\Big) \exp(-\bar{z}_m\ z) f(z) , \tag{3.6}$$

where $f(z)$ is an arbitrary entire function such that the series (3.6) is converging.

To establish the relation between q-WH algebra and theta functions, we write $q = q_m = e^{\varsigma_m}$, with ς_m a vector on the lattice L and, by setting $z_m = (q_m - 1)z$, we have [3]

$$\theta_{\epsilon}(q_m z) = [a_{q_m}, \hat{a}_{q_m}]\ \theta_{\epsilon}(z) , \tag{3.7}$$

$$[a_{q_m}, \hat{a}_{q_m}]\ \theta_{\epsilon}(z) = \exp(i\pi F_{\epsilon}(-m)) \exp\Big(-(1-\bar{q}_m)(1+q_m)\frac{|z|^2}{2}\Big)\theta_{\epsilon}(z) , \tag{3.8}$$

$$\theta_{\epsilon}(z) = \sum_m \exp(-i\pi F_{\epsilon}(m)) \exp\Big((1-\bar{q}_m)(1+q_m)\frac{|z|^2}{2}\Big)[a_{q_m}, \hat{a}_{q_m}]\ f(z) . \tag{3.9}$$

Eqs. (3.7-9) show that theta functions span indeed a space of representations for the q-algebra (2.5).

Finally, we study the relation of q-WH algebra with squeezing. Let $\hat{p}_z = -i\frac{d}{dz}$ and $[\hat{z}, \hat{p}_z] = i$, over a Hilbert space of states $\psi(z)$ identified with the space of entire analytic functions $\mathcal{F}$. Introduce the operators $\alpha = \frac{1}{\sqrt{2}}(\hat{z} + i\hat{p}_z)$, $\alpha^\dagger = \frac{1}{\sqrt{2}}(\hat{z} - i\hat{p}_z)$, $[\alpha, \alpha^\dagger] = I$. It is immediate to observe that

$$[a_q, \hat{a}_q]\; \psi(z) \;=\; \exp\Big(\varsigma z \frac{d}{dz}\Big)\; \psi(z) \;=\; \frac{1}{\sqrt{q}} \exp\Big(\frac{\varsigma}{2}(\alpha^2 - \alpha^{\dagger 2})\Big)\; \psi(z) \tag{3.10}$$

$$= \frac{1}{\sqrt{q}} \hat{S}(\varsigma)\psi(z) = \frac{1}{\sqrt{q}} \psi_s(z) \, ,$$

with $\hat{S}(\varsigma)$ denoting the squeezing generator [11], $\varsigma = \log q$ the squeezing parameter and $\psi_s(z)$ the squeezed state. We therefore conclude that the operator $[a_q, \hat{a}_q]$ *is* the squeezing generator for CS in the FBR, thus confirming the conjecture previously [10] formulated whereby q-groups are the natural candidates to study the squeezed CS.

4 Quantum mechanics on the lattice

Our purpose is now to show that q-WH algebra is underlying the physics of lattice quantum systems. Lattice Quantum Mechanics (LQM) is characterized by the E(2) commutator algebra, which in the momentum space is written as [3] [12]

$$\begin{aligned} [\hat{x}_\epsilon, \hat{p}_\epsilon] &= [i\frac{d}{dk}, \epsilon^{-1}\sin(k\epsilon)] = i\cos(k\epsilon) \, , \\ [\hat{x}_\epsilon, \cos(k\epsilon)] &= [i\frac{d}{dk}, \cos(k\epsilon)] = -i\epsilon\sin(k\epsilon) \, . \\ [\hat{p}_\epsilon, \cos(k\epsilon)] &= [\epsilon^{-1}\sin(k\epsilon), \cos(k\epsilon)] = 0 \, . \end{aligned} \tag{4.1}$$

where $\hat{x}_\epsilon$ and $\hat{p}_\epsilon$, denote the (one-dimensional) lattice position operator and the lattice momentum operator, respectively (extension to higher dimensions is straightforward). The corresponding uncertainty relations are

$$\Delta^2(\hat{x}_\epsilon)\Delta^2(\hat{p}_\epsilon) \geq \frac{1}{4}(\langle\cos(k\epsilon)\rangle^2) \, , \tag{4.2}$$

$$\Delta^2(\hat{x}_\epsilon)\Delta^2(\cos(k\epsilon)) \geq \frac{1}{4}(\epsilon^2\langle\sin(k\epsilon)\rangle^2) \, , \tag{4.3}$$

where $\Delta^2(\hat{A}) = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ with $= \langle\hat{A}\rangle = \int dk \Psi^*(k)\hat{A}\Psi(k)$. We observe that these relations go, in the continuum limit $\epsilon \to 0$, to the usual ones. In this connection we observe that the continuum limit is, in fact, an isometric and conformal mapping of the torus on the plane.

Following the usual procedure [13], the states minimizing the uncertainties (4.2) and (4.3) are found to be, in momentum space

$$\Psi(k) = G^{-\frac{1}{2}} \exp\left[\bar{\gamma}\cos(\epsilon k) - i\bar{\lambda}\epsilon k\right] \, . \tag{4.4}$$

The normalization constant G is given by $G = \frac{2\pi}{\epsilon} I_0(2\bar{\gamma})$, I_0 denoting the modified Bessel function of the first kind of order 0. We adopted the notation: $\bar{\lambda} = \lambda\epsilon^{-1}$, $\bar{\gamma} = \gamma\epsilon^{-2}$, $\lambda = \langle\hat{x}_\epsilon\rangle + i\gamma\langle\hat{p}_\epsilon\rangle$, and γ is connected with the mean square roots of position and momentum.

In the continuum limit, i.e. for small ϵ, one recovers in the space of configurations, $\tilde{\Psi}(x) = (\gamma\pi)^{-\frac{1}{4}} \exp\left\{-\left[(2\gamma)^{-1}(x - \langle\hat{x}\rangle)^2 + i\langle\hat{p}\rangle(x - \langle\hat{x}\rangle)\right]\right\}$, which is the minimum uncertainty wave-function given by Schrödinger [14]. The $\Psi(k)$'s are the lattice coherent states.

In order to see the relation with the q-algebra we consider the conformal image $\tilde{\mathcal{H}}$ of the Hilbert space obtained upon introducing the variable $z = e^{i\phi}$ ($\phi = k\epsilon$, $-\pi \le \phi \le \pi$), such that $-i\frac{d}{dk} = -i\epsilon\frac{d}{d\phi} = \epsilon z\frac{d}{dz}$. The functions in $\tilde{\mathcal{H}}$ are assumed to be entire square-integrable analytic functions. We have

$$L_3 f(\phi) = -i\frac{d}{d\phi}f(\phi) = z\frac{d}{dz}\tilde{f}(z) = N\tilde{f}(z) \quad , \quad \tilde{f} \in \tilde{\mathcal{H}} \quad , \tag{4.5}$$

$$f(\phi + \epsilon) = e^{i\epsilon L_3} f(\phi) = q^N \tilde{f}(z) = \tilde{f}(qz) \quad , \tag{4.6}$$

with $q = e^{i\epsilon}$. The realization (2.4) has been adopted in the FBR, with z restricted to the unit circle. The $E(2)$ algebra (4.1) is realized by

$$[L_1, L_3]\tilde{f}(z) = -iL_2\tilde{f}(z) \ , \quad [L_2, L_3]\tilde{f}(z) = iL_1\tilde{f}(z) , \quad [L_1, L_2]\tilde{f}(z) = 0 , \tag{4.7}$$

with $\tilde{f} \in \tilde{\mathcal{H}}$, and the identifications

$$L_1 = \frac{z + \bar{z}}{2} \ , \quad L_2 = \frac{z - \bar{z}}{2i} \ , \quad L_3 = z\frac{d}{dz} \ , \quad L_+ = z \ , \quad L_- = \bar{z} \ . \tag{4.8}$$

One can see that $[a_q, \hat{a}_q]$ is nothing but the group element $e^{i\epsilon L_3}$ of $E(2)$. The algebraic structure of LQM is thus intimately related with the q-WH algebra, the deformation parameter q being determined by the discrete lattice length $\epsilon = -i \log q$.

We finally note that $z^n = e^{in\phi}$, n integer, is the eigenfunction of L_3 associated with the eigenvalue n of the number operator in the FBR: $L_3 z^n = N z^n = n z^n$.

The functions $z = e^{i\phi}$ play also a rôle in the Bloch functions theory. Suppose we have a periodic potential $V(x_n) = V(x_n + \epsilon)$ on the lattice. Bloch theorem ensures the existence of solutions of the related Schrödinger equation of the form $\psi(x_n) = e^{\pm ikx_n} v_k(x_n)$, with $v_k(x_n) = v_k(x_n + \epsilon)$. $\psi(x_n)$ is the Bloch function. Let us limit ourself to consider for simplicity the plus sign in the exponentials. $\psi(x_n)$ has the property

$$\psi(x_n + \epsilon) = e^{ik\epsilon}\psi(x_n) = z\psi(x_n) \ . \tag{4.9}$$

The choice of the variable $z = e^{ik\epsilon}$ turns out to be natural in the case of periodic potentials:

$$\psi(x_n) = z^n v_k(x_n) \ , \quad \psi(x_n + \epsilon) = z^{n+1} v_k(x_n) \ . \tag{4.10}$$

Since $z^n = (z_n)^k$ and $q^N (z_n)^k = (qz_n)^k = e^{ik\epsilon(n+1)} = z^{n+1}$,

$$\psi(x_n + \epsilon) = [a_q, \hat{a}_q](z_n)^k v_k(x_n) = [a_q, \hat{a}_q]\psi(x_n) \ , \tag{4.11}$$

which shows that the Bloch functions provide indeed a representation for the q-WH algebra.

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