N95- 13907

SQUEEZING IN A 2-D GENERALIZED OSCILLATOR

Octavio Castaños and Ramón López-Peña
Instituto de Ciencias Nucleares, UNAM
Circuito Exterior, C.U., Apdo. Postal 70-543
04510 México, D. F., México

Vladimir I. Man'ko
Physical Institute of the Russian Republic
Leninsky Prospect 53. Moscow, Russia

Abstract

A two-dimensional generalized oscillator with time-dependent parameters is considered to study the two-mode squeezing phenomena. Specific choices of the parameters are used to determine the dispersion matrix and analytic expressions, in terms of standard hermite polynomials, of the wavefunctions and photon distributions.

1 Introduction

In the middle of the sixties and beginning of the seventies a set of quantum states of the electromagnetic field were observed which have less uncertainty in one quadrature than a coherent state [1-3]. These one-mode squeezed states have generated big expectations in optical communication systems [4]. In some quantized fields, the interaction hamiltonians occur only between pairs of modes and then to understand the main features of the system, one restricts to study one and two normal modes . In the last decade two-mode squeezing phenomena have attracted attention to study properties of noise and correlations [5-8]. Recently the accidental degeneracy of a two-dimensional (2-D) harmonic oscillator with frequency ω_0 plus an interaction proportional to the z-th projection of the angular momentum was studied [9]. This system was called the generalized 2-D harmonic oscillator because presents a bigger accidental degeneracy depending on the strength λ of the angular momentum interaction. This model was generalized [10] to include time-dependent parameters, $m = m_0 f(t)$ and $\lambda = \omega_0 \lambda_0(t)$. If we take $f(0) = \lambda_0(0) = 1$; the hamiltonian, for $t = 0$, represents a charged particle moving in a constant magnetic field .

The aim of this work is to study two-mode squeezing phenomena with this model because it demonstrates the change of dispersions due to variation of the mass and coupling constant during the evolution. In the framework of quantum optics the hamiltonian is built by: the operator $(1/f + f)\vec{a}^\dagger \cdot \vec{a}$, that causes a time-dependent exchange of kinetic and potential energies within each mode; the interaction $1/2(1/f - f)(\vec{a}^\dagger \cdot \vec{a}^\dagger + \vec{a} \cdot \vec{a})$, which describes a degenerate two-photon interaction; and the potential $i\lambda_0(t)(a_2^\dagger a_1 - a_1^\dagger a_2)$, that is a mode mixing operator.

The solution of the corresponding time dependent Schroedinger equation is obtained through the theory of integrals of motion [11]. By means of Noether's theorem, using a special variation

[10], we construct the linear time dependent integrals of the motion. The resulting quantum invariants are given in terms of the positions and momenta operators [10,11] by

$$\vec{P}(t) = \lambda_1 \vec{p} + \lambda_2 \vec{q}, \quad \vec{Q}(t) = \lambda_3 \vec{p} + \lambda_4 \vec{q}, \tag{1}$$

with the initial conditions $\vec{P}(0) = \vec{p}$ and $\vec{Q}(0) = \vec{q}$, so that the 2×2 matrices previously introduced satisfy $\lambda_1(0) = \lambda_4(0) = I_2$ and $\lambda_3(0) = \lambda_2(0) = 0$. The operators $\vec{A}(t) = 1/\sqrt{2}\,[\vec{Q}(t)/l + il/\hbar \vec{P}(t)]$ and its hermitean conjugate, can be constructed with the matrices

$$\lambda_p = \frac{1}{l}\lambda_3 + \frac{il}{\hbar}\lambda_1, \quad \lambda_q = \frac{1}{l}\lambda_4 + \frac{il}{\hbar}\lambda_2, \tag{2}$$

with $l = \sqrt{\frac{\hbar}{m_0 \omega_0}}$ defining the oscillator length. These integrals of motion also are given in terms of the creation and annihilation photon operators

$$\vec{A}(t) = M_1 \vec{a} + M_2 \vec{a}^\dagger, \quad \vec{A}^\dagger(t) = \quad M_3 \vec{a} + M_4 \vec{a}^\dagger . \tag{3}$$

With the initial conditions $\vec{A}(0) = \vec{a}$ and $\vec{A}^\dagger(0) = \vec{a}^\dagger$, the matrices defined in (3) comply with $M_1(0) = M_4(0) = I_2$ and $M_3(0) = M_2(0) = 0$. The λ_k's, M_k's, λ_p and λ_q are entries of symplectic matrices in four dimensions because the invariants (1) and (3) satisfy the commutation relations of Heisenberg-Weyl algebras.

In the present work, we study the behavior of the model for $\lambda_0(t)$ an arbitrary function of time and considering two kinds of varying masses, *i.e.*, two choices for the function $f(t)$, namely:

$$f(t) = \exp(\gamma t) \quad ; \tag{4}$$

$$f(t) = \begin{cases} 1 & , \quad t \le 0 \\ \cosh^2 \Omega_0 t & , \quad 0 \le t \le T \\ \{\Omega_0 (t-T) \sinh \Omega_0 T + \cosh \Omega_0 T\}^2 & , \quad T \le t \end{cases} . \tag{5}$$

For these two cases the λ_k matrices take the general form

$$\lambda_k = \mu_k \, \mathbf{R} = \mu_k \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \quad k = 1, 2, 3, 4; \tag{6}$$

where the definition $\theta = \int_0^t \omega_0 \lambda_0(\tau) d\tau$ was used. The analytic expressions for the μ_k's functions are given in Ref. [10]. In the next sections we determine the coherent and Fock-like states, the photon distributions and the dispersion matrices in terms of these μ_k's.

2 Squeezed Coherent and Fock States

The coherent-like states are obtained by solving the differential equation $\vec{A}(t)\Phi_0(\vec{q}, t) = 0$ with $\vec{A}(t)$ given in Eq. (3). This solution yields the vacuum state of the physical system, and its phase is chosen to guarantee that satisfies the time dependent Schroedinger equation. The expression for the ground state wavefunction is

$$\Phi_0(\vec{q}, t) = \frac{1}{\sqrt{2\pi\hbar^2}\,\mu_p} \exp\left\{ -\frac{i}{2\hbar} \frac{\mu_q}{\mu_p} \vec{q} \cdot \vec{q} \right\} . \tag{7}$$

To get the last expression the relation (2) was used and the functions $\mu_p = \frac{1}{\sqrt{2}}\left(\frac{il}{\hbar}\mu_1 + \frac{1}{l}\mu_3\right)$ and $\mu_q = \frac{1}{\sqrt{2}}\left(\frac{il}{\hbar}\mu_2 + \frac{1}{l}\mu_4\right)$ were defined. To obtain the general expression for the eigenstates in the coordinate representation one needs to apply the unitary operator $\hat{D}(\alpha) = \exp\{\vec{\alpha}\cdot\vec{A}^\dagger - \vec{\alpha}^*\cdot\vec{A}\}$, which is an invariant, to the vacuum wavefunction (7), *i.e.*,

$$\Phi_\alpha(\vec{q},t) = \exp\left\{-\frac{|\alpha|^2}{2} + \frac{1}{2}\frac{\mu_p^*}{\mu_p}\vec{\alpha}\cdot\vec{\alpha} + \frac{i}{\hbar\mu_p}\vec{q}\,\tilde{\mathbf{R}}\vec{\alpha}\right\}\Phi_0(\vec{q},t) \quad . \tag{8}$$

These are expressed in terms of multi-dimensional Hermite polynomials [12] through the relation

$$\exp\left(-\frac{1}{2}u\,\vec{\alpha}^*\cdot\vec{\alpha}^* + v\,\vec{\alpha}^*\,\tilde{\mathbf{R}}\,\vec{\gamma}\right) = \sum_{n_1,n_2=0}^{\infty}\frac{\alpha_1^{*n_1}}{n_1!}\frac{\alpha^{*n_2}}{n_2!}\mathbf{H}_{n_1,n_2}^{\{u\mathbf{I}_2\}}\left(\frac{v}{u}\tilde{\mathbf{R}}\vec{\gamma}\right) \quad . \tag{9}$$

Substituting the last expression into (8) and using the form of the coherent-like states in the Fock-like representation, we get the Fock-like eigenstates in the coordinate representation:

$$\langle\vec{q}|n_1 n_2\rangle = \Phi_0(\vec{q},t)\;\mathbf{H}_{n_1,n_2}^{\left\{-\frac{\mu_p^*}{\mu_p}\mathbf{I}_2\right\}}\left(-\frac{i}{\hbar\mu_p^*}\mathbf{R}\vec{q}\right) \quad . \tag{10}$$

These multi-dimensional Hermite polynomials are rewritten as a product of two standard one-dimensional Hermite polynomials [12] as follows:

$$\begin{aligned}\mathbf{H}_{n_1,n_2}^{\left\{\frac{\mu_p^*}{\mu_p}\mathbf{I}_2\right\}}\left(-\frac{i}{\hbar\mu_p^*}\mathbf{R}\vec{q}\right) &= \left(-\frac{\mu_p^*}{2\mu_p}\right)^{(n_1+n_2)/2} H_{n_1}\left(\frac{1}{\sqrt{2\hbar}|\mu_p|}[\cos\theta\; q_1 + \sin\theta\; q_2]\right)\\ &\quad\times H_{n_2}\left(\frac{1}{\sqrt{2\hbar}|\mu_p|}[-\sin\theta\; q_1 + \cos\theta\; q_2]\right) ,\end{aligned} \tag{11}$$

where we use the explicit expression of matrix $\mathbf{R}$. These Fock (10) and coherent (8) -like states represent squeezed and correlated eigenstates of the system as it will be shown further.

3 Propagator

The propagator in the coherent state representation is given by the matrix elements of the evolution operator $U(t)$, which will be obtained by means of the theory of time dependent integrals of motion [11]. If $\vec{I}(t)$ is an integral of motion then satisfies $\vec{I}(t)\hat{U}(t) = \hat{U}(t)\vec{I}(0)$. Taking its matrix elements with respect to the coherent states, we get a linear system of differential equations, which can be solved. Thus the propagator takes the form

$$G(\vec{\alpha}^*,\vec{\gamma},t) = \frac{\exp\left(-|\vec{\alpha}|^2/2 - |\vec{\gamma}|^2/2\right)}{\sqrt{detM_1}}\exp\left(-\frac{1}{2}\vec{\alpha}^* M_1^{-1}M_2\vec{\alpha}^* + \vec{\alpha}^* M_1^{-1}\vec{\gamma} + \frac{1}{2}\vec{\gamma}M_3M_1^{-1}\vec{\gamma}\right) \quad . \tag{12}$$

For the cases (4) and (5), the following relations are satisfied

$$\sqrt{detM_1} = \frac{1}{\sqrt{2}}\left(l\mu_q - \frac{i\hbar}{l}\mu_p\right) \equiv g_1, \quad M_1^{-1}M_2 = \frac{1}{\sqrt{2}g_1}\left(l\mu_q + \frac{i\hbar}{l}\mu_p\right)\mathbf{I_2} \equiv g_2\mathbf{I_2} \quad , \tag{13}$$

$$M_1^{-1} = \frac{1}{g_1}\tilde{\mathbf{R}} \ , \quad M_3 M_1^{-1} = \frac{1}{\sqrt{2}g_1}\Big(l\mu_q^* + \frac{i\hbar}{l}\mu_p^*\Big) I_2 \equiv g_4 \mathbf{I_2} \ . \tag{14}$$

Substituting these relations into the Eq.(12) we get the propagator, which through Eq. (9) can be expressed in terms of multi-dimensional Hermite polynomials. If we compare with the power series expansion of the propagator we get the probability amplitude for having n_1 and n_2 photons in the coherent-like state $|\vec{\gamma}, t\rangle$, *i.e.*,

$$\langle n_1 n_2 | \vec{\gamma}, t\rangle = \frac{1}{g_1\sqrt{n_1! N_2!}} \exp\left(-\frac{|\vec{\gamma}|^2}{2} + \frac{1}{2} g_4 \vec{\gamma}\cdot\vec{\gamma}\right) \mathbf{H}_{n_1,n_2}^{\{g_2 \mathbf{I_2}\}}\Big(\frac{1}{g_1 g_2}\tilde{\mathbf{R}}\vec{\gamma}\Big) \ . \tag{15}$$

By means of the Eq.(12) this amplitude can be rewritten in terms of standard Hermite polynomials [12]. The squared absolute value of this amplitude yields the photon distribution function of the system, $W_{n_1 n_2}(\vec{\gamma}, t) = |\langle n_1 n_2|\vec{\gamma}, t\rangle|^2$. This will let us calculate, at least formally, the mean, $\langle N_k\rangle$, and the mean squared fluctuation of the number of photons, $(\Delta N_k)^2$, in direction k, which are present in the coherent state $|\vec{\gamma}, t\rangle$. The expectation values of N_k and N_k^2 are evaluated directly using the expressions of the creation and annihilation photon operators in terms of the integrals of the motion (3), and the commutation properties for these invariants. For the vacuum state one has

$$\langle N_k\rangle = \frac{1}{4}\Big\{(\mu_1 - \mu_4)^2 + \Big(m_0\omega_0\mu_3 + \frac{1}{m_0\omega_0}\mu_2\Big)^2\Big\}, \tag{16}$$

$$\begin{aligned} \langle N_k^2\rangle \ &= \ \frac{1}{8}\Big\{(\mu_1 - \mu_4)^2 + (m_0\omega_0\mu_3 + \frac{1}{m_0\omega_0}\mu_2)^2\Big\} \\ &\Big\{(\mu_1 + \mu_4)^2 + (m_0\omega_0\mu_3 - \frac{1}{m_0\omega_0}\mu_2)^2\Big\} \ + \frac{1}{16}\Big\{(\mu_1 - \frac{1}{m_0\omega_0}\mu_2)^2\Big\}^2 . \end{aligned} \tag{17}$$

With these expressions, we evaluate the ratio of the mean squared fluctuation $(\Delta N_k)^2$ and the mean number of photons $\langle N_k\rangle$, which determines the nature of the distribution function of the system:

$$\frac{(\Delta N_k)^2}{\langle N_k\rangle} = \frac{1}{2}\Big\{(\mu_1 + \mu_4)^2 + \Big(m_0\omega_0\mu_3 - \frac{1}{m_0\omega_0}\mu_2\Big)^2\Big\} \tag{18}$$

For the cases (4) and (5) the ratio is greater than one when $t > 0$, which implies that we have a super-Poissonian photon distribution function. For $t = 0$, there is a discontinuity in the ratio, which is obtained by comparing the following limiting procedures: making $t \to 0$ and then $\vec{\alpha} \to 0$, and conversely.

4 Dispersion Matrices

The dispersion matrix can be written in terms of 2×2 matrices characterizing the dispersions in the positions and momenta operators and the correlation between them. Besides for the cases under study, due to (6), they take the form

$$\sigma_{pp}^2(t) = \frac{1}{2}\hbar m_0\omega_0\Big(\frac{1}{(m_0\omega_0)^2}\mu_2^2 + \mu_4^2\Big) \ \mathbf{I_2} \ , \tag{19}$$

$$\sigma^2_{qp}(t) = -\frac{\hbar}{2}\Big(\frac{1}{m_0\omega_0}\mu_1\mu_2 + m_0\omega_0\mu_3\mu_4\Big)\ \mathbf{I_2}\ , \tag{20}$$

$$\sigma^2_{qq}(t) = \frac{1}{2}\frac{\hbar}{m_0\omega_0}(\mu_1^2 + (m_0\omega_0)^2\mu_3^2)\ \mathbf{I_2}\ . \tag{21}$$

The corresponding correlation matrices for the creation and annihilation operators are obtained immediately from the last expressions; they are given by

$$\sigma^2_{aa} = \frac{1}{4}\Big\{\mu_1^2 - \mu_4^2 + (m_0\omega_0)^2\mu_3^2 - \frac{\mu_2^2}{(m_0\omega_0)^2} - 2i\Big(\frac{1}{m_0\omega_0}\mu_1\mu_2 + m_0\omega_0\mu_3\mu_4\Big)\Big\}\ , \tag{22}$$

$$\sigma^2_{a^\dagger a} = \frac{1}{4}\Big(\mu_1^2 + \mu_4^2 + (m_0\omega_0)^2\mu_3^2 + \frac{\mu_2^2}{(m_0\omega_0)^2}\frac{m_0\omega_0}{\hbar}\Big), \tag{23}$$

$$\sigma^2_{a^\dagger a^\dagger} = \frac{1}{4}\Big\{\mu_1^2 - \mu_4^2 + (m_0\omega_0)^2\mu_3^2 - \frac{\mu_2^2}{(m_0\omega_0)^2} + 2i\Big(\frac{1}{m_0\omega_0}\mu_1\mu_2 + m_0\omega_0\mu_3\mu_4\Big)\Big\}\ . \tag{24}$$

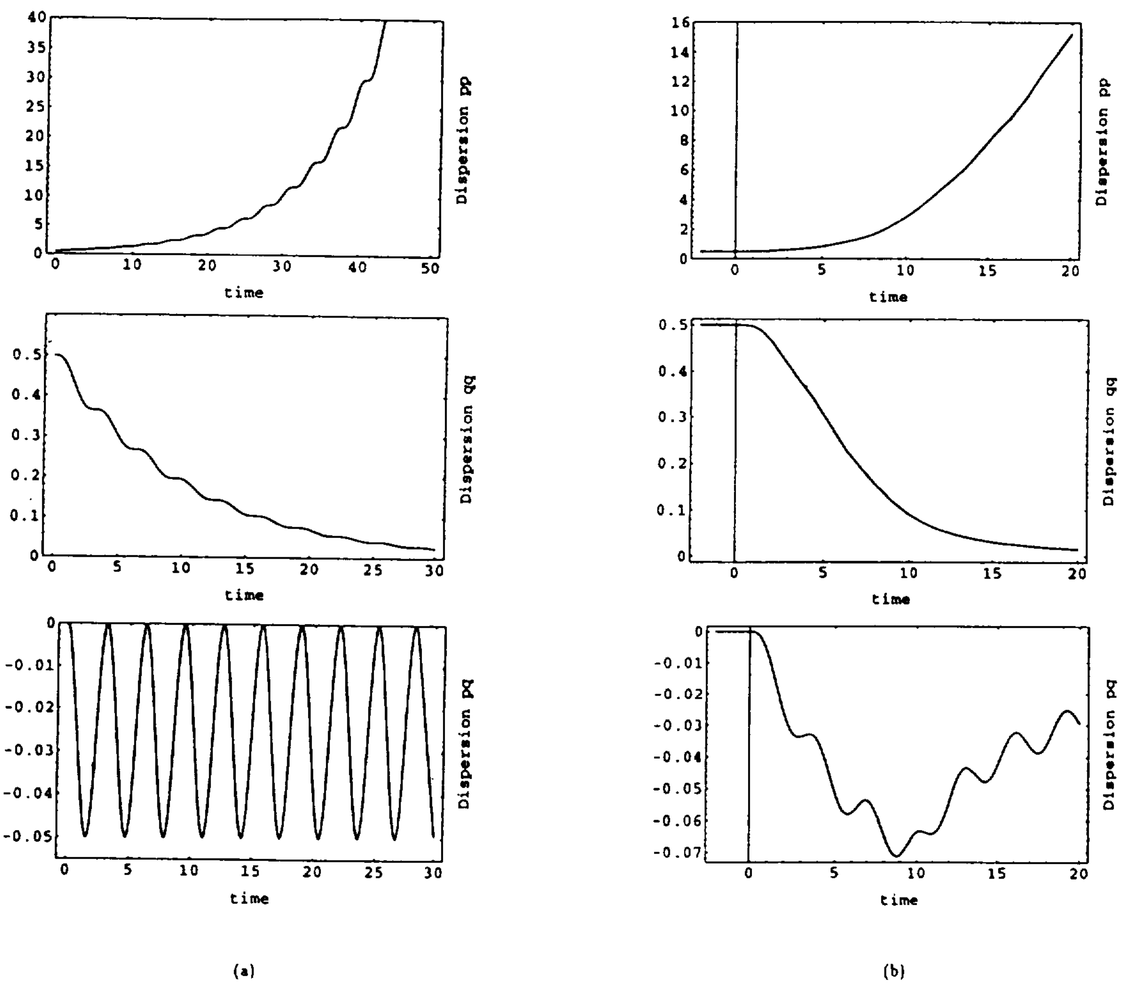

Fig. 1. Dispersion and correlation matrices behavior in positions and momenta space for the studied cases in this paper: (a) corresponds to Eq.(4), and (b), to Eq.(5).

The behavior of the dispersion matrices is illustrated in Fig. 1. For the case (4), we choose the parameters $\gamma = 0.1$ and $m_0 = \omega_0 = 1$. It is seen that there is squeezing for the coordinates and stretching for the momenta. Also one notes that σ_{pq} is a negative function and therefore there are one-mode correlations between the coordinates and the momenta. If we reverse the sign of γ, the roles between the dispersion for coordinates and momenta are interchanged, and σ_{pq} becomes positive. In the case (5), we use the parameters $\Omega_0 = 0.15$, $T = 10$, and $m_0 = \omega_0 = 1$. In spite of the mass is different that in the previous example, the general trends are similar. For example, the σ_{pp} is an increasing function of time starting from its minimum value at $t \leq 0$, and there is squeezing for the σ_{qq}. The main difference appears in the correlation σ_{pq}: in this case, it can be positive for large times, while in the previous one is negative or zero for any time.

5 Acknowledgments

Work supported in part by projects UNAM-DGAPA IN-103091 and CONACYT 1570-E9208.

References

[1] D. R. Robinson, *Commun. Math. Phys.* **1**, 159 (1965).

[2] D. Stoler, *Phys. Rev. D* **4**, 1925 (1971).

[3] E. Y. C. Lu, *Lett. Nuovo Cimento* **2**, 1458 (1971); **4**, 585 (1972).

[4] H. P. Yuen and J. H. Shapiro, *IEEE Trans. Inform. Theory IT* **24**, 657 (1978); **26**, 78 (1980).

[5] C. M. Caves and B. L. Schumaker, *Phys. Rev. A* **31**, 3068 (1985); B. L. Schumaker and C. M. Caves. *Phys. Rev. A* **31**, 3093 (1985).

[6] B. Yurke, S. L. McCall and J. R. Klauder. *Phys.Rev. A* **33**, 4033 (1986).

[7] Y. S. Kim and V. I. Man'ko, *Phys. Lett. A* **157** (1991), 226.

[8] R. Glauber, in *Proc. of Quantum Optics Conference*, Hyderabad, January 5-10, 1991, Eds. G. S. Agarwal and R. Ingua, Plenum Presss, 1993.

[9] O. Castaños and R. López-Peña, *J. Phys. A: Math. Gen.* **25**, 6685 (1992).

[10] O. Castaños, R. López-Peña, and V. I. Man'ko: *Noether's Theorem and Time-Dependent Quantum Invariants.* Submitted for publication.

[11] V. V. Dodonov and V. I. Man'ko, in: Proc. P. N. Lebedev Physical Institute, Vol. 183. *Invariants and evolution of non-stationary quantum systems*, Ed. M. A. Markov (Nova Science, Commack, N. Y., 1987).

[12] V. V. Dodonov, V. I. Man'ko, and V. V. Semjonov, *Nuovo Cimento B* **83**, 145 (1984).