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SIMULTANEOUS TWO COMPONENT SQUEEZING IN GENERALIZED q–COHERENT STATES

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Abstract

Using a generalization of the q-commutation relations, we develop a formalism in which it is possible to define generalized q-bosonic operators. This formalism includes both types of the usual q-deformed bosons as special cases. The coherent states of these operators show interesting and novel noise reduction properties including simultaneous squeezing in both field components, unlike the conventional case in which squeezing is permitted in only one component. This also contrasts with the usual quantum group deformation which also only permits one component squeezing.

1 Deformed Commutation Relations

Consider the single particle deformed commutation relation [1]

$$aa^{\dagger} - f(N)a^{\dagger}a = 1 \tag{1}$$

where $a^{\dagger}$ and a are generalized creation and annihilation operators, N is the number operator such that $N|n\rangle = n|n\rangle$, f is a real function, and the vacuum $|0\rangle$ is defined by $a|0\rangle = 0$. We define a normalized one-particle state by $a^{\dagger}|0\rangle = |1\rangle$. This formalism incorporates the deformation schemes previously encountered in the literature as special cases.

Examples:

1. $f(N) = 1$.
   This is the usual commutation relation of the Heisenberg–Weyl algebra and describes ordinary quantum mechanical bosonic systems such as the the harmonic oscillator.

2. $f(N) = q$.
   The so–called q–oscillator, first suggested by Arik and Coon [2]. It has since been studied in detail by several authors e.g. Jannussis et al [3], Kuryshkin [4], Kulish and Damaskinsky [5].

3. $f(N) = \frac{q^{N+2}+1}{q(q^{N}+1)}$.
   This gives a deformed commutation relation equivalent to that of the q–boson first discovered by Macfarlane [6] and Biedenharn in connection with the representation theory of quantum groups.

4. This form of deformed commutation relation can also be related to the extensive work of Bonatsos, Daskaloyannis and others,[7] and refs. therein, on the generalized oscillator formalism as well as the recent work of Jannussis [8].

Building up normalized eigenstates of the number operator N by repeated application of the generalized creation operators in (1), we obtain

$$|n\rangle = \frac{(a^\dagger)^n}{([n]!)^{\frac{1}{2}}}|0\rangle. \tag{2}$$

where the function $[n]$ is defined recursively by

$$[n+1] = 1 + f(n)[n] \tag{3}$$

with initial condition $[0] = 0$.

Explicitly, we see

$$\begin{aligned} [n] &= 1 + f(n-1) + f(n-1)f(n-2) + f(n-1)f(n-2)f(n-3) \\ &\quad + \cdots + f(n-1)f(n-2)\cdots f(2)f(1) \end{aligned} \tag{4}$$

$$= \sum_{k=0}^{n-1} \frac{f(n-1)!}{f(k)!}. \tag{5}$$

The functions $[n]$ can be thought of as generalizations of the basic numbers of q-analysis [9]. They obey a highly non-linear arithmetic but for appropriate choice of the function f, they tend in some limit to the ordinary integers.

2 Coherent States

Conventional coherent states of the oscillator obeying the undeformed commutation relation ($f(N) = 1$) may be defined by

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{6}$$

or equivalently

$$|\lambda\rangle = \frac{1}{\exp(|\lambda|^2)} \exp(\lambda a^\dagger)|0\rangle \tag{7}$$

where the exponentional function, by definition, has the property

$$\frac{d}{dx}\exp(\lambda x) = \lambda \exp(\lambda x) \tag{8}$$

These definitions of coherent states have been used to generalize the concept to the cases where the commutation relations have been deformed.

Given the q–commutation relation $aa^\dagger - q\,a^\dagger a = 1$, we may define coherent states $|\lambda\rangle$ by

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{9}$$

To achieve the alternative definition given by (7), it is necessary to introduce a q–derivative operator [9], ${}_qD_x$ such that

$${}_qD_x E_q(\lambda x) = \lambda E_q(\lambda x) \tag{10}$$

where $E_q(x)$ is the Jackson q–exponential. When this is done, we see that

$$|\lambda\rangle = \frac{1}{E_q(|\lambda|^2)} E_q(\lambda a^\dagger)|0\rangle \tag{11}$$

The same procedure can also be used to define q–coherent states for the Macfarlane–Biedenharn oscillator (although in this case the generalization of the exponential function is different from that of Jackson).

For $[n]$ (defined by (5)), an analytic function of the variable n, it is possible to extend the above analysis to the case of bosonic creation and annihilation operators obeying the general commutation relations (1).

We define an operator D_x such that

$$D_x = \frac{1}{x}\left[x\frac{d}{dx}\right]. \tag{12}$$

This acts as a generalized differential operator e.g.

$$D_x\, x^n = [n]x^{n-1}. \tag{13}$$

The eigenfunctions of D_x given by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}. \tag{14}$$

are well–defined provided the function f satisfies the appropriate convergence criteria. If $f(n) \geq 1$ as $n \to \infty$ then $E(x)$ converges for all real values of x. If $f(n) < 1$ as $n \to \infty$ then convergence is ensured for a certain range of x dependent on the precise nature of the function f.

Since $a\, E(\lambda a^\dagger)|0\rangle = \lambda E(\lambda a^\dagger)|0\rangle$, we can use $E(x)$ to define analogues of coherent states as normalized eigenstates of the generalized annihilation operator.

$$|\lambda\rangle = \{E(|\lambda|^2)\}^{-\frac{1}{2}} E(\lambda a^\dagger)|0\rangle \tag{15}$$

3 Noise Reduction Properties

We consider conventional (undeformed) bosons.

The electromagnetic field components x and p are given by

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad \text{and} \quad p = \frac{1}{i\sqrt{2}}(a - a^\dagger). \tag{16}$$

As usual, we define the variances (Δx) and (Δp) by

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad \text{and} \quad (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2. \tag{17}$$

In the vacuum state

$$(\Delta x)_0 = \frac{1}{\sqrt{2}} \quad \text{and} \quad (\Delta p)_0 = \frac{1}{\sqrt{2}}. \tag{18}$$

and so

$$(\Delta x)_0(\Delta p)_0 = \frac{1}{2}. \tag{19}$$

The commutation relation for a and $a^\dagger$ leads to the following uncertainty principle

$$(\Delta x)(\Delta p) \geq \frac{1}{2}|\langle [x, p] \rangle| = \frac{1}{2}. \tag{20}$$

Thus the vacuum state attains the lower bound for the uncertainty, as do the coherent states.

While it impossible to lower the product $(\Delta x)(\Delta p)$ below the vacuum uncertainty value, it is nevertheless possible to define *squeezed* states [11] for which (at most) one quadrature lies below it's vacuum value, i.e.

$$(\Delta x) < (\Delta x)_0 = \frac{1}{\sqrt{2}} \quad \text{or} \quad (\Delta p) < (\Delta p)_0 = \frac{1}{\sqrt{2}} \tag{21}$$

If we now consider the generalized bosonic operators given by (1), using the same definitions for the the field quadratures, x and p, as in (16) we find that, just as in the conventional case, the vacuum uncertainty product $(\Delta x)_0(\Delta p)_0 = \frac{1}{2}$ is a lower bound for all *number* states.

However, unlike the conventional case, it is not a global lower bound.

Consider the quadrature values in eigenstates of the generalized annihilation operator.

Then

$$\langle x \rangle_\lambda = \langle \lambda | \frac{1}{\sqrt{2}}(a^\dagger + a) | \lambda \rangle = \frac{1}{\sqrt{2}}(\lambda + \bar{\lambda}) \tag{22}$$

and

$$\begin{aligned} \langle x^2 \rangle_\lambda &= \langle \lambda | \, \frac{1}{2}((a^\dagger)^2 + a^2 + a^\dagger a + a a^\dagger) \, | \lambda \rangle & (23) \\ &= \frac{1}{2}\{(\bar{\lambda} + \lambda)^2 + 1 - \varepsilon_{f,\lambda}|\lambda|^2\} & (24) \end{aligned}$$

where

$$\varepsilon_{f,\lambda} = 1 - \langle f(N+1) \rangle_\lambda \tag{25}$$

If we choose $0 < f(n) < 1$, then it can be shown that $\varepsilon_{f,\lambda}|\lambda|^2 \in (0,1)$ for λ within the radius of convergence of the generalized exponential.

Hence

$$(\Delta x)^2_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} \tag{26}$$

Evaluating the variance for the other component, we find that $(\Delta p)^2_\lambda = (\Delta x)^2_\lambda$ so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} < \frac{1}{2} \tag{27}$$

However, it can also be shown that

$$\frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} = \frac{1}{2}|\langle[x,p]\rangle_\lambda| \tag{28}$$

so

$$(\Delta x)_\lambda(\Delta p)_\lambda = \frac{1}{2}|\langle[x,p]\rangle_\lambda| \tag{29}$$

Thus we see that these generalized q–coherent states satisfy a restricted form of the Minimum Uncertainty Property (M.U.P.) of the conventional coherent states. Additionally we see that there is a general noise reduction in both quadratures compared to their vacuum value. In conventional coherent states there is no noise reduction relative to the vacuum value. In conventional squeezed states, there is noise reduction in only one component.

4 Special Cases

We can apply the preceding analysis to the q-deformed bosons recently studied in connection with quantum groups (e.g. [5]).

a). 'Physics' q–bosons
First consider the q-bosons described by Macfarlane and Biedenharn [6].
These use the definition of the generalised number, $[n]$, recently discussed in the Physics literature and so will be termed '*physics*' q–bosons. They are characterised by the deformed commutation relation

$$aa^\dagger - q\,a^\dagger a = q^{-N}. \tag{30}$$

This can be rewritten [1] as

$$aa^\dagger - f(N)\,a^\dagger a = 1 \tag{31}$$

where $f(N) = \frac{q^{N+2}+1}{q(q^N+1)}$
In this case, for normalizable eigenstates, the function $\varepsilon_{f,\lambda}$ is negative and so noise reduction does not take place. This is in agreement with the findings of Katriel and Solomon [12].

b). 'Maths' q–bosons

We now consider the q-boson described by Arik and Coon [2]. This uses the generalized number function found in classical q–analysis and will therefore be termed a '*maths*' q–boson. It is characterised by the deformed commutation relation

$$aa^{\dagger} - q\, a^{\dagger} a = 1 \tag{32}$$

For $q \in (0,1)$, the Jackson q-exponential $E_q(|\lambda|^2)$ converges, provided $\varepsilon_q|\lambda|^2 = (1-q)|\lambda|^2 < 1$. Given this condition on λ, we have normalizable q-analogue coherent states satisfying (6) in which

$$(\Delta\, x)^2_\lambda = (\Delta\, p)^2_\lambda = (\Delta\, x)_\lambda (\Delta\, p)_\lambda = \frac{1}{2}\{1 - \varepsilon_q|\lambda|^2\} < \frac{1}{2} \tag{33}$$

Hence, for this type of q–boson, we do obtain noise reduction in both quadratures with respect to the vacuum value.

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References

[1] A.I. Solomon and J.L. Birman *Symmetries in Science VII* ed. B. Gruber, (Plenum Press, N.Y., 1993).

[2] M. Arik and D.D. Coon *J. Math. Phys.* **17** 524 (1976).

[3] A. Jannussis, L.C. Papaloucas and P.D. Siafarikas *Had. J.* **3** 1622 (1980)

[4] V.V. Kuryshkin *Ann. Fond. L. de Broglie* **5** 111 (1980).

[5] P.P. Kulish and E.V. Damaskinsky *J. Phys. A.* **23** 415 (1990).

[6] A.J. Macfarlane *J. Phys. A.* **22** 4581 (1989).
L.C. Biedenharn *J. Phys. A.* **22** L873 (1989).

[7] C. Daskaloyannis *J. Phys. A.* **25** 2261 (1992).
D. Bonatsos and C. Daskaloyannis *J. Phys. A.* **26** 1589 (1993).

[8] A. Jannussis *J. Phys. A.* **26** L233 (1993)

[9] F.H. Jackson *Mess. Math.* **38** 62 (1909).
H. Exton *q-Hypergeometric Functions and Applications*, (Ellis Horwood, Chichester, 1983)

[10] R.J. Glauber *Phys. Rev.* **131** 27 66 (1963).

[11] H.P. Yuen *Phys. Rev. A.* **13** 2226 (1976).

[12] J. Katriel and A.I. Solomon *J. Phys. A.* **24** 2093 (1991).