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SQUEEZING GENERATED BY A NONLINEAR MASTER EQUATION AND BY AMPLIFYING-DISSIPATIVE HAMILTONIANS

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Abstract

In the first part of this contribution we show that the master equation derived from the generalized version of the nonlinear Doebner-Goldin equation leads to the squeezing of one of the quadratures. In the second part we consider two familiar Hamiltonians, the Bateman – Caldirola – Kanai and the optical parametric oscillator; going back to their classical Lagrangian form we introduce a stochastic force and a dissipative factor. From this new Lagrangian we obtain a modified Hamiltonian that treats adequately the simultaneous amplification and dissipation phenomena, presenting squeezing, too.

1 The nonlinear master equation

Our search for a new class of equations was inspired by the model proposed recently by Doebner and Goldin [1]. They looked for the most general Schrödinger equation compatible with the Fokker-Planck equation for the probability density $\rho(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2$,

$$\rho_t + \nabla \cdot \mathbf{j} = D\,\nabla^2 \rho \tag{1}$$

(where D is a constant positive diffusion coefficient), and derived the nonlinear equation

$$\begin{aligned} i\hbar\partial\psi(\mathbf{x},t)/\partial t &= [-(\hbar^2/2m)\nabla^2 + V(\mathbf{x})]\psi(\mathbf{x},t) \\ &+ i\hbar D\left[\frac{\nabla^2 + |\nabla\psi(\mathbf{x},t)|^2}{|\psi(\mathbf{x},t)|^2}\right]\psi(\mathbf{x},t) \quad , \end{aligned} \tag{2}$$

for a particle with mass m moving in a scalar potential $V(\mathbf{x})$. The advantage of eq. (2), comparatively to other ones [2], is its group theoretical origin, the nonlinear term was not simply added to the usual Schrödinger equation *ad hoc*, but its structure was derived from the analysis of representations of the $Diff(\mathbf{R}^3)$ group, proposed as a "universal quantum kinematical group" [1]. The only drawback of eq. (2) is its limited range of applicability, since it can be used only in the coordinate representation. It is desirable to have a more general equation valid in arbitrary representations. To obtain such a generalization (heuristically), first, we remove the Cartesian coordinates dependence, introducing an arbitrary set of states $\mid z\rangle$, which form a complete system with respect to some measure $d\mu(z)$, i.e.,

$$\int \mid z\rangle\langle z \mid d\mu(z) = \mathbf{\hat{1}}. \tag{3}$$

Secondly, we replace the two ∇ - operators in the additional term of eq. (2) by two *arbitrary* (linear) operators $\hat{A}$ and $\hat{B}$. By this way we arrive at the equation, whose nonlinear part, in general, is neither Hermitian, nor anti-Hermitian,

$$i\hbar\frac{\partial}{\partial t}\langle z \mid \psi\rangle \;=\; \langle z \mid \hat{H} \mid \psi\rangle + i\hbar D\left[\langle z \mid \psi\rangle\frac{\langle\psi \mid \hat{B} \mid z\rangle\langle z \mid \hat{A} \mid \psi\rangle}{\langle\psi \mid z\rangle\langle z \mid \psi\rangle} - \langle z \mid \hat{B}\hat{A} \mid \psi\rangle\right]. \tag{4}$$

Multiplying both sides of this equation by $\langle\psi \mid z\rangle$ and integrating over $d\mu(z)$ with account of (3), one can check that the normalization of the wave function is preserved in time since

$$\int\langle\psi \mid z\rangle\langle z \mid \Omega\{\psi\}\rangle\; d\mu(z) = 0, \tag{5}$$

where $\langle z \mid \Omega\{\psi\}\rangle$ is the non-Hamiltonian term in the right-hand side of eq. (4). Evidently, the new equation satisfies the homogeneity condition. Moreover, the "separability property" also holds: the wave function of noninteracting particles is factorized for all times, if it was factorized at the initial moment. Nonetheless, the form (4) is not the most general, in [3] we showed that the more general form for $\Omega\{\psi\}$ is

$$\Omega\{\mid\psi\rangle\} \;=\; i\hbar D\left[\frac{\langle\psi \mid \hat{B} \mid z\rangle\langle z \mid \hat{A} \mid \psi\rangle}{\langle\psi \mid z\rangle\langle z \mid \psi\rangle} \;-\; r\,\frac{\langle\psi \mid \hat{B}\hat{A} \mid z\rangle}{\langle\psi \mid z\rangle} \;-\; \lambda\hat{B}\hat{A} \mid \psi\rangle\right], \tag{6}$$

where λ and r may be arbitrary complex numbers satisfying the restriction $\lambda + r = 1$. In papers [4, 5] we investigated eq. (4) with $\hat{B} = \hat{A}^{+}$, $\hat{A}$, $\hat{B}$ being lowering and rising operators for the two-level system and for the harmonic oscillator. It was demonstrated that this equation, written in the discrete energy (Fock) basis $\mid n\rangle$, describes the relaxation to "pseudothermal" stationary states, possessing the Planck distribution for the populations of energy levels, but nonzero off-diagonal elements of the density matrix. This is not surprising, since eq. (4) relates to *pure quantum states*. However, it is more natural to describe relaxation processes in terms of *density matrices*, in order to investigate the evolution of *mixed quantum states*. Therefore, our next goal is to obtain *nonlinear master equations* originating from eq. (6) and its special cases.

Starting from equation (6) and considering the evolution of the pure state density matrix $\langle z \mid \rho_\psi \mid z'\rangle = \langle z \mid \psi\rangle\langle\psi \mid z'\rangle$ governed by this equation and replacing $\mid\psi\rangle\langle\psi\mid$ by $\hat{\rho}$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\langle z \mid \hat{\rho} \mid z'\rangle \;=\;& D\left\{\frac{\langle z \mid \hat{A}\,\hat{\rho}\,\hat{B} \mid z\rangle}{\langle z \mid \hat{\rho} \mid z\rangle} + \frac{\langle z' \mid \hat{B}^{+}\,\hat{\rho}\,\hat{A}^{+} \mid z'\rangle}{\langle z' \mid \hat{\rho} \mid z'\rangle}\right.\\
&- \left. r\,\frac{\langle z' \mid \hat{\rho}\,\hat{B}\,\hat{A} \mid z\rangle}{\langle z' \mid \hat{\rho} \mid z\rangle} - r^{*}\frac{\langle z' \mid \hat{A}^{+}\,\hat{B}^{+}\,\hat{\rho} \mid z\rangle}{\langle z' \mid \hat{\rho} \mid z\rangle}\right\}\langle z \mid \hat{\rho} \mid z'\rangle\\
&- D\lambda\,\langle z \mid \hat{B}\,\hat{A}\,\hat{\rho} \mid z'\rangle - D\lambda^{*}\langle z \mid \hat{\rho}\,\hat{A}^{+}\,\hat{B}^{+} \mid z'\rangle - \frac{i}{\hbar}\langle z \mid \left[\hat{H}\,,\,\hat{\rho}\right] \mid z'\rangle,
\end{aligned} \tag{7}$$

which preserves the trace, normalization and hermiticity for an arbitrary initial density matrix.

2 Application: The harmonic oscillator in the coordinate representation

This section is devoted to investigating a special case of the general equation derived in Sect. 1. We shall limit ourselves to one dimension, identifying the ket-vector $| z\rangle \equiv | x\rangle$ and apply the new equation to the harmonic oscillator Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2 - fx \quad , \tag{8}$$

we choose operators $\hat{A}$ and $\hat{B}$ as $\hat{A} = \hat{B}^{+} = \frac{\partial}{\partial x} + sx$, where s, D, r, λ, are assumed to be real constants. Now, designating the elements of the density matrix in the coordinate representation as $\rho(x,y) = \rho^*(y,x)$ and introducing the notation

$$\begin{aligned}\mathcal{H}\{\rho\} &= -\frac{i}{\hbar}\langle x \mid \left[\hat{H},\hat{\rho}\right] \mid y\rangle \\ &= \frac{i\hbar}{2m}\left(\rho_{xx}(x,y) - \rho_{yy}(x,y)\right) - \frac{i}{\hbar}\left(V(x) - V(y)\right) \quad ,\end{aligned} \tag{9}$$

$$\rho_x \equiv \frac{\partial\rho}{\partial x}, \; \rho_y \equiv \frac{\partial\rho}{\partial y}, \; \rho_{xx} \equiv \frac{\partial^2\rho}{\partial x^2}, \ldots \tag{10}$$

we can write eq. (7) as

$$\begin{aligned}\frac{\partial\rho(x,y)}{\partial t} &= \mathcal{H}\{\rho\} + D\Big\{\frac{\rho_{xy}(x,x) + sx\left[\rho_x(x,x) + \rho_y(x,x)\right]}{\rho(x,x)} \\ &+ \frac{\rho_{xy}(y,y) + sy\left[\rho_x(y,y) + \rho_y(y,y)\right]}{\rho(y,y)} + 2s \\ &+ r\,\frac{\rho^*_{xx}(x,y) + \rho^*_{yy}(x,y)}{\rho^*(x,y)}\Big\}\rho(x,y) + \lambda D\left[\rho_{xx}(x,y) + \rho_{yy}(x,y)\right] .\end{aligned} \tag{11}$$

All derivatives in this equation must be calculated for the *independent* variables x and y, and only after that one should consider $x = y$ in the functions ρ_x, ρ_y, and ρ_{xy}.

For the probability density $P(x) \equiv \rho(x,x)$ eq. (11) results in the following Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = D\left\{\frac{\partial^2 P}{\partial x^2} + 2s\,\frac{\partial}{\partial x}\,(xP)\right\} \quad , \tag{12}$$

where $J(x) = -\frac{i\hbar}{2m}\left[\rho_x(x,x) - \rho_y(x,x)\right]$ is the current density, $2sDx$ is the drift velocity, and D is the diffusion coefficient. Eq. (12) clearly shows that the total probability is conserved in time. Eq. (11) admits a special class of solutions in the form of Gaussian exponentials,

$$\rho(x,y) = \exp\left[-\frac{1}{2}ax^2 + cxy - \frac{1}{2}a^*y^2 + bx + b^*y + \Phi\right] \quad , \tag{13}$$

with c and Φ being real functions of time, while a and b may be, in general, complex. Putting expression (13) into eq. (11) we get the equations for the coefficients $a(t)$ and $c(t)$, which do not contain neither the force $f(t)$, nor the function $b(t)$. Therefore, let us assume, for simplicity, that $f = b = 0$. Then, instead of eq. (11) we have the following set of ordinary differential equations for the *real* functions $\alpha = \mathrm{Re}a$, $\beta = \mathrm{Im}a$, c, and Φ,

$$\dot{\alpha} = 2\frac{\hbar}{m}\alpha\beta + 4D\left\{\alpha c - c^2 - \alpha^2 + s(\alpha - c)\right\} \quad , \tag{14}$$

$$\dot{\beta} = \frac{m}{\hbar}\omega^2 + \frac{\hbar}{m}(c^2 - \alpha^2 + \beta^2) + 4\xi D\alpha\beta \quad , \tag{15}$$

$$\dot{c} = 2\frac{\hbar}{m}\beta c - 4D\alpha c \quad , \tag{16}$$

$$\dot{\Phi} = \frac{\hbar}{m}\beta + 2D(c - \alpha + s) \quad . \tag{17}$$

Here the parameter $\xi = r - \lambda$ is introduced .

The trace of the Gaussian density matrix equals

$$Tr\hat{\rho} = \left[\frac{\pi}{\alpha - c}\right]^{1/2} e^{\Phi} \quad , \tag{18}$$

consequently, the condition of its conservation is the equation

$$\dot{\Phi} - \frac{1}{2}\frac{\dot{\alpha} - \dot{c}}{\alpha - c} = 0 \quad , \tag{19}$$

and it is easy to check that this equation is fulfilled. The difficulty of treating the equations for the coefficients α, β, c is connected with their nonlinearity even in the absence of nonlinear terms in the master equations (i.e., when $D = 0$). It is well known, however, that in the latter case the equations of motion for *average values* and the *second order moments* are *linear* for any quantum system with quadratic Hamiltonian. Therefore we replace the equations for coefficients α, β, c, by the equivalent equations for the variances

$$\sigma_{xx} = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2 \, , \; \sigma_{pp} = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 \, , \; \sigma_{xp} = \frac{1}{2}\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle - \langle\hat{x}\rangle\langle\hat{p}\rangle \, , \tag{20}$$

these quantities are related to the coefficients of the Gaussian density matrix (13) by

$$\sigma_{xx} = \frac{1}{2(\alpha - c)} \quad , \tag{21}$$

$$\sigma_{xp} = -\hbar\beta\sigma_{xx} = -\frac{\hbar\beta}{2(\alpha - c)} \quad , \tag{22}$$

$$\sigma_{pp} = \frac{\hbar^2}{2}(\alpha + c) + \frac{\sigma_{xp}^2}{\sigma_{xx}} = \frac{\hbar^2}{2}\frac{\alpha^2 + \beta^2 - c^2}{\alpha - c} \quad . \tag{23}$$

Another quantity characterizing the quantum state is

$$\Delta = \frac{4}{\hbar^2}\left(\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2\right) = \frac{\alpha + c}{\alpha - c} \quad . \tag{24}$$

Its importance is explained by two factors. First, any quantum state must satisfy the *generalized Robertson – Schrödinger uncertainty relation* [6] $\Delta \geq 1$. Second, any positively definite density operator must satisfy the inequality

$$\frac{Tr(\hat{\rho}^2)}{[Tr(\hat{\rho})]^2} \leq 1 \quad , \tag{25}$$

but for any Gaussian state [7]

$$\frac{Tr(\hat{\rho}^2)}{[Tr(\hat{\rho})]^2} = \Delta^{-1/2} \quad . \tag{26}$$

Consequently, the parameter Δ characterizes the "degree of coherence" of the Gaussian state: $\Delta = 1$ for pure states, and $\Delta > 1$ for mixed states.

The relations inverse to eqs. (21) – (23) read

$$\alpha = \frac{1+\Delta}{4\sigma_{xx}}, \; c = \frac{\Delta - 1}{4\sigma_{xx}}, \; \beta = -\frac{\sigma_{xp}}{\hbar\sigma_{xx}} \quad , \tag{27}$$

so one can check that eqs. (14) – (16) result in the following equations for the variances,

$$\dot{\sigma}_{xx} = \frac{2}{m}\sigma_{xp} - 4Ds\,\sigma_{xx} + 2D \quad , \tag{28}$$

$$\dot{\sigma}_{xp} = \frac{1}{m}\sigma_{pp} - \omega^2 m\,\sigma_{xx} - 4Ds\,\sigma_{xp} + D\frac{\sigma_{xp}}{\sigma_{xx}}\left[2 + \xi(1 + \Delta)\right] , \tag{29}$$

$$\begin{aligned} \dot{\sigma}_{pp} &= -2\omega^2 m\,\sigma_{xp} + \frac{Ds}{\sigma_{xx}}\left(\hbar^2 - 4\sigma_{xp}^2\right) \\ &+ \frac{2D}{\sigma_{xx}^2}\left\{\sigma_{xp}^2\left[1 + \xi(1+\Delta)\right] - \frac{\hbar^2}{8}\left[1 + \Delta^2\right]\right\} . \end{aligned} \tag{30}$$

As a consequence, the equation for the parameter Δ is

$$\dot{\Delta} = -D(\Delta - 1)\left[4s + \frac{\Delta - 1}{\sigma_{xx}}\right] \quad . \tag{31}$$

We perceive that Δ is conserved in time if $D = 0$. Alternatively, if $D \neq 0$, then Δ tends to the unit value at $t \to \infty$, provided both coefficients, D and s, are positive. This means that any initial state exhibits the relaxation to a *pure state*!

The advantage of the equations for the variances is clear: The nonlinear terms are multiplied by the diffusion coefficients, which are small in all reasonable situations. Therefore we may use the solutions corresponding to $D = 0$ as the zero approximation, and develop some perturbative scheme.

If $D = 0$, getting rid of σ_{xp} and σ_{pp}, one arrives at a single third-order equation for the coordinate variance,

$$\frac{d^3\sigma_{xx}}{dt^3} + 4\omega^2\frac{d\sigma_{xx}}{dt} = 0 \quad , \tag{32}$$

whose solution reads

$$\sigma_{xx}(t) = A + B\,e^{2i\omega t} + B^*\,e^{-2i\omega t} \quad , \tag{33}$$

where A and B are constant coefficient. It is natural to suppose that for $D \to 0$ the evolution of σ_{xx} is given by the same expression, but with the *slowly varying in time* coefficients:

$$\sigma_{xx}(t) = A(t) + B_+(t) \quad , \tag{34}$$

where

$$B_\pm(t) = B(t)\, e^{2i\omega t} \pm B^*(t)\, e^{-2i\omega t} \quad . \tag{35}$$

Now inserting (33) into eq. (28) and neglecting terms of the order of $O(D^2)$ (guessing that the n-th derivatives of $A(t)$ and $B(t)$ are proportional to D^n), we can express the covariance σ_{xp} as follows,

$$\sigma_{xp} = i\omega\, m\, B_- + \frac{m}{2}\left(\dot{A} + \dot{B}e^{2i\omega t} + \dot{B}^*\, e^{-2i\omega t}\right) + 2mDs\,(A + B_+) - mD\,, \tag{36}$$

and with the same accuracy we find the momentum variance for eq. (30):

$$\begin{aligned}\sigma_{pp} &= (\omega m)^2(A - B_+) + 8i\omega m^2 DsB_- + 2i\omega m^2\left(\dot{B}\, e^{2i\omega t} - \dot{B}^*\, e^{-2i\omega t}\right) \\ &- \frac{i\omega m^2 DB_-}{A + B_+}\,[2 + \xi(1+\Delta)] \quad .\end{aligned} \tag{37}$$

Since in this expression the function Δ is multiplied by the small parameter D, we must calculate it in the zeroth approximation. Thus we get

$$\Delta = \left(\frac{2\omega m}{\hbar}\right)^2\left(A^2 - 4\,|\,B\,|^2\right) \equiv R^2 \quad , \tag{38}$$

being both, R and Δ, slowly varying functions of time.

Now, we make the crucial step first proposed in [8, 9]: we put the expression for σ_{pp} in terms of A and B into eq. (30) and average both sides with respect to the fast oscillations of frequency ω . Then we arrive at the equation, which contains functions A and R only

$$\begin{aligned}\dot{A} &= 2\Gamma\omega\left\{\left[-4\sigma_0 s + \left(\xi - \frac{1}{2}\right)R + \frac{1+\xi}{R} - \frac{1}{2R^3}\right]A\right. \\ &+ \left.\left(1 + R^2\right)\left(\frac{2\sigma_0 s}{R} - \xi\right)\sigma_0\right\} \quad ,\end{aligned} \tag{39}$$

and where $\sigma_0 = \hbar/(2\omega m)$ is the coordinate variance in the oscillator ground state, while $\Gamma = mD/\hbar$ is the dimensionless diffusion coefficient.

Now, averaging eq. (31) we arrive at a *closed* equation for $R(t)$,

$$\dot{R} = -\Gamma\omega\left(R - \frac{1}{R}\right)\left(4\sigma_0\, s + R - \frac{1}{R}\right) \quad . \tag{40}$$

Consequently, eq. (39) actually is the *first order linear nonuniform equation* with respect to function *A(t)*, which can be easily solved provided the solution to eq. (40) is known.

Thus, averaging the equations for the variances over the fast oscillations leads to the effective linearization of the equations governing the evolution of the coefficients of the Gaussian density matrix. To get the equation for $B(t)$, one should first multiply both sides of the equation for $\dot{\sigma}_{pp}$, by $\exp(-2i\omega t)$, and then average over the fast oscillations. We confine ourselves, however, to the evolution of functions $\Delta(t)$ and $A(t)$ (the absolute value of $B(t)$ can be extracted from eq. (38)). It is worth to mention that $A(t)$ is nothing but the energy of quantum fluctuations (up to the constant factor and small corrections of the order of $O(D)$):

$$\delta E(t) = \frac{\omega^2 m}{2}\sigma_{xx} + \frac{1}{2m}\sigma_{pp} = \omega^2 m A(t) . \tag{41}$$

Now if we consider a pure state, $R = 1$, then the solution for eq. (39) is

$$A(t) = \sigma_0 + [A(0) - \sigma_0]\exp[-4\Gamma\omega(2\sigma_0 s - \xi)t] \quad . \tag{42}$$

We see that the energy of fluctuations can increase or decrease, depending on the sign of the term in the exponential:

1)If $2\sigma_0 s - \xi > 0$, we verify that $A(\infty) = \sigma_0$, $\delta E(\infty) = \hbar\omega/2$ and $B(\infty) = 0$, therefore asymptoticaly σ_{xx} and σ_{pp} do not oscillate.

2)If $2\sigma_0 s - \xi < 0$ and since reasonably $\Gamma\omega \ll 1$, $A(t)$ becomes a slowly increasing function with time (the same for $\delta E(t)$). From eq. (38), asymtoticaly we have $A(t) \simeq 2B(t) \gg \sigma_0^2$. If $B(t)$ is real at $t = 0$ then it is real for any $t > 0$, so from eqs. (34) and (35) $\sigma_{xx}(t) = A(t) + 2B(t)\cos(2\omega t)$ or $A(t) - 2B(t) \leq \sigma_{xx} \leq A(t) + 2B(t)$. But from eq. (38) $A(t) - 2B(t) = \sigma_0^2/(A(t) + 2B(t)) \sim \sigma_0^2/(2A(t))$. Therefore asymtoticaly σ_{xx} oscillates between two values,

$$\frac{\sigma_0^2}{2A(t)} \leq \sigma_{xx} \leq 2A(t) \quad , \tag{43}$$

thence, as time goes on the solution of the nonlinear Schrödinger equation becomes highly squeezed. Note that a similar behavior of the variances (i.e., an exponential increase of the squeezing coefficient) is observed in the case of the usual Schrödinger equation for the parametrically excited oscillator, when its frequency changes in time [7]. However, in the present case all the coefficients in the generalized Schrödinger equation with the functional (6) (or its master equation counterpart eq.(7)) do not depend on time, and the increase of fluctuations is caused by the nonlinear terms.

3 Amplifying – Dissipative Hamiltonians

Here we shall consider the theory developed many years ago by P. Havas [10], which is quite suited to construct Hamiltonians that take into account dissipation and apply it to two examples, confining ourselves to the one-dimensional case.

3.1 The Bateman-Caldirola-Kanai (BCK) Hamiltonian

The harmonic oscillator with an exponential time-dependent mass $m(t) = m_0\exp(\zeta t)$, is known as the BCK Hamiltonian [11] and we shall introduce the phenomenon of friction in it. According to [10] when friction is present the Lagrangian that describes the motion, (that reproduces the classical equation of motion) is

$$L(q,\dot{q};t) = \Big(\frac{1}{2}\, m(t)\,\dot{q}^2 - V(q;t) + qF(t)\Big)\exp[\gamma(t)] \quad . \tag{44}$$

Here $V(q;t) = \frac{1}{2}m(t)\omega_0^2\, q^2$ and $F(t)$ is a time-dependent external force. The exponential factor introduces the friction effects (also external) on the system and for the specific Lagrangian one has

$$\gamma(t) = \frac{\gamma_0}{\zeta}(1 - \exp(-\zeta t)) \quad , \tag{45}$$

as $\lim_{t\to\infty}\gamma(t) = \gamma_0/\zeta$ and $\lim_{\zeta\to 0}\gamma(t) = \gamma_0 t$, so, recovering previous results [12]. We shall consider $F(t)$ as being a stochastic force whose mean in an *ensemble* is null, $\langle F(t)\rangle = 0$, and that its correlation is *Markovian*, $\langle F(t)F(t')\rangle = 2d\delta(t-t')$.

According to the quantization procedure of [12] one obtains the system Hamiltonian

$$\hat{H}(\hat{P},\hat{Q};t) = \frac{\hat{P}^2}{2m_0}\exp(-\zeta t) + \frac{1}{2}m_0\omega_0^2\hat{Q}^2\exp(\zeta t) + \frac{\gamma_0}{4}\{\hat{Q},\hat{P}\}\exp(-\zeta t) + \hat{Q}F(t)\exp(\gamma(t)/2) \tag{46}$$

where $\hat{P},\hat{Q}$ are canonicaly conjugated operators, $[\hat{Q},\hat{P}] = i\hbar$. The physical position and momentum operators are related to those through

$$\hat{q}_{phys} = \hat{Q}\exp(-\gamma(t)/2) \quad , \quad \hat{p}_{phys} = \hat{P}\exp(-\gamma(t)/2) \tag{47}$$

In order to obtain the equations of motion for the operators $\hat{P}$ and $\hat{Q}$ in the Heisenberg picture we first do an unitary transformation in Schrödinger equation with operator $\mathbf{S}(\zeta t/2) = \exp(-\frac{\zeta t}{4}\{\hat{Q},\hat{P}\})$, which leads to the new Hamiltonian

$$\begin{aligned}\hat{K}(\hat{\pi},\hat{x};t) &= \frac{\hat{\pi}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2\hat{x}^2 + \frac{1}{4}(\gamma_0\exp(-\zeta t) + \zeta)\{\hat{x},\hat{\pi}\} \\ &+ \hat{x}F(t)\exp\left[\frac{1}{2}(\gamma(t) - \zeta t)\right]\end{aligned} \tag{48}$$

with

$$\begin{aligned}\hat{q}_{phys} &= \hat{x}\exp\left[-\frac{1}{2}(\gamma(t)+\zeta t)\right] \\ \hat{p}_{phys} &= \hat{\pi}\exp\left[-\frac{1}{2}(\gamma(t)-\zeta t)\right] \quad .\end{aligned} \tag{49}$$

The equations of motion for $\hat{x}(t)$ and $\hat{\pi}(t)$ are solved assuming $\hat{x}_H = u(t)\hat{x}_0 + v(t)\hat{\pi}_0 + w(t)$, leading to the set of linear differential equations

$$\frac{d^2u}{dt^2} - \left[\Omega_0^2 + \frac{\gamma_0^2}{4}\exp(-2\zeta t)\right]u = 0 \tag{50}$$

$$\frac{d^2v}{dt^2} - \left[\Omega_0^2 + \frac{\gamma_0^2}{4}\exp(-2\zeta t)\right]v = 0 \tag{51}$$

$$\frac{d^2w}{dt^2} - \left[\Omega_0^2 + \frac{\gamma_0^2}{4}\exp(-2\zeta t)\right]w = -\frac{F(t)}{m_0}\exp\left[\frac{1}{2}(\gamma(t) - \zeta t)\right] \tag{52}$$

where $\Omega_0^2 = \frac{\zeta^2}{4} - \omega_0^2$ and with the following initial conditions

$$u(0) = 1 \quad \text{and} \quad \dot{u}(0) = \frac{1}{2}(\gamma_0 + \zeta) \tag{53}$$

$$v(0) = 0 \quad \text{and} \quad \dot{v}(0) = \frac{1}{m_0} \tag{54}$$

$$w(0) = \dot{w}(0) = 0 \quad . \tag{55}$$

The exact solutions of the above differential equations are :

$$
\begin{aligned}
u(t) &= \left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathbf{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right) \\
&- \left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right) - \frac{\gamma_0}{2\zeta}\mathbf{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right) \qquad (56) \\
v(t) &= \frac{1}{m_0\zeta}\left[\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right) - \mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right)\right] \qquad (57) \\
w(t) &= \frac{1}{m_0\zeta}\int_0^t \exp\left\{\frac{1}{2}\left[\gamma(t_1) - \zeta t_1\right]\right\}\left[\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right)\mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t_1)\right)\right. \\
&- \left.\mathbf{K}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t)\right)\mathbf{I}_\nu\left(\frac{\gamma_0}{2\zeta}\exp(-\zeta t_1)\right)\right] F(t_1)\,dt_1 \qquad , \qquad (58)
\end{aligned}
$$

where $\mathbf{I}_\nu(.)$ and $\mathbf{K}_\nu(.)$ are the modified Bessel functions of the first and third kind, respectively [13]. The parameter ν is

$$
\nu = \begin{cases} \frac{\Omega_0}{\zeta} & \text{if } \frac{\zeta}{2} > \omega_0 \\ 0 & \text{if } \frac{\zeta}{2} = \omega_0 \\ \imath\frac{\Omega_0}{\zeta} & \text{if } \frac{\zeta}{2} < \omega_0 \end{cases} \qquad . \qquad (59)
$$

Similarly the momentum operator in the Heisenberg picture is $\hat{\pi}_H(t) = \mu(t)\hat{x}_0 + \beta(t)\hat{\pi}_0 + \xi(t)$, where

$$
\begin{aligned}
\mu(t) &= m_0\left[\dot{u}(t) - \frac{1}{2}\left[\gamma_0\exp(-\zeta t) + \zeta\right]u(t)\right] \\
\beta(t) &= m_0\left[\dot{v}(t) - \frac{1}{2}\left[\gamma_0\exp(-\zeta t) + \zeta\right]v(t)\right] \\
\xi(t) &= m_0\left[\dot{w}(t) - \frac{1}{2}\left[\gamma_0\exp(-\zeta t) + \zeta\right]w(t)\right] \qquad . \qquad (60)
\end{aligned}
$$

Considering that the energy increasing prevails over energy dissipation, $\gamma_0/\zeta \ll 1$, one has the approximative solutions,

$$
\begin{aligned}
\hat{x}_H(t) &= \left(\cosh(\Omega_0 t) + \frac{\gamma_0+\zeta}{2\Omega_0}\sinh(\Omega_0 t)\right)\hat{x}_0 + \left(\frac{1}{m_0\Omega_0}\sinh(\Omega_0 t)\right)\hat{\pi}_0 \\
&- \frac{1}{m_0\Omega_0}\exp\left(\frac{\gamma(t)}{2}\right)\int_0^t \exp\left(\frac{-\zeta t_1}{2}\right)\sinh[\Omega_0(t-t_1)]\,F(t_1)\,dt_1 \qquad (61) \\
\hat{\pi}_H(t) &= \left[-\frac{m_0\omega_0^2}{\Omega_0}\sinh(\Omega_0 t) + \frac{m_0\gamma_0}{2}\left(\cosh(\Omega_0 t) - \frac{\zeta}{2\Omega_0}\sinh(\Omega_0 t)\right)\right]\hat{x}_0 \\
&+ \left(\cosh(\Omega_0 t) - \frac{\zeta}{2\Omega_0}\sinh(\Omega_0 t)\right)\hat{\pi}_0 \\
&+ \exp\left(\frac{\gamma(t)}{2}\right)\int_0^t \exp\left(\frac{-\zeta t_1}{2}\right)\left[\frac{\zeta}{2\Omega_0}\sinh[\Omega_0(t-t_1)] - \cosh[\Omega_0(t-t_1)]\right]F(t_1)dt_1 \qquad (62)
\end{aligned}
$$

and the variances for $\hat{q}_{phys}$ and $\hat{p}_{phys}$ become

$$\lim_{t\to\infty}\langle\Delta\hat{X}^2_{1,H}\rangle \sim \frac{\zeta}{16\Omega_0}\exp[-(\zeta-2\Omega_0)t]\left\{\frac{\gamma_0}{\zeta}\left(\frac{\zeta}{2\Omega_0}-1\right)\coth\left(\frac{\hbar\omega_0}{2k_BT}\right)+\exp\left(-\frac{\gamma_0}{\zeta}\right)\left(\frac{\zeta}{2\Omega_0}+1\right)\left(1+\frac{\gamma_0}{\zeta}\right)\right\}\to 0\left(\frac{\zeta}{2\Omega_0}>1\right) \tag{63}$$

$$\lim_{t\to\infty}\langle\Delta\hat{X}^2_{2,H}\rangle \sim \frac{\zeta}{16\Omega_0}\exp[(\zeta+2\Omega_0)t]\left\{\frac{\gamma_0}{\zeta}\left(\frac{\zeta}{2\Omega_0}+1\right)\coth\left(\frac{\hbar\omega_0}{2k_BT}\right)+\exp\left(-\frac{\gamma_0}{\zeta}\right)\left(\frac{\zeta}{2\Omega_0}-1\right)\left(1+\frac{\gamma_0}{\zeta}\right)\right\}\to \infty\left(\frac{\zeta}{2\Omega_0}>1\right) \quad , \tag{64}$$

verifying squeezing even in occurence of a weak dissipation.

3.2 The optical parametric oscillator (OPO)

Using the same method we can treat the OPO introducing in the Hamiltonian the dissipation of the cavity, hence,

$$H(A^+,\,A,\,t) := f_1(t)A^+A + \left\{f_2(t)(A^+)^2 + f_3(t)A^+ + h.c.\right\} \tag{65}$$

where the mathematical operators A^+ and A are related to the physical creation and destruction operators by $a = A\exp(-\lambda t/2)$ and $a^+ = A^+\exp(-\lambda t/2)$. Moreover, $f_1(t) = \omega_0$ is the mode frequency in the cavity, $f_2 = \kappa\exp(-2i\omega t) + i\lambda/4$, where κ and ω are, respectively, the intensity and the frequency of the pumping field, while λ is the damping constant of the cavity. $f_3(t) = F(t)\,e^{\lambda t}/(2\omega_0)^{1/2}$. The force $F(t)$ is assumed to be a *Markovian* stochastic force: $< F(t) > = 0$, $< F(t)F^*(t') > = < F(t)^*F(t') > = 2d\delta(t-t')$ and $< F(t)F(t') > = 0$, and the parameter d is related to the temperature of the cavity,

$$d = \frac{\lambda\omega_0}{2}\coth\left(\frac{\omega_0}{2k_BT}\right) \quad . \tag{66}$$

Considering the pumping at resonance, $\omega_0 = \omega$, and for $\kappa\lambda/\omega_0 \ll 1$, the solution of Heisenberg equations for operator $A_H(t)$ is

$$A_H(t) = u(t)A(t) + v(t)A^+(t) + w(t) \tag{67}$$

where

$$u(t) = e^{-\imath\omega_0 t}\cosh(2\kappa t) + \left[e^{-\imath\omega_0 t}\left(\sinh^2\gamma\,\cosh(2\kappa t) - \imath\,\sinh\gamma\cosh\gamma\sinh(2\kappa t)\right) - c.c.\right] \tag{68}$$

$$v(t) = \imath\,\sinh(2\kappa t)\left(\cosh^2\gamma\, e^{-\imath\omega_0 t} - \sinh^2\gamma\, e^{\imath\omega_0 t}\right) - \imath\,\sinh(2\gamma)\sinh(2\kappa t)\cos(2\omega_0 t) \tag{69}$$

$$w(t) = -\cosh(\gamma - 2\kappa t)\left[\int_0^t f_3(t')\exp[-\imath\omega_0(t-t')]\,dt' - c.c.\right] \tag{70}$$

and $\tanh(2\gamma) = \lambda/(2\omega_0)$.

With the above solution we present the asymptotic ($t \to \infty$) mean values of several quantities, after taking the average over the high frequency oscillations of the field (ω_0) and considering $4\kappa/\lambda < 1$:

1)Energy,

$$\lim_{t\to\infty} \langle \hat{E} \rangle = \omega_0 e^{-\lambda t} \langle A_H^{+} A_H \rangle_{\omega_0} \sim \frac{d}{\lambda} \frac{1}{1-(4\kappa/\lambda)^2} \tag{71}$$

2)Variances of the two quadratures,

$$\lim_{t\to\infty} \langle \Delta \hat{X}_{1,2} \rangle_t \sim \frac{d}{2\lambda\omega_0} \frac{1}{1 - (4\kappa/\lambda)^2} \tag{72}$$

Although the asymptotic values of the variances of the quadratures are both above the value 1/2, at initial times squeeze is seen in the fast oscillations (before averaging over the mode frequency). In eqs. (71) and (72) one verifies the effects of the dissipation-amplification process through the quantity $4\kappa/\lambda$:

i) If $4\kappa/\lambda \ll 1$ (very weak pumping compared with the cavity dissipation), the thermalization, represented by the parameter d, dominates in the physiçal expressions at equilibrium.

ii) On the other side, when $4\kappa/\lambda$ is close to 1 (strong pumping) the factor $(1-(4\kappa/\lambda)^2)^{-1}$ dominates the strenghts of the asymptotic values, increasing dramatically the energy and fluctuations, as is expected to occur at resonance.

4 Summary

We have presented two different formulations of quantum dynamical equations that show squeezing in the variances of the conjugate canonical operators. In the first one we considered a generalization of the Doebner-Goldin nonlinear extension of the Schrödinger equation and we verified that although the parameters that enter the nonlinear part of the equation are constant in time, squeezing occurs, essentially due to the nonlinearity. Moreover, the master equation shows the surprising feature that any initial mixed state relax to a pure state!

In the other approach we introduced the dissipation phenomenon into the Hamiltonian formalism by starting with a conveniently defined Lagrangian, as proposed by P. Havas [10]. We considered two familiar time-dependent Hamiltonians, the BCK and the OPO. The BCK Hamiltonian has a time-dependent mass and it displays amplification of energy and squeezing of variance of momentum or of position, although uncertainty is preserved. The dissipation was introduced and the effects are seen in eqs. (63)-(64).

The second Hamiltonian is the OPO, describing a single mode in an electromagnetic cavity with pumping at resonance. Dissipation is introduced to take into account the loss in the cavity walls. As an expressive result we verify that the asymtotic physical expressions depend, essentially, on the factor $4\kappa/\lambda$, κ representing the pumping and λ, the dissipation.

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