

THE WAVE FUNCTION AND MINIMUM UNCERTAINTY FUNCTION OF THE BOUND QUADRATIC HAMILTONIAN SYSTEM

Kyu Hwang Yeon

*Department of Physics, Chungbuk National University
Cheongju, Chungbuk, 360-763, Korea*

Chung In Um

*Department of Physics, College of Science, Korea University
Seoul, 136-701, Korea*

T.F. George

*Department of Chemistry and Physics, Washington State University
Pullman, Washington 99164-1046*

Abstract

The bound quadratic Hamiltonian system is analyzed explicitly on the basis of quantum mechanics. We have derived the invariant quantity with an auxiliary equation as the classical equation of motion. With the use of this invariant it can be determined whether or not the system is bound. In bound system we have evaluated the exact eigenfunction and minimum uncertainty function through unitary transformation.

1 Introduction

In recent years an extensive effort has been devoted to obtaining an exact solution for the oscillator systems with time-dependent Hamiltonian ¹⁾ and especially dissipative system, *i.e.*, damped free particle ²⁾, damped ³⁾ or damped driven harmonic oscillator ⁴⁾ and driven time-dependent harmonic oscillator⁵⁾. After Lewis and Riesenfeld⁶⁾ first derived the relation between the eigenstates of the dynamical invariant and the solution of the Schrödinger equation, many authors have applied the dynamical invariant method to investigate the time-dependent oscillator system. The dynamical invariant is related directly to an auxiliary equation as the classical equation of motion for the Hamiltonian system, which is given as nonlinear second-order differential equation. Therefore the dynamical invariant can be determined by the particular solution to the auxiliary equation.

In this paper, employing the operator method, we derive the wave function and the minimum uncertainty function for the quadratic Hamiltonian system which includes canonical variables with time-dependent coefficients. Recently, we have investigated this system to obtain the wave function and propagator through path integral method ⁷⁾. In Sec. 2, we derive the dynamical invariant from the equation of motion. We classify whether or not the system is bound in the consideration of our system as classical system and then find the conditions for bound and unbound. In Sec. 3, using the quantum invariant operator, we define the creation and annihilation operators and then evaluate the wave function and propagator of our system. In Sec. 4, we introduce new creation and annihilation operators from the old ones in Sec. 3, and evaluate eigenfunction of the unitary transformed system and minimum uncertainty function. Finally, we give the summary in Sec. 5.

2 The bound quadratic Hamiltonian system

The quadratic Hamiltonian of the system is given as

$$H = \frac{1}{2}[A(t)p^2 + B(t)(pq + qp) + C(t)q^2] \quad (1)$$

where p and q are canonical variables. $A(t)$, $B(t)$ and $C(t)$ is continuously differentiable function but $A(t)$ is nonzero. The classical equation of motion can be obtained from the Hamilton's equation of motion:

$$\ddot{q} + \zeta(t)\dot{q} + \xi(t)q = 0 \quad (2)$$

with

$$\zeta(t) = -\frac{\dot{A}(t)}{A(t)} \quad (3)$$

$$\xi(t) = A(t)C(t) + \frac{\dot{A}(t)B(t)}{A(t)} - B^2(t) - \dot{B}^2(t). \quad (4)$$

The general solution for Eq. (2) can not be found, but we may take the solution in the following form:

$$q = \rho(t)e^{i\gamma(t)} \quad (5)$$

here, $\rho(t)$ and $\gamma(t)$ are the functions to be determined from Eq. (2). The functions are real and depend only on time. Substitution of Eq. (5) in Eq. (2) offers the real and imaginary parts of this equation as

$$\ddot{\rho} - \rho\dot{\gamma}^2 + \zeta(t)\dot{\rho} + \xi(t)\rho = 0. \quad (6)$$

and

$$\ddot{\gamma} + 2\dot{\rho}\dot{\gamma} + \zeta(t)\dot{\gamma}\rho = 0. \quad (7)$$

The time invariant quantity can be found from Eq. (7) in the form:

$$\Omega = \frac{\rho^2\dot{\gamma}}{A(t)} \quad (8)$$

with auxiliary condition as the classical equation of motion, Eq. (2). With the use of Eq. (8), the nonlinear differential equation [Eq.(6)] can be written as

$$\ddot{\rho} + \zeta(t)\dot{\rho} + \xi(t)\rho = \frac{\Omega^2}{\rho^3}A^2(t). \quad (9)$$

We may find another classical time invariant quantity with an auxiliary equation as classical equation of motion. We assume that this invariant quantity depends on p , q and t . Then, from Hamilton's equation of motion, the time derivative of $I(p, q, t)$ becomes

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial I}{\partial p} \frac{\partial H}{\partial q} = 0. \quad (10)$$

Combining Eq. (10) with Eq. (1) we may obtain the time invariant quantity as

$$I = \left(\frac{\Omega}{\rho}q\right)^2 + \left[\left(\frac{B(t)}{A(t)}\rho - \frac{1}{A(t)}\dot{\rho}\right)q + \rho p\right]^2. \quad (11)$$

$I(q, p, t)$ is an invariant quantity and thus we can express it in phase space. For $\Omega = 0$, Eq. (11) becomes a linear line in phase space and canonical variables q and p can occupy every region in phase space. Therefore the motion of the system is unbound. On the other and, for $\Omega \neq 0$, Eq. (11) becomes ellipse in phase space because the coefficient matrix of it has positive real eigenvalues. The canonical variables q and p can occupy some finite region in phase space, and thus the motion of the system is bound.

3 The Wave Function, Propagator and Uncertainty Values

For quantum mechanical treatment of our system, we may replace the canonical variables with the corresponding quantum operators in Eq. (7) and then we may also obtain the quantum invariant operator of the system as the same form of Eq. (11).

In order to obtain the eigenfunctions and eigenvalues of the invariant operator, we define the creation and annihilation operator, a and a^\dagger with auxiliary equations, Eq. (8) and Eq. (9) as

$$a = \sqrt{\frac{A}{2\hbar\dot{\gamma}}} \left\{ \frac{1}{A} \left[\dot{\gamma} + i \left(B - \frac{\dot{\rho}}{\rho} \right) \right] q + ip \right\} \quad (12)$$

$$a^\dagger = \sqrt{\frac{A}{2\hbar\dot{\gamma}}} \left\{ \frac{1}{A} \left[\dot{\gamma} - i \left(B - \frac{\dot{\rho}}{\rho} \right) \right] - ip \right\}. \quad (13)$$

The invariant operator [Eq. (11)] can be expressed in terms of a and a^\dagger as

$$I = \hbar\Omega \left(a^\dagger a + \frac{1}{2} \right). \quad (14)$$

Since the a and a^\dagger satisfy the commutation relation, the normalized eigenstates and eigenvalues of Eq. (14) are given by

$$a^\dagger a |n\rangle = n |n\rangle \quad n = 0, 1, 2, \dots \quad (15)$$

$$\lambda = \hbar\Omega \left(n + \frac{1}{2} \right) \quad n = 0, 1, 2, \dots \quad (16)$$

The ground state must satisfy the condition that

$$a u_0 = 0. \quad (17)$$

Solving Eq. (17) for u_0 we obtain

$$u_0 = \left(\frac{\dot{\gamma}}{\pi\hbar A} \right)^{1/4} e^{-\frac{1}{2\hbar\dot{\gamma}} \left[\dot{\gamma} - i \left(\frac{\dot{\rho}}{\rho} - B \right) \right] q^2}. \quad (18)$$

Operating a^\dagger continuously to Eq. (18) we may obtain the n th excited states:

$$\begin{aligned} u_n &= \frac{1}{\sqrt{n!}}(a^\dagger)^n u_0 \\ &= \sqrt{\frac{1}{2^n n!}} \left(\frac{\dot{\gamma}}{\pi \hbar A}\right)^{1/4} H_n \left(\sqrt{\frac{\dot{\gamma}}{\hbar A}} q\right) e^{-\frac{1}{2\hbar A}[\dot{\gamma} - i(\frac{\dot{\rho}}{\rho} - B)]q^2} \end{aligned} \quad (19)$$

where H_n is a n th order Hermite polynomial.

Eq. (19) is an eigenfunction of the invariant operator [Eq. (14)] with an auxiliary equation as classical equation of motion, but is not the solution of the Schrödinger equation;

$$i\hbar \frac{\partial \phi}{\partial t} = \frac{1}{2}[-\hbar^2 A(t) \frac{\partial^2}{\partial q^2} + \frac{\hbar}{i} B(t) \left(q \frac{\partial}{\partial q} + \frac{\partial}{\partial q} q\right) + C(t)q^2] \phi. \quad (20)$$

Comparison of Eq. (19) with (20) offers the exact wave function of the system:

$$\begin{aligned} \phi_n(q, t) &= e^{i\beta_n} u_n(q, t) \\ &= \left(\frac{1}{2^n n!}\right)^{1/2} \left(\frac{\dot{\gamma}}{\pi \hbar A}\right)^{1/4} e^{-i(n+\frac{1}{2})\gamma} H_n \left(\sqrt{\frac{\dot{\gamma}}{\hbar A}} q\right) e^{-\frac{1}{2\hbar A}[\dot{\gamma} - i(\frac{\dot{\rho}}{\rho} - B)]q^2}. \end{aligned} \quad (21)$$

Making use of the Mehler's formula together with Eq. (21), we can easily evaluate the propagator of the system given by

$$\begin{aligned} K(q, t; q', t') &= \left[\frac{\dot{\gamma}^{1/2} \dot{\gamma}'^{1/2}}{2\pi i \hbar \sin(\gamma - \gamma') A^{1/2} A'^{1/2}} \right]^{1/2} \\ &\times \exp \left\{ \frac{i}{2\hbar A} \left[\frac{\dot{\rho}}{\rho} + \dot{\gamma} \cot(\gamma - \gamma') - B \right] q^2 \right. \\ &\left. + \frac{i}{2\hbar A'} \left[-\frac{\dot{\rho}'}{\rho'} + \dot{\gamma}' \cot(\gamma - \gamma') + B' \right] q'^2 + \frac{i}{\hbar} \sqrt{\frac{\dot{\gamma} \dot{\gamma}'}{A A'} \frac{q q'}{\sin(\gamma - \gamma')}} \right\} \end{aligned} \quad (22)$$

where $\rho' = \rho(t')$, $\gamma' = \gamma(t')$, $A' = A(t')$, and $B' = B(t')$. Eq. (22) is the result that we obtained previously⁷⁾.

The uncertainty relation is defined by

$$(\Delta q \Delta p)_{m,n} = [|(\langle m|q^2|n\rangle - \langle m|q|n\rangle^2)(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)|]^{1/2}. \quad (23)$$

With the help of Eq. (21), we can express Eq. (23) as

$$(\Delta q \Delta p)_{n,n} = \hbar \left(n + \frac{1}{2}\right) \sqrt{1 + \frac{1}{\dot{\gamma}^2} \left(\frac{\dot{\rho}}{\rho} - B\right)^2}. \quad (24)$$

4 The Minimum Uncertainty Function

The minimum value for $n = 0$ in Eq. (24) is larger than $\hbar/2$ and thus the coherent state of our system is not a minimum uncertainty state. To obtain minimum uncertainty state, we introduce the new creation and annihilation operators defined by

$$b = \mu a + \nu a^\dagger \quad (25)$$

for a pair of c numbers μ, ν obeying

$$|\mu|^2 - |\nu|^2 = 1. \quad (26)$$

The canonical transformation [Eq. (25)] which keeps the commutator invariant, is a unitary transformation. The properties of the b and b^\dagger are the same as those of a and a^\dagger ⁸⁾.

Performing the same procedures in Sec. 3, we can obtain the wave function for n th excited states:

$$\begin{aligned} \psi_n = & \left(\frac{1}{2^n n!} \right)^{1/2} \left(\frac{\dot{\gamma}}{\pi \hbar A |\mu - \nu|^2} \right)^{1/4} \left(\frac{\mu^* - \nu^*}{|\mu - \nu|} \right)^n H_n \left(\sqrt{\frac{\dot{\gamma}}{\hbar |\mu - \nu|^2 A}} q \right) \\ & \times \exp \left\{ -\frac{q^2}{2 \hbar A |\mu - \nu|^2} \left[\dot{\gamma} - i \left[\left(B - \frac{\dot{\rho}}{\rho} \right) |\mu - \nu|^2 + i(\mu \nu^* - \nu \mu^*) \dot{\gamma} \right] \right] \right\}. \end{aligned} \quad (27)$$

Substituting Eq. (26) into Eq. (24) and evaluating the diagonal element, the uncertainty relation for (n, n) states can be obtained:

$$(\Delta q \Delta p)_{n,n} = \left(n + \frac{1}{2} \right) \hbar \left\{ 1 + \left[\frac{1}{\dot{\gamma}} \left(B - \frac{\dot{\rho}}{\rho} \right) |\mu - \nu|^2 + i(\mu \nu^* - \nu \mu^*) \right]^2 \right\}^{1/2}. \quad (28)$$

From Eq. (27) we can also find the condition of μ and ν for the minimum uncertainty

$$\mu = \frac{k}{\sqrt{k^2 - 1}} \quad (29)$$

$$\nu = \frac{1}{\sqrt{k^2 - 1}} e^{i\theta} \quad (30)$$

where

$$k = \frac{\alpha \pm \sqrt{\alpha^2 - \frac{4}{\dot{\gamma}^2} \left(B - \frac{\dot{\rho}}{\rho} \right)^2}}{2B} \quad (31)$$

$$\theta = \tan^{-1} \left[\frac{\frac{4}{\dot{\gamma}} \left(B - \frac{\dot{\rho}}{\rho} \right) \pm |\alpha| \sqrt{\frac{4}{\dot{\gamma}^2} \left(B - \frac{\dot{\rho}}{\rho} \right)^2 - \alpha^2 + 4}}{\alpha^2 - 4} \right] \quad (32)$$

and

$$\frac{4}{\dot{\gamma}^2} \left(B - \frac{\dot{\rho}}{\rho} \right)^2 \leq \alpha^2 \leq \frac{4}{\dot{\gamma}^2} \left(B - \frac{\dot{\rho}}{\rho} \right)^2 + 4. \quad (33)$$

Here, k is real and positive, and α must have the same sign of $\frac{1}{\dot{\gamma}} \left(B - \frac{\dot{\rho}}{\rho} \right)$. We can confirm that the minimum uncertainty is a function of one continuous parameter in the finite region.

5 Summary

Introducing the quadratic Hamiltonian system given in Eq. (1), we have derived the classical invariant quantity with an auxiliary equation as the classical equation of motion. With the use of this invariant, we can distinguish whether or not the system is bound. We transformed the invariant into an operator in the replacement of the creation and annihilation operator [Eq. (14)] and then evaluated the corresponding eigenfunction and eigenvalues. However, this eigenfunction is not the Schrödinger solution of the system. Though we obtain the exact wave function of the system [Eq. (21)] and propagator [Eq. (22)] the minimum uncertainty constructed by this wave function is larger than $\hbar/2$ and thus the coherent states of the system is not minimum uncertainty state. To obtain the minimum uncertainty we introduce the canonical transformation, which keeps the commutator invariant. Through this unitary transformation we obtained the eigenfunction and minimum uncertainty state of the system.

Acknowledgments

This work was supported by the Center for Thermal and Statistical Physics, KOSEF under Contact No. 93-08-00-05 and by the BSRI Program, Ministry of Education, Republic of Korea, and by the U.S. National Science Foundation under Grant No. CHE-9196214.

References

- (1) J. G. Hartley and J. R. Ray, *Phys. Rev. D* **25**, 382(1982); D. C. Khandekar and S. S. Lawande, *J. Math. Phys.* **20**, 1870(1979); P. Camiz, A. Gerardi, C. Marehioso, E. Pre-sutti and E. Scaeciatelli, *J. Math. Phys.* **12**, 2040(1971); P. G. L. Leach, *J. Math. Phys.* **21**,300(1980).
- (2) K. H. Yeon and C. I. Um, *J. Korean Phys. Soc.* **23**, 82(1990).
- (3) K. H. Yeon, C. I. Um and T. F. George, *Phys. Rev. A* **36**,5287(1987); V. V. Dodonov, T. F. George, O. V. Man'ko, C. I. Um and K. H. Yeon, *J. Soviet Laser Research* **12**(5), 385(1991); L. F. Landovitz, A. M. Lavine. E. Ozizmis and W. M. Schreiber, *J. Chem. Phys.* **78**, 291(1983); **78**, 6133(1983).
- (4) V. V. Dodonov and V. I. Man'ko, *Phys. Rev. A* **20**, 550(1979); C. I. Um, K. H. Yeon and W. H. Kahng, *J. Phys. A: Math. Gen.* **20**, 611(1987).
- (5) C. F. Lo, *Phys. Rev. A* **47**, 115(1993).
- (6) H. R. Lewis, Jr. and W. B. Riesenfeld, *J. Math. Phys.* **10**, 1458(1969).
- (7) K. H. Yeon. K. K. Lee, C. I. Um, T. F. George and L. N. Pandey, *Phys. Rev. A* in print.
- (8) H. P. Yuen, *Phys. Rev. A* **13**, 2226(1976); J. von Neumann, *Math. Ann.* **104**, 570(1931)