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QUANTUM NOISE LIMITS TO MATTER-WAVE INTERFEROMETRY

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Abstract

We derive the quantum limits for an atomic interferometer in which the atoms obey either Bose-Einstein or Fermi-Dirac statistics. It is found that the limiting quantum noise is due to the uncertainty associated with the particle sorting between the two branches of the interferometer. As an example, the quantum-limited sensitivity of a matter-wave gyroscope is calculated and compared with that of laser gyroscopes.

1. Introduction

Matter-wave interferometry dates from the inception of quantum mechanics, i.e., the early electron diffraction experiments [1]. More recent neutron interferometry experiments have yielded insight into many fundamental aspects of quantum mechanics [2]. Presently, atom interferometry has been demonstrated and holds promise as a new field of optics — matter-wave optics [3]. This field is particularly interesting since the potential sensitivity of matter-wave interferometers [4] far exceeds that of their light-wave or "photon" antecedents [5].

However, as was emphasized at the recent Solvay conference on quantum optics, there is at present no paradigm available for calculating or estimating the quantum noise limits to matter-wave interferometers, and therefore we have no basis for estimating the potential sensitivity of devices based on matter-wave interferometry (e.g., gyroscopes) [6].

In order to motivate the analysis and derive the quantum limits, we proceed as follows: First, we "set the stage" by considering a simple gyroscope and deriving the rotation-induced signal in matter-wave optics. Next, we proceed to develop the theory for atomic interferometers, cast in an operator formalism that is well suited to a quantum noise analysis, and then we obtain the quantum noise limits for matter-wave interferometry. Finally, we compare current laser gyroscope sensitivity to that of near-term, matter-wave devices.

We begin by considering an idealized atom interferometer used as a rotation detector or gyroscope, as shown in Fig. 1. From this diagram it is easy to see that the atomic path difference between the upper branch α and the lower branch β is given by $\delta\ell = 2r\Omega t$, where Ω is the angular velocity of the interferometer, r is the radius of the circle,

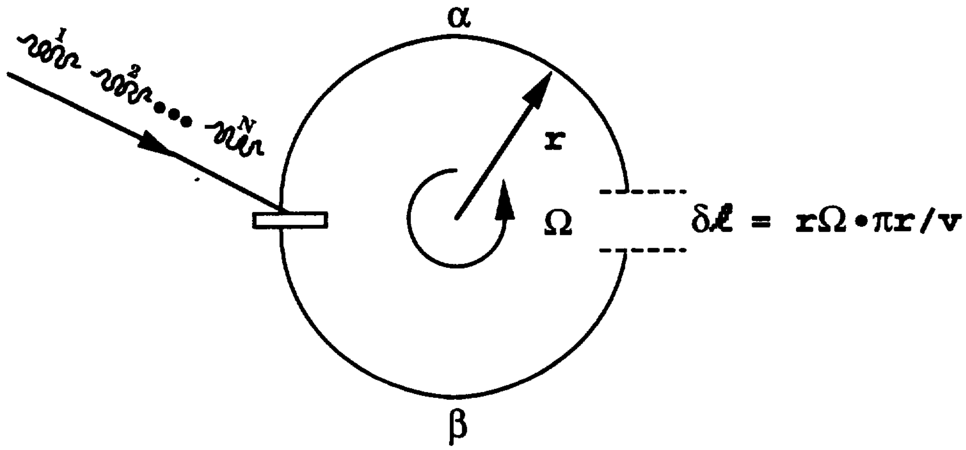

FIG. 1. A schematic illustration of an interferometer with semicircular arms to be used as a rotation sensor or gyroscope. The loop rotates with an angular frequency Ω about an axis through its center and normal to the loop plane. The path difference between counter- and copropagating beams can be easily seen to be $\delta\ell = r\Omega \bullet \frac{\pi r}{v}$, where v is the atomic velocity. From these considerations the phase shift of Eq. (21) follows immediately. We may then use this result to estimate the minimum detectable rotation rate $\Omega^{\min}$, Eq. (23).

v the particle velocity, and $t = \pi r / v$ is the particle transit time through the interferometer. This readily translates into a Sagnac phase difference of $\delta\varphi_{\alpha\beta} = k\left(\ell_\alpha - \ell_\beta\right) = 2\pi r^2\Omega/\lambda\!\!\!^{-}\, v = 2A\Omega/\lambda\!\!\!^{-}\, v$, where $\lambda\!\!\!^{-} \equiv \hbar / mv$ is the atomic de Broglie wavelength [7] and A the area enclosed by the arms. The phase signal is then given by $\varphi^{\text{signal}} = 2A\,m\Omega/\hbar$; independent of the interferometer shape as long as A is the total area enclosed. This expression holds for both atom and light interferometers, if, in the photon case, we define an effective photon mass m_γ implicitly by $m_\gamma c^2 = \hbar\,\omega$. Now, since the "mass" of a photon is governed by optical energies of a few electron volts — and atomic masses are of order 10^3 MeV — we see that matter-wave gyroscopes potentially have a signal that is enhanced by many orders of magnitude, compared to light (laser) gyroscopes. Thus motivated, we next consider a detailed analysis of phase sensitivity in matter-wave interferometry.

2. A Simple Model

In accordance with current experiments [3], let us consider the model illustrated in Fig. 2. There, we see a stream of N atoms passing one-at-a-time through a beam splitter into a simple interferometer with upper and lower branches labelled α and β, respectively. Upon recombining the two beams, we inspect the resultant interference pattern for phase shifts induced, say, by a gravitational potential between the two branches or a net rotation of the system. As in the optical dual [5], one might expect that the overall sensitivity of the device will be limited by the quantum limits imposed by particle number fluctuations ΔN on the phase noise $\Delta\varphi$ in the interferometer. It is

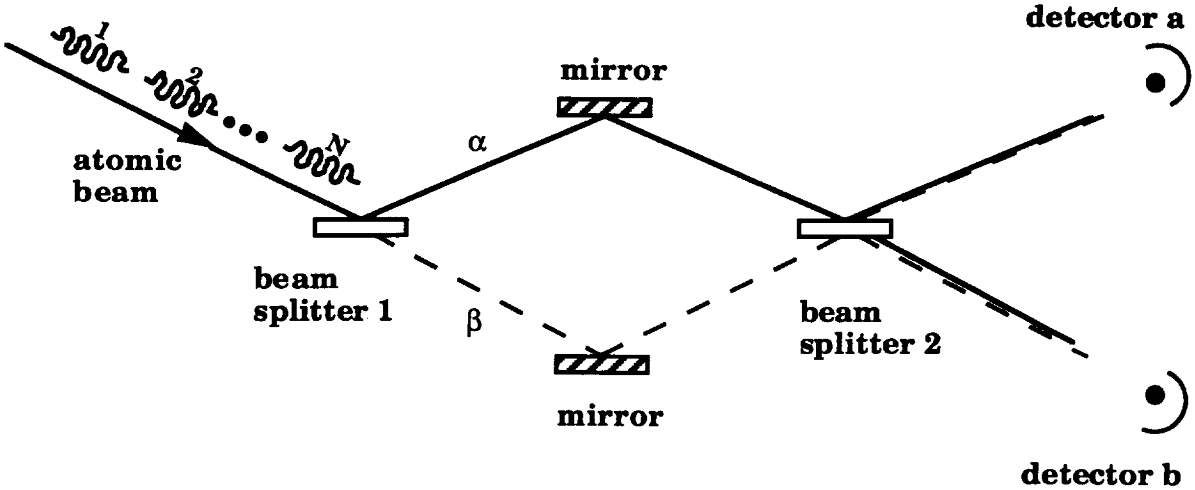

FIG. 2. We illustrate a scheme whereby a stream of N atoms is sent through a simple interferometer during a measurement time t_m. The atoms are split at beam splitter 1, follow paths α or β, are reflected off the mirrors, and are then recombined at beam splitter 2. The recombined atoms are detected at upper detector a or lower detector b where interference fringes are recorded.

often stated that ΔN is to be associated with the fluctuations in the arrival time of atoms in the input beam, i.e., $\Delta N \sim \sqrt{\bar{n}}$ where $\bar{n}$ is the mean number of particles. However, we shall show that the particle number noise arises not from fluctuations in the input beam intensity but rather from beam splitter uncertainties pertaining to the lack of knowledge of which path, α or β , the atom has taken through the interferometer.

3. The Quantum Signal

Let us continue developing our simple model depicted in Fig. 2. We assume that, upon reflection from a beam splitter surface, the particles undergo an unimportant phase shift that we take to be $\pi/2$, but that in reality depends upon the structure of the beam splitter. Upon passage *through* a beam splitter, however, the atom undergoes a phase shift of φ_i, $i = 1, 2$, for the first and second beam splitter, respectively. The cumulative effect in the interferometer of these various processes on the atomic wave function ψ is depicted in Fig. 3, and leads to a wave function ψ_a corresponding to the upper detector and ψ_b for the lower detector, namely

$$\psi_a = \frac{\psi}{2}\, e^{i\theta_a}\left[1 - e^{-ik\left(\ell_\alpha - \ell_\beta\right)}\right] \quad , \quad \psi_b = \frac{\psi}{2}\, e^{i\theta_b}\left[1 + e^{-ik\left(\ell_\alpha - \ell_\beta\right)}\right] \tag{1}$$

where $\theta_a \equiv \pi/2 + k\ell_\alpha + \varphi_2$, and $\theta_b \equiv k\ell_\alpha + \varphi_1 + \varphi_2$, and where, without loss of generality, we let $\varphi_1 = \varphi_2 = \pi$. Here, k is the atomic wave number and ℓ_α and ℓ_β are the

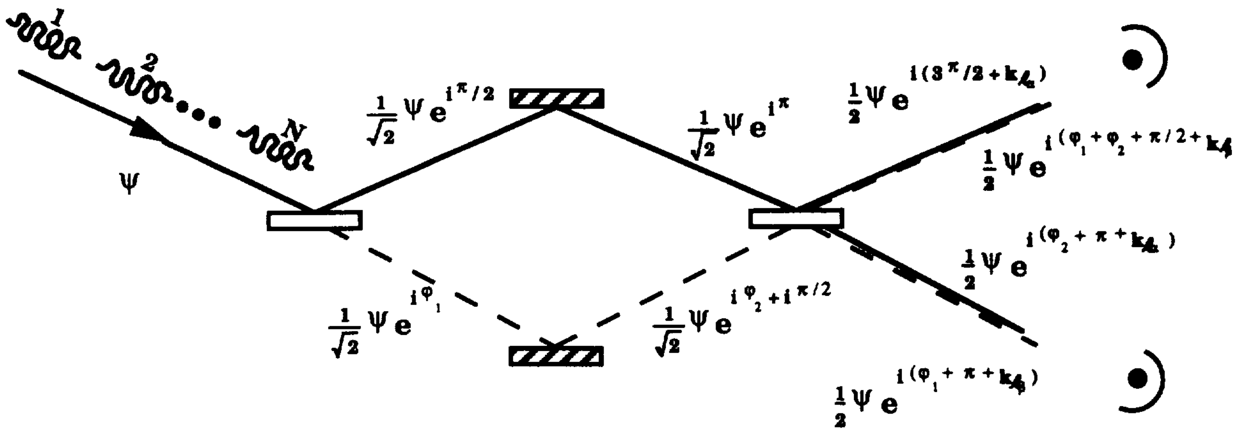

FIG. 3. Chasing phases through the interferometer accounts for accumulated phase shifts in the upper or lower detectors. The phase shift upon reflection is arbitrary, but we choose it here to be $\pi/2$ for simplicity. Upon transmission, a phase shift of φ_1 or φ_2 is assumed for beam splitter one or two, respectively, and without loss of generality we take $\varphi_1 = \varphi_2 = \pi$.

path lengths through the upper and lower branches, respectively. We imagine now that the beam is recombined by the second beam splitter and then the detectors a and b shown in Fig. 2 count the number of atoms as they arrive in the recombined upper beam or lower beam, respectively. If we label N atoms with the index $i = 1, \ldots, N$, as those sent through the interferometer during a measurement time t_m, then the appropriate state vector $|\varphi\rangle_i$ for the i^{th} atom in the interferometer, after recombination, is given by

$$|\varphi\rangle_i = \frac{e^{i\theta_a}}{2}\left(1 - e^{-i\varphi_{\alpha\beta}}\right)|1_a, 0_b\rangle_i + \frac{e^{i\theta_b}}{2}\left(1 + e^{-i\varphi_{\alpha\beta}}\right)|0_a, 1_b\rangle_i \ , \tag{2}$$

where here $\varphi_{\alpha\beta} \equiv k(\ell_\alpha - \ell_\beta)$. We see that this state is an appropriate superposition of the number states $|1_a, 0_b\rangle$ and $|0_a, 1_b\rangle$ corresponding to an atom incident on the upper or lower detectors, respectively. The state vector $|\Phi\rangle_N$ for the N-atom state is then constructed via a direct product of the individual atomic states, namely

$$|\Phi\rangle_N \equiv \prod_{i=1}^{N} |\varphi\rangle_i \ . \tag{3}$$

Let $\hat{c}^\dagger_{\sigma,i}$ and $\hat{c}_{\sigma,i}$, where $\sigma = a,b$, be the creation and annihilation operators, respectively, for the number states $|n_a, n_b\rangle_i$, where, corresponding to number operators $\hat{n}_{\sigma i} \equiv \hat{c}^\dagger_{\sigma,i}\,\hat{c}_{\sigma,i}$, the eigenvalues n_a and n_b are 0 or 1. Then the number operator $\hat{N}_\sigma$ for the number of upper or lower atoms is determined by

$$\hat{N}_\sigma = \sum_{i=1}^{N} \hat{n}_{\sigma,i} \qquad (\sigma = a, b) \quad , \tag{4}$$

and the operators $\hat{c}$ obey the commutation relations

$$\left[\hat{c}_{\sigma,i}\, \hat{c}^\dagger_{\sigma,j} \pm \hat{c}^\dagger_{\sigma,j}\, \hat{c}_{\sigma,i}\right] = \delta_{ij} \ , \tag{5}$$

where the plus or minus sign indicates Bose or Fermi statistics, respectively. The statistical nature of the atoms will be important in circumstances where the density of particles in the interferometer is so large that there is more than one atom at a time within a single coherence length, or if the atoms are injected in a correlated manner into the input ports [9]. The expectation values $\langle \hat{N}_\sigma \rangle_N$ of these number operators, Eq. (4), are given by

$$_N\langle\Phi|\, \hat{N}_a\, |\Phi\rangle_N = \sum_{i=1}^{N} \left|\frac{1 - e^{-i\varphi_{\alpha\beta}}}{2}\right|^2 {}_i\langle 1_a, 0_b|\, \hat{n}_{a,i}\, |1_a, 0_b\rangle_i \ , \tag{6a}$$

$$_N\langle\Phi|\, \hat{N}_b\, |\Phi\rangle_N = \sum_{i=1}^{N} \left|\frac{1 + e^{-i\varphi_{\alpha\beta}}}{2}\right|^2 {}_i\langle 0_a, 1_b|\, \hat{n}_{b,i}\, |0_a, 1_b\rangle_i \ . \tag{6b}$$

This yields the expression for the mean number of atoms in the α and β branches as

$$\langle \hat{N}_a \rangle_N = N \sin^2 \varphi_{\alpha\beta}/2 \quad , \qquad \langle \hat{N}_b \rangle_N = N \cos^2 \varphi_{\alpha\beta}/2 \ . \tag{7}$$

These expectations constitute the signal; we proceed to calculate the noise.

4. A Calculation of Poisson Noise

As noted earlier, it is frequently stated that number fluctuations ΔN in the interferometer should be just noise of the form $\sqrt{\bar{n}}$ due to fluctuations in the input beam. Let us briefly investigate this hypothesis. A reasonable assumption is that the distribution of the N atoms in the input beam is Poissonian with a distribution function P_n , given by

$$P_n \equiv \frac{\bar{n}^n}{n!} e^{-\bar{n}} \ , \tag{8}$$

where $\bar{n}$ is the mean number of particles in the beam. The signal or expectation value of the operators $\hat{N}_\sigma$ is then given by

$$\langle \hat{N}_\sigma \rangle = \sum_{n=1}^{N} {}_n\langle \Phi | \hat{N}_\sigma | \Phi \rangle_n \, P_n = \bar{n} \begin{Bmatrix} \sin^2 \varphi_{\alpha\beta}/2 \\ \cos^2 \varphi_{\alpha\beta}/2 \end{Bmatrix} \tag{9}$$

where $\bar{n} = \sum_n n \, P_n$, and the upper and lower terms in braces are associated with detectors $\sigma = a$ and $\sigma = b$, respectively — a convention we shall use throughout. Hence, the fluctuations accompanying this signal are determined by the root variance $\langle \Delta\hat{N}_\sigma \rangle$, which is found to be

$$\langle \Delta N_\sigma \rangle = \sqrt{\bar{n}} \begin{Bmatrix} \sin^2 \varphi_{\alpha\beta}/2 \\ \cos^2 \varphi_{\alpha\beta}/2 \end{Bmatrix} \tag{10}$$

Now, to get a determination of minimum detectable phase shift, one usually equates the signal, Eq. (9), to the noise, given by Eq. (10). Regardless of the choice of $\varphi_{\alpha\beta}$, we see from Eqs. (9) and (10) that upon equating signal to error, the phase dependence cancels out and we have no determination of the minimum detectable phase. The point is that N is not a random number, since we have the constraint $N_a + N_b = N = constant$, and fluctuations in the incoming atomic beam do not determine sensitivity. In other words, precise knowledge of the value of N obviates the need for a Poisson analysis, since N is clearly *not* a random variable. What, then, is the limiting noise mechanism in the interferometer?

5. The Quantum Noise

We compute the quantum noise fluctuations using the second quantized formalism developed earlier. Recalling the definitions for the number operator $\hat{N}$, Eq. (4), and the state vector $|\Phi\rangle_N$, Eq. (3), and using the commutation relations, Eq (5), we may write,

$$\langle \Delta\hat{N}_\sigma \rangle^2 = {}_N\langle \Phi | \hat{N}_\sigma^2 | \Phi \rangle_N - \left[{}_N\langle \Phi | \hat{N}_\sigma | \Phi \rangle_N \right]^2$$

$$= {}_N\langle \Phi | \sum_{i=1}^{N} \hat{n}_{\sigma,i} \sum_{j=1}^{N} \hat{n}_{\sigma,j} | \Phi \rangle_N - \left[{}_N\langle \Phi | \sum_{i=1}^{N} \hat{n}_{\sigma,i} | \Phi \rangle_N \right]^2$$

$$= \sum_{i=1}^{N} {}_i\langle\varphi|\, \hat{n}_{\sigma,i}\,|\varphi\rangle_i \sum_{\substack{j=1\\ j\neq i}}^{N} {}_j\langle\varphi|\, \hat{n}_{\sigma,j}\,|\varphi\rangle_j$$

$$+ \sum_{i=1}^{N} {}_i\langle\varphi|\, \hat{c}^{\dagger}_{\sigma,i} \left[1 \pm \hat{n}_{\sigma,i}\right] \hat{c}_{\sigma,i}\,|\varphi\rangle_i - \left[\sum_{i=1}^{N} {}_i\langle\varphi|\, \hat{n}_{\sigma,i}\,|\varphi\rangle_i\right]^2$$

$$= \frac{N}{4}\sin^2\varphi_{\alpha\beta} \pm \sum_{i=1}^{N} {}_i\langle\varphi|\, \hat{c}^{\dagger}_{\sigma,i}\, \hat{c}^{\dagger}_{\sigma,i}\, \hat{c}_{\sigma,i}\, \hat{c}_{\sigma,i}\,|\varphi\rangle_i \tag{11}$$

where, as before, the upper and lower terms in braces correspond to $\sigma = a$ or b, respectively, and the $\pm$ sign refers to the statistics of the particles: a plus sign for bosons and a minus sign for fermions. We note that the last, statistics-dependent term of Eq. (11) is the sum of non-negative matrix elements and so itself is non-negative or non-positive, according to the plus sign or negative sign, respectively. A quantitative analysis of the contribution of this statistics-dependent term requires a specific model of the coherences between atoms in a dense beam. However, one can qualitatively state that for sufficiently high densities the use of fermionic atoms will tend to *lower* the quantum noise limit. This is because the last term will be negative. Bosons will have the opposite effect. Detailed analysis of the statistics-dependent contribution is beyond the scope of this letter, and will be left to a later work. Hence, since, in current experiments, the beam intensity is so low that there is only one atom at a time within a single coherence length. In this case, the statistics-dependent second term in the last line of Eq. (11) is zero, and we are left with the result

$$\langle \Delta N_\alpha \rangle = \frac{\sqrt{N}}{2}\sin\varphi_{\alpha\beta} \tag{12}$$

We notice that this result depends on the total number of atoms N and *not* the mean number $\bar{n}$ as in the Poisson-distribution argument given before. Now, the signal in either branch N_σ is given by Eq. (7).

The quantum fluctuations in phase $\Delta\varphi_{\alpha\beta}$ in the measured phase difference $\varphi_{\alpha\beta}$ may be determined by [8]

$$\left|\Delta\varphi_{\alpha\beta}\right| \equiv \frac{\langle \Delta N_\sigma \rangle}{\left|\partial\langle N_\sigma\rangle/\partial\varphi_{\alpha\beta}\right|}$$

$$= \frac{1}{\sqrt{N}}\,, \tag{13}$$

a result that is *independent* of $\varphi_{\alpha\beta}$. This independence might appear surprising at first, but it is a direct result of the fact that the *quantum* number state noise $\langle \Delta N_\sigma \rangle$ is proportional to the slope of the signal $\langle N_\sigma \rangle$ for the upper/lower number states considered here. (See, in particular, reference 8.) Again, we stress that N is not the expectation value but rather the total number of atoms detected in the measurement time t_m. This is *not* then the expression one would expect from application of the uncertainty principle, for in that case N would have to be replaced by $\langle N \rangle$. We reemphasize that it has not been clear what form of the uncertainty principle one should even use in an atom interferometer [6]. For light, the so called number-phase uncertainty principle, $\Delta\varphi\Delta N \gtrsim 1$, yields for a coherent state $\Delta\varphi \cong 1/\langle N \rangle$ – where the expectation $\langle N \rangle$ and not the total number N is used. For atoms it is not obvious at all what the relationship should be, and we have shown that the result is unexpected in that Eq. (13) depends on the total number N , that is precisely known for the atom interferometer, and where $\langle N \rangle$ has no meaning. In contradistinction, in a laser interferometer, it is impossible to know the total number of photons and only the mean can be specified. Hence, the atom result, Eq. (13), is quantitatively, qualitatively, and philosophically different from the optical result. Hence, Eq. (13) is indeed a novel result. We note that B. Yurke obtained a similar result for Fermions, using spin algebra techniques [9].

6. Comparing Laser and Matter Gyros

We conclude by applying this result to the gyroscope problem. Let us note that the atom number N is given by $j\,t_m$, where j is the atomic flux (in atoms per second) hitting the detector. We have from Eq. (14) the minimum detectable phase shift, $\varphi_{\min} = 2/\sqrt{j t_m}$, and equating this to the signal derived earlier, $\varphi^{\text{signal}} = 2Am\Omega/\hbar$, we find the minimum detectable rotation rate $\Omega^{\min}$ is given by

$$\Omega^{\min} \cong \frac{\hbar}{Am}\frac{1}{\sqrt{j t_m}} \qquad \text{(matter).} \tag{14}$$

This should be compared to the same result obtained from using an optical interferometer in which the flux j is given by the power P divided by the photon energy $\hbar\omega$ [5,7], in other words

$$\Omega^{\min} \cong \frac{\hbar}{Am_\gamma}\frac{1}{\sqrt{\frac{P}{\hbar\omega}t_m}} \qquad \text{(light)}, \tag{15}$$

where m_γ is the effective photon mass, defined by $m_\gamma \equiv \hbar\omega/c^2$. In Table 1 we compare and contrast properties of the matter-wave and laser light interferometers in order to gauge their effectiveness in measuring $\Omega^{\min}$. As mentioned before, we note that the typical photon effective mass gives an increase in sensitivity of 10^{10}. This mass factor, however, is offset by the low particle flux available for atoms. This fact increases the laser gyroscope sensitivity over that of matter-wave devices by a factor of around 10^2. In addition, the atoms make about one "round trip" through an interferometer, whereas in a ring laser gyroscope the photons make many ($\approx 10^4$) circuits around the ring and yield an additional sensitivity factor of 10^4 in favor of the laser system. This still leaves the matter-wave device 10^4 times more sensitive.

In summary then, we conclude that the phase uncertainty arising in an atomic interferometer arises from atomic number fluctuations associated with the sorting of the particles between the two arms of the interferometer. Applying our results to an interferometer used as a gyroscope, we find that a matter-wave gyroscope can be expected to be more sensitive to rotation by some *four orders of magnitude* than present laser devices.

	Matter	Laser	Matter Over Light Sensitivity Factor
Mass Factor	$\sim 10^4$ MeV	~ 1 eV	$\sim 10^{10}$
Flux	$\rho v A \sim 10^{10} \bullet 10^4 \bullet 10^{-2}$ $= 10^{12} \frac{\text{particles}}{\text{sec}}$	$\frac{P}{\hbar \nu} \sim \frac{10^{-3}}{10^{-19}}$ $= 10^{16} \frac{\text{photons}}{\text{sec}}$	$\sim 10^{-2}$
Round Trips	~ 1	$\sim 10^4$	$\sim 10^{-4}$

TABLE I. Compared and contrasted are different properties of matter-wave and optical gyroscopes in terms of their sensitivity to phase differences — or equivalently — rotation rates. We see that the high mass of atoms initially contributes an increase of sensitivity of 10^{10}, but that the low atomic beam intensity, compared to photon beams, removes some of this advantage, as does the reduced number of round trips possible in an atom interferometer. Nevertheless — a typical factor of a 10^4 increase in rotation sensitivity can still be expected using atoms rather than photons.

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