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Nonlinear instability of a uni-directional transversely sheared mean flow

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It is well known that the presence of a weak cross flow in an otherwise two-dimensional shear flow results in a spanwise variation in the mean streamwise velocity profile that can lead to an amplification of certain three-dimensional disturbances through a kind of resonant-interaction mechanism (Goldstein & Wundrow 1994). The spatial evolution of an initially linear, finite-growth-rate, instability wave in such a spanwise-varying shear flow is considered. The base flow, which is governed by the three-dimensional parabolized Navier–Stokes equations, is initiated by imposing a spanwise-periodic cross-flow velocity on an otherwise two-dimensional shear flow at some fixed streamwise location. The resulting mean-flow distortion initially grows with increasing streamwise distance, reaches a maximum and eventually decays through the action of viscosity. This decay, which coincides with the viscous spread of the shear layer, means that the local growth rate of the instability wave will eventually decrease as the wave propagates downstream. Nonlinear effects can then become important within a thin spanwise-modulated critical layer once the local instability-wave amplitude and growth rate become sufficiently large and small, respectively. The amplitude equation that describes this stage of evolution is shown to be a generalization of the one obtained by Goldstein & Choi (1989) who considered the related problem of the interaction of two oblique modes in a two-dimensional shear layer.

1. Formulation

To fix ideas, we consider an incompressible shear flow formed at the interface between two parallel streams of differing velocity or alternatively between a single parallel stream and a flat plate. The Cartesian coordinate system (x, y, z) is attached to the interface with x in the direction of the external flow, y normal to the interface, and z in the spanwise direction. All lengths are non-dimensionalized by δ_* where δ_* characterizes the local shear-layer thickness at $x = 0$. The time t, velocity $\boldsymbol{u} = \boldsymbol{i}u + \boldsymbol{j}v + \boldsymbol{k}w$, and pressure variation p from the external value P_* are non-dimensionalized by δ_*/U_*, U_* and $\rho_* U_*^2$, respectively, where U_* characterizes the velocity of the external flow and ρ_* is the density. With this non-dimensionalization, the Navier–Stokes equations become

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1}$$

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = R^{-1} \nabla^2 \boldsymbol{u}, \tag{1.2}$$

where $\nabla \equiv \boldsymbol{i}\partial/\partial x + \boldsymbol{j}\partial/\partial y + \boldsymbol{k}\partial/\partial z$ is the gradient operator,

$$R \equiv \delta_* U_*/\nu_* \gg 1 \tag{1.3}$$

is the local Reynolds number, ν_* is the kinematic viscosity and an independent variable used as a subscript denotes differentiation with respect to that variable.

The solutions to (1.1) and (1.2) that are of interest here can be represented as the sum of a steady base flow plus a time-dependent perturbation,

$$\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) + \epsilon \acute{\boldsymbol{u}}(\boldsymbol{x}, t), \tag{1.4}$$

$$p = P(\boldsymbol{x}) + \epsilon \acute{p}(\boldsymbol{x}, t), \tag{1.5}$$

where ϵ characterizes the local amplitude of the perturbation at $x = 0$. Substituting (1.4) and (1.5) into (1.1) and (1.2) gives

$$\nabla \cdot \boldsymbol{U} = 0, \tag{1.6}$$

$$\boldsymbol{U} \cdot \nabla \boldsymbol{U} + \nabla P = R^{-1} \nabla^2 \boldsymbol{U}, \tag{1.7}$$

for the base flow and

$$\nabla \cdot \acute{\boldsymbol{u}} = 0, \tag{1.8}$$

$$\acute{\boldsymbol{u}}_t + \boldsymbol{U} \cdot \nabla \acute{\boldsymbol{u}} + \acute{\boldsymbol{u}} \cdot \nabla (\boldsymbol{U} + \epsilon \acute{\boldsymbol{u}}) + \nabla \acute{p} = R^{-1} \nabla^2 \acute{\boldsymbol{u}}, \tag{1.9}$$

for the perturbation.

The steady spanwise-periodic base flow $\{\boldsymbol{U}, P\}$ evolves over the long streamwise scale,

$$x_2 \equiv x/R, \tag{1.10}$$

and has an $O(\delta_*)$ wavelength in the spanwise direction. This implies that the base-flow solution expands like

$$\boldsymbol{U} = \boldsymbol{i} U_0(x_2, y, z) + R^{-1} \boldsymbol{V}_0(x_2, y, z) + \cdots, \tag{1.11}$$

$$P = R^{-2} P_0(x_2, y, z) + \cdots, \tag{1.12}$$

where $\boldsymbol{V}$ denotes the base-flow velocity in the transverse (or y–z) plane. Substituting (1.11) and (1.12) into (1.6) and (1.7) shows that the leading-order base-flow solution is determined by the parabolized Navier–Stokes equations (Rudman & Rubin 1968),

$$U_{0x_2} + \nabla_T \cdot \boldsymbol{V}_0 = 0, \tag{1.13}$$

$$U_0 (\boldsymbol{i} U_0 + \boldsymbol{V}_0)_{x_2} + \boldsymbol{V}_0 \cdot \nabla_T (\boldsymbol{i} U_0 + \boldsymbol{V}_0) + \nabla_T P_0 = \nabla_T^2 (\boldsymbol{i} U_0 + \boldsymbol{V}_0), \tag{1.14}$$

where $\nabla_T \equiv \boldsymbol{j} \partial/\partial y + \boldsymbol{k} \partial/\partial z$ is the gradient operator in the transverse plane.

It is assumed that the initial amplitude of the perturbation is small enough so that $\epsilon\acute{u} \ll U_0$ over the streamwise region of interest. Substituting (1.11) into (1.8) and (1.9) then yields

$$\nabla\cdot\acute{u} = 0, \tag{1.15}$$

$$\mathrm{D}\acute{u} + \imath(\nabla_T U_0\cdot\acute{u}) + \nabla\acute{p} = O(R^{-1}) \tag{1.16}$$

where $\mathrm{D} \equiv \partial/\partial t + U_0\partial/\partial x$ is the leading-order convective derivative relative to the base flow. These equations are just the familiar equations for the linear perturbations about a uni-directional transversely sheared base flow (Goldstein 1976; Henningson 1987). It is well known that the velocity fluctuations can be eliminated between (1.15) and (1.16) (see Goldstein 1976, pp. 6–10 for a detailed derivation) to obtain the following equation for the pressure fluctuation

$$\mathrm{D}\nabla^2\acute{p} - 2\nabla_T U_0\cdot\nabla_T\acute{p}_x = O(R^{-1}). \tag{1.17}$$

Attention will be restricted to perturbations that are spatially growing and periodic in time with, at least initially, a single angular frequency, say F_*. The relevant solutions to (1.15)–(1.17) then form a spanwise periodic instability wave that propagates in the streamwise direction. The local amplitude of the instability wave increases as the wave propagates downstream, but its local growth rate will ultimately decrease owing to the combined effects of the viscous spread of the basic shear layer and the viscous decay of the mean streamwise vorticity. Nonlinear effects can then become important first within a thin critical layer located at the transverse position where the phase speed of the instability wave equals the base-flow velocity U_0 (once the instability-wave amplitude and growth rate become sufficiently large and small respectively). In this stage of development, the unsteady flow outside the critical layer remains essentially linear but the instability-wave amplitude is completely determined by the nonlinear motion inside the critical layer.

With this in mind, the origin of the x axis is chosen so that the deviation,

$$\sigma S_1 \equiv S - S_0 < 0, \tag{1.18}$$

of the local Strouhal number (or non-dimensional angular frequency) $S \equiv \delta_* F_* / U_*$ from its neutral (or zero-growth) value S_0 is $O(\sigma)$ where $\sigma \ll 1$. The precise relationship between ϵ and σ will be specified below when the flow in the critical layer is analyzed. The relevant solutions to (1.16) and (1.17) are then of the form

$$\acute{u} = \mathrm{Re}\left(A\hat{u}\mathrm{e}^{\mathrm{i}X}\right) + \ldots, \tag{1.19}$$

$$\acute{p} = \mathrm{Re}\left(A\hat{p}\mathrm{e}^{\mathrm{i}X}\right) + \ldots, \tag{1.20}$$

where $A(x_1)$ is an amplitude function that accounts for the slow growth of the instability wave,

$$x_1 \equiv \sigma x \tag{1.21}$$

is the streamwise scale over which the wave growth occurs,

$$X \equiv \alpha_0 x - St, \tag{1.22}$$

is a normalized streamwise coordinate in a reference moving with the wave, and α_0 is the neutral wavenumber. The ellipses in (1.19) and (1.20) indicate harmonics of the fundamental instability wave that are generated by the critical-layer nonlinearity. Since these harmonics do not interact outside the critical layer (to the order of accuracy considered here), their outer solutions can be determined *a posteriori.*

Substituting (1.20) into (1.17) shows that, outside the critical layer, the function $\hat{p}$ of x_1, y and z is determined to the required order of accuracy by

$$\nabla_T \cdot \left[\frac{\nabla_T \hat{p}}{(U_0 - c)^2}\right] - \frac{\alpha^2 \hat{p}}{(U_0 - c)^2} = 0 \tag{1.23}$$

where

$$\alpha \equiv \alpha_0 - \sigma \mathrm{i} A'/A, \tag{1.24}$$

and

$$c \equiv S/\alpha, \tag{1.25}$$

are the generalized wavenumber and phase speed, respectively, and a prime denotes differentiation with respect to the argument. It follows from (1.15), (1.16) and (1.19) that the velocity fluctuations are determined in terms of $\hat{p}$ by

$$\boldsymbol{i} \cdot (\mathrm{i}\alpha \hat{\boldsymbol{u}}) + \nabla_T \cdot \hat{\boldsymbol{u}} = 0, \tag{1.26}$$

and

$$\mathrm{i}\alpha (U_0 - c)\hat{\boldsymbol{u}} + \boldsymbol{i}(\nabla_T U_0 \cdot \hat{\boldsymbol{u}} + \mathrm{i}\alpha \hat{p}) + \nabla_T \hat{p} = 0. \tag{1.27}$$

The solution to (1.23) that satisfies

$$\left.\begin{array}{l} \hat{p}_y = 0 \text{ at } \quad y = 0 \quad \text{; boundary layer} \\ \hat{p} \to 0 \text{ as } y \to -\infty \text{ ; free-shear layer} \end{array}\right\}, \tag{1.28}$$

and

$$\hat{p} \to 0 \quad \text{as} \quad y \to \infty, \tag{1.29}$$

is analyzed in the following section.

2. Unsteady flow outside the critical layer

Outside the critical layer, the shape functions $\{\hat{u}, \hat{p}\}$ expand like

$$\hat{u} = \hat{u}_0(y, z) + \sigma \hat{u}_1(x_1, y, z) + \cdots, \tag{2.1}$$

$$\hat{p} = \hat{p}_0(y, z) + \sigma \hat{p}_1(x_1, y, z) + \cdots, \tag{2.2}$$

as $\sigma \to 0$, where the Reynolds number R has been assumed to be large enough so that the coefficients $\{\hat{u}_m, \hat{p}_m\}$ depend only parametrically on the slow streamwise variable x_2, i.e. x_2 plays the role of a constant. Substituting (2.2), (1.24) and (1.25) into (1.23) and equating like powers of σ leads to

$$\nabla_T \cdot \left[\frac{\nabla_T \hat{p}_0}{(U_0 - c_0)^2}\right] - \frac{\alpha_0^2 \hat{p}_0}{(U_0 - c_0)^2} = 0, \tag{2.3}$$

and

$$\nabla_T \cdot \left[\frac{\nabla_T \hat{p}_1}{(U_0 - c_0)^2}\right] - \frac{\alpha_0^2 \hat{p}_1}{(U_0 - c_0)^2} = 2\alpha_1 \frac{\alpha_0 \hat{p}_0}{(U_0 - c_0)^2} + 2c_1 \frac{\nabla_T U_0 \cdot \nabla_T \hat{p}_0}{(U_0 - c_0)^4}, \tag{2.4}$$

where $c_0 \equiv S_0/\alpha_0$, $\alpha_1 \equiv -\mathrm{i}A'/A$, and $c_1 \equiv (S_1 - \alpha_1 c_0)/\alpha_0$.

Equations (2.3) and (2.4) must, of course, be solved numerically subject to the boundary conditions (1.28) and (1.29). However, for the present analysis, it is only necessary to know the behavior of the solutions near the critical level. This is most easily determined by first expressing (2.3) and (2.4) in orthogonal curvilinear coordinates, say (η, ζ), with one set of coordinate surfaces corresponding to surfaces of constant base-flow velocity U_0 - as was done, for example, by Goldstein (1976, pp. 6–10). The functions η and ζ of y and z are chosen so that

$$U_0 = U_0(x_2, \eta), \tag{2.5}$$

$$\eta = y_0 \quad \text{at} \quad y = y_0, \quad \eta \to \infty \quad \text{as} \quad y \to \infty, \tag{2.6}$$

and

$$\nabla_T U_0 \cdot \nabla_T \zeta = 0, \tag{2.7}$$

$$\zeta = 0 \quad \text{at} \quad z = 0, \quad \zeta = 2\pi/\beta \quad \text{at} \quad z = 2\pi/\beta, \tag{2.8}$$

where

$$y_0 = \begin{cases} 0 & ; \text{boundary layer} \\ -\infty & ; \text{free-shear layer} \end{cases}, \tag{2.9}$$

β is the (non-dimensional) spanwise wavenumber of the base flow and (2.8) requires (without loss of generality) that $z = 0$ and $z = 2\pi/\beta$ be the planes of symmetry of U_0. In terms of η and ζ, the gradient operator in the transverse plane is

$$\nabla_T = l\frac{1}{g}\frac{\partial}{\partial\eta} + m\frac{1}{h}\frac{\partial}{\partial\zeta}, \tag{2.10}$$

where $(l, m) \equiv (g\nabla\eta, h\nabla\zeta)$ are the unit vectors and $(g, h) \equiv (|\nabla\eta|^{-1}, |\nabla\zeta|^{-1})$ are the scale factors corresponding to the coordinates (η, ζ), respectively.

It follows from (2.5) that the critical-level position is given by $\eta = \eta_c$ where

$$U_0(x_2, \eta) = c_0 \quad \text{at} \quad \eta = \eta_c. \tag{2.11}$$

The near-critical-level expansions of $\hat{p}_0$ and $\hat{p}_1$ can now be found by the method of Frobenius (Hall & Horseman 1991; Horseman 1991; and Hall & Smith 1991). To the required level of approximation, these expansions are

$$\begin{aligned} \hat{p}_0 = {} & a_{00} + a_{02}(\eta - \eta_c)^2 + (a_{03}^{(L)} \ln|\eta - \eta_c| + b_{03}^{\pm})(\eta - \eta_c)^3 \\ & + (a_{04}^{(L)} \ln|\eta - \eta_c| + a_{04} + b_{04}^{\pm})(\eta - \eta_c)^4 + O[(\eta - \eta_c)^5 \ln|\eta - \eta_c|], \end{aligned} \tag{2.12}$$

and

$$\hat{p}_1 = a_{10} + d_{11}(\eta - \eta_c) + (d_{12}^{(L)} \ln|\eta - \eta_c| + a_{12} + d_{12}^{\pm})(\eta - \eta_c)^2$$

$$+\ [(a_{13}^{(L)} + d_{13}^{(L)}) \ln|\eta - \eta_c| + b_{13}^{\pm}](\eta - \eta_c)^3 + O[(\eta - \eta_c)^4 \ln|\eta - \eta_c|], \tag{2.13}$$

where the $\pm$ superscript denotes differing values for $\eta \gtrless \eta_c$ and the fact that the pressure is continuous across the critical layer to $O(\sigma\epsilon)$ (see (3.13), (3.17) and (B 3) below) has been used. At this point, the coefficients a_{m0} and $b_{m3}^{\pm}$ are arbitrary functions of ζ. Expressions for the remaining coefficients in terms of these functions are given in appendix A.

The boundary-value problem (1.28), (1.29), and (2.4) only possesses solutions for certain values of α_1 since $\hat{p}_0$ is a homogeneous solution to (2.3). These values can be found without explicitly solving for $\hat{p}_1$ by integrating the difference between $\hat{p}_0$ times (2.4) and $\hat{p}_1$ times (2.3) over the transverse domain, applying the divergence theorem to the simply connected regions and then making use of (1.28), (1.29), the z periodicity of $\hat{p}$ and the expansions (2.12) and (2.13) to arrive at a solvability condition. For definiteness, we consider the simplest case where the critical level forms a single closed or open curve that divides the transverse domain into two simply connect regions. In this case, the solvability condition becomes

$$\int_0^{2\pi/\beta} \Phi_0 \left[\left(2a_{00}\frac{c_1\Phi_1}{U_{0\eta_c}\Phi_0} + a_{10}\right)(b_{03}^{+} - b_{03}^{-}) - a_{00}(b_{13}^{+} - b_{13}^{-})\right] \mathrm{d}\zeta$$
$$= \frac{2U_{0\eta_c}}{3\alpha_0}\left(I_P\frac{\alpha_1}{\alpha_0} + J_P\frac{c_1}{c_0}\right), \tag{2.14}$$

where the functions Φ_0 and Φ_1 of ζ are given by (A 26), the c subscript denotes evaluation at $\eta = \eta_c$,

$$I_P \equiv \int_0^{2\pi/\beta} ⨍_{y_0}^{\infty} \frac{\alpha_0^2\hat{p}_0^2}{(U_0 - c_0)^2} gh\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{2.15}$$

$$J_P \equiv \int_0^{2\pi/\beta} ⨍_{y_0}^{\infty} \frac{c_0}{(U_0 - c_0)^3}\left(\nabla_T\hat{p}_0\cdot\nabla_T\hat{p}_0 + \alpha_0^2\hat{p}_0^2\right) gh\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{2.16}$$

and $⨍$ denotes the Cauchy principal value.

For purposes of analyzing the nonlinear flow within the critical layer, it is convenient to express the velocity perturbation as

$$\acute{u} = i\frac{\acute{\bar{u}}}{gh} + l\frac{\acute{\bar{v}}}{h} + m\frac{\acute{\bar{w}}}{g}. \tag{2.17}$$

The near-critical-level expansions of the shape functions corresponding to $\acute{\bar{u}}$, $\acute{\bar{v}}$ and $\acute{\bar{w}}$ are given in appendix A where it is shown that the discontinuities in (2.12) and (2.13) lead to a jump in the streamwise velocity component

$$\Delta\hat{\bar{u}} = -\frac{3\Phi_0}{\alpha_0}\left[b_{03}^+ - b_{03}^- + \sigma(b_{13}^+ - b_{13}^-) - 2\sigma\left(\frac{c_1\Phi_1}{U_{0\eta_c}\Phi_0} + \frac{\alpha_1}{\alpha_0}\right)(b_{03}^+ - b_{03}^-)\right] + \cdots \tag{2.18}$$

across the critical layer. Matching this jump with the one induced by the flow in the critical layer determines the functions $b_{m3}^{\pm}$. However, when determining $b_{13}^{\pm}$, it is more convenient to express the jump condition as

$$\Delta\left[\hat{\bar{v}}_\eta - \frac{U_{0\eta\bar{\eta}_c}}{U_{0\eta_c}}\hat{\bar{v}} - \left(e_{02}^{(L)} + 2e_{02}^{\pm} - \frac{U_{0\eta\bar{\eta}_c}}{U_{0\eta_c}}e_{01}^{\pm}\right)(\eta - \eta_c)\right]$$
$$= \mathrm{i}3\Phi_0\left[(b_{03}^+ - b_{03}^-) + \sigma(b_{13}^+ - b_{13}^-) + \sigma\left(2\frac{c_1\bar{g}_{\eta_c}}{U_{0\eta_c}\bar{g}_c} - \frac{\alpha_1}{\alpha_0}\right)(b_{03}^+ - b_{03}^-)\right] + \cdots, \tag{2.19}$$

which follows directly from (A 29) and (A 30).

3. Unsteady flow inside the critical layer

As already noted, nonlinear effects first come into play locally within the so-called critical layer once the deviation of the local Strouhal number from its neutral value becomes sufficiently small. The thickness of the critical layer, which is determined by the balance of wave-growth and base-flow-convection effects, turns out to be order σ on the η scale so the appropriate scaled coordinate for this region is

$$\bar{\eta} \equiv (\eta - \eta_c)/\sigma. \tag{3.1}$$

The nonlinear terms in (1.9) produce a critical-layer velocity jump at the same order as the linear-growth effects when the scale of the frequency deviation σ, which was introduced in (1.18), is chosen to be

$$\sigma = \epsilon^{\frac{1}{3}} \tag{3.2}$$

(Goldstein & Choi 1989). Viscous effects will enter into the dominant balance for the critical-layer while making only insignificant modifications to the outer flow when the Benney–Bergeron parameter

$$\lambda \equiv 1/\sigma^3 R \tag{3.3}$$

(Benney & Bergeron 1969) is order one. In the present analysis, λ is assumed to be small enough so that viscous effects, which may arise from the x_2 dependence of the base-flow solution as well as the viscous-diffusions terms in (1.9), are negligible.

Since the flow inside the critical layer depends on x and t only through the variables (1.21) and (1.22), the appropriate governing equations for this region are obtained by expressing (1.8) and (1.9) in terms of x_1, X, $\bar{\eta}$ and ζ. Upon introducing (2.17), these equations become

$$\sigma^2 \acute{u}_{x_1} + \sigma\alpha_0 \acute{u}_X + \acute{v}_{\bar{\eta}} + \sigma \acute{w}_\zeta = 0, \tag{3.4}$$

and

$$\mathcal{L}\acute{u} + ghU_{0\bar{\eta}}\frac{\acute{v}}{\sigma} + g^2h^2(\sigma \acute{p}_{x_1} + \alpha_0 \acute{p}_X) = -\sigma^3 gh\mathcal{N}\left(\frac{\acute{u}}{gh}\right), \tag{3.5}$$

$$\sigma\mathcal{L}\acute{v} + h^2\acute{p}_{\bar{\eta}} = -\sigma^3\left[\sigma\frac{h}{g}\mathcal{N}\left(\frac{g\acute{v}}{h}\right) - \frac{g_{\bar{\eta}}}{g}\acute{v}^2 - \frac{hh_{\bar{\eta}}}{g^2}\acute{w}^2\right], \tag{3.6}$$

$$\mathcal{L}\acute{w} + g^2\acute{p}_\zeta = -\sigma^3\left[\frac{g}{h}\mathcal{N}\left(\frac{h\acute{w}}{g}\right) - \frac{gg_\zeta}{h^2}\acute{v}^2 - \frac{h_\zeta}{h}\acute{w}^2\right], \tag{3.7}$$

where

$$\mathcal{L} \equiv \sigma g h U_0 \frac{\partial}{\partial x_1} + \alpha_0 g h \left(U_0 - c_0 - \sigma \frac{S_1}{\alpha_0} \right) \frac{\partial}{\partial X}, \tag{3.8}$$

$$\mathcal{N} \equiv \sigma \acute{\hat{u}} \frac{\partial}{\partial x_1} + \alpha_0 \acute{\hat{u}} \frac{\partial}{\partial X} + \frac{\acute{\hat{v}}}{\sigma} \frac{\partial}{\partial \bar{\eta}} + \acute{\hat{w}} \frac{\partial}{\partial \zeta}. \tag{3.9}$$

Introducing (3.1) into the expressions for $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{p}$ obtained from (A 18), (A 29)–(A 34), (2.2), (2.12) and (2.13) and re-expanding the result shows that the unsteady flow in the critical layer should expand like

$$\acute{\hat{u}} = \sigma^{-1} \bar{u}_0 + \bar{u}_1 + \sigma \bar{u}_2 + \cdots, \tag{3.10}$$

$$\acute{\hat{v}} = \bar{v}_0 + \sigma \bar{v}_1 + \sigma^2 \bar{v}_2 + \cdots, \tag{3.11}$$

$$\acute{\hat{w}} = \sigma^{-1} \bar{w}_0 + \bar{w}_1 + \sigma \bar{w}_2 + \cdots, \tag{3.12}$$

$$\acute{\hat{p}} = p_0 + \sigma p_1 + \sigma^2 p_2 + \cdots, \tag{3.13}$$

where, in general, the functions $\bar{u}_m$, $\bar{v}_m$, $\bar{w}_m$ and p_m of x_1, X, $\bar{\eta}$ and ζ have an implicit σ dependence of the form

$$\bar{u}_m = \bar{u}_m^{(L)} \ln \sigma + \bar{u}_m^{(o)}. \tag{3.14}$$

In this region, the known functions U_0, g and h are given by their Taylor series expansions about $\eta = \eta_c$ when expressed in terms of $\bar{\eta}$.

Substituting (3.10)–(3.13) into (3.4)–(3.7) and equating like powers of σ leads to the following set of equations at leading order

$$\alpha_0 \bar{u}_{0X} + \bar{v}_{0\bar{\eta}} + \bar{w}_{0\zeta} = 0, \tag{3.15}$$

$$\mathcal{L}_0 \bar{u}_0 + U_{0\eta_c} \bar{v}_0 + \alpha_0 g_c h_c p_{0X} = 0, \tag{3.16}$$

$$p_{0\bar{\eta}} = 0, \tag{3.17}$$

$$\mathcal{L}_0 \bar{w}_0 + \frac{g_c}{h_c} p_{0\zeta} = 0, \tag{3.18}$$

where

$$\mathcal{L}_0 \equiv c_0 \frac{\partial}{\partial x_1} + (\alpha_0 U_{0\eta_c} \bar{\eta} - S_1) \frac{\partial}{\partial X}. \tag{3.19}$$

The solutions to (3.15)–(3.18) must reduce to the appropriate linear solutions as $x_1 \to -\infty$, they must be periodic in X, and they must match with the outer solutions discussed in §2. It follows from (2.12), (2.13) and (A 29)–(A 34) that the last condition implies that

$$\{\bar{u}_0, \bar{v}_0, \bar{w}_0, p_0\} \to \mathrm{Re}\left(\{\mathrm{i} f_{0\text{-}1\zeta}/\alpha_0 \bar{\eta},\ e_{00},\ f_{0\text{-}1}/\bar{\eta},\ a_{00}\} A \mathrm{e}^{\mathrm{i}X}\right), \tag{3.20}$$

as $\bar{\eta} \to \pm\infty$, where the functions e_{00} and $f_{0\text{-}1}$ of ζ are given in terms of a_{00} by (A 35) and (A 44). It is easy to show that the appropriate solutions to (3.15)–(3.18) are

$$\{\bar{u}_0, \bar{v}_0, \bar{w}_0, p_0\} = \mathrm{Re}\left(\{-U_{0\eta_c} f_{0\text{-}1\zeta} E,\ e_{00} A \mathrm{e}^{\mathrm{i}X},\ \mathrm{i}\alpha_0 U_{0\eta_c} f_{0\text{-}1} E,\ a_{00} A \mathrm{e}^{\mathrm{i}X}\}\right), \tag{3.21}$$

where the function $E(x_1, X, \bar{\eta})$ is determined by

$$\mathcal{L}_0 E = A \mathrm{e}^{\mathrm{i}X} \tag{3.22}$$

together with the condition that $E \to 0$ as $x_1 \to -\infty$ and that E be periodic in X. Therefore

$$E = \frac{1}{c_0} \int_{-\infty}^{x_1} A(\xi) \mathrm{e}^{\mathrm{i}[X + \bar{Y}(\xi - x_1)]} \mathrm{d}\xi \tag{3.23}$$

where $\bar{Y} \equiv (\alpha_0 U_{0\eta_c} \bar{\eta} - S_1)/c_0$.

The higher-order critical-layer problems are derived in appendix B. There it is shown that the relevant solutions to the order-σ problem can be expressed as

$$\bar{u}_{1\bar{\eta}} = \bar{u}^{\dagger}_{1\bar{\eta}} + \bar{u}^{\ddagger}_{1\bar{\eta}} + \left(\frac{\bar{u}_0 \bar{u}_{0\bar{\eta}}}{g_c h_c U_{0\eta_c}}\right)_{\bar{\eta}}, \tag{3.24}$$

$$\bar{w}_1 = \bar{w}^{\dagger}_1 + \bar{w}^{\ddagger}_1 + \left(\frac{\bar{u}_0 \bar{w}_0}{g_c h_c U_{0\eta_c}}\right)_{\bar{\eta}}, \tag{3.25}$$

where the linear components $\bar{u}^{\dagger}_{1\bar{\eta}}$ and $\bar{w}^{\dagger}_{1}$ are given by (B 18)–(B 20) and the nonlinear components $\bar{u}^{\ddagger}_{1\bar{\eta}}$ and $\bar{w}^{\ddagger}_{1}$ must be determined from

$$\alpha_0\mathcal{L}_0\bar{u}^{\ddagger}_{1\bar{\eta}} = \alpha_0 U_{0\eta_c}\bar{w}^{\ddagger}_{1\zeta} - (\gamma_{3\zeta} - 2\gamma_4)\mathrm{Re}\left(\mathrm{i}A\mathrm{e}^{\mathrm{i}X}\right)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}\bar{\eta}}), \tag{3.26}$$

$$\mathcal{L}_0\bar{w}^{\ddagger}_1 = (\gamma_1 - 2\gamma_2)\mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}X}\right)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}}) - \gamma_3\mathrm{Re}\left(\mathrm{i}A\mathrm{e}^{\mathrm{i}X}\right)\mathrm{Re}(E_{\bar{\eta}}) - 2\gamma_2\alpha_0 U_{0\eta_c}\mathrm{Re}(E)^2, \tag{3.27}$$

where the functions $\gamma_n(\zeta)$ are given as

$$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\} \equiv \frac{g_c}{2\alpha_0 h_c U_{0\eta_c}}\left\{\left(\frac{a^2_{00\zeta}}{h^2_c}\right)_\zeta, \frac{1}{g_c}\left(\frac{g_c a^2_{00\zeta}}{h^2_c}\right)_\zeta, \left(\alpha^2_0 a^2_{00}\right)_\zeta, \alpha^2_0 a^2_{00\zeta}\right\}, \tag{3.28}$$

and it has been assumed (without loss of generality) that a_{00} is purely real. The solutions to (3.26) and (3.27) turn out to be

$$\begin{aligned}\alpha_0\bar{u}^{\ddagger}_{1\bar{\eta}} &= -\tfrac{1}{2}(\gamma_1-\gamma_3)_\zeta\mathrm{Re}(F_{\bar{\eta}} - G) - (\gamma_{3\zeta} - 2\gamma_4)\mathrm{Re}(\mathrm{i}G_X + G) - \gamma_{2\zeta}\mathrm{Re}(G)\\ &\quad + \tfrac{1}{2}(\gamma_1+\gamma_3)_\zeta\mathrm{Re}(H) + \gamma_{2\zeta}\mathrm{Re}(E)\mathrm{Re}(E_{\bar{\eta}\bar{\eta}}),\end{aligned} \tag{3.29}$$

$$\bar{w}^{\ddagger}_1 = \gamma_1\mathrm{Re}(F_X - \mathrm{i}F) - \gamma_3\mathrm{Re}(\mathrm{i}F) - 2\gamma_2\mathrm{Re}(E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}}), \tag{3.30}$$

where the functions F, G and H of x_1, X and $\bar{\eta}$ are determined by

$$\mathcal{L}_0\{F,G,H\} = \{A\mathrm{e}^{\mathrm{i}X}\mathrm{Re}(E_{\bar{\eta}}),\ A\mathrm{e}^{\mathrm{i}X}\mathrm{Re}(E_{\bar{\eta}\bar{\eta}}),\ \alpha_0 U_{0\eta_c}(F_X - \mathrm{i}2F)\} \tag{3.31}$$

together with the condition that $\{F,G,H\} \to 0$ as $x_1 \to -\infty$ and that $\{F,G,H\}$ be periodic in X. Therefore

$$\begin{aligned}F = \mathrm{i}\frac{\alpha_0 U_{0\eta_c}}{2c_0^3}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_2}(\xi_2-\xi_1)A(\xi_2)&\left\{A^*(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(\xi_2-\xi_1)}\right.\\ &\left. - A(\xi_1)\mathrm{e}^{\mathrm{i}[2X+\bar{Y}(\xi_1+\xi_2-2x_1)]}\right\}\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned} \tag{3.32}$$

$$\begin{aligned}G = -\frac{\alpha_0^2 U^2_{0\eta_c}}{2c_0^4}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_2}(\xi_2-\xi_1)^2 A(\xi_2)&\left\{A^*(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(\xi_2-\xi_1)}\right.\\ &\left. + A(\xi_1)\mathrm{e}^{\mathrm{i}[2X+\bar{Y}(\xi_1+\xi_2-2x_1)]}\right\}\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned} \tag{3.33}$$

and

$$H=\frac{\alpha_0^2 U_{0\eta_c}^2}{c_0^4}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_2}(x_1-\xi_2)(\xi_2-\xi_1)A(\xi_2)A^*(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(\xi_2-\xi_1)}\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{3.34}$$

where the asterisk denotes complex conjugation.

In appendix B, it is shown that the relevant solution to the order-σ^2 problem can be expressed as

$$\begin{aligned}\bar{v}_{2\bar{\eta}\bar{\eta}}=\bar{v}^{\dagger}_{2\bar{\eta}\bar{\eta}}+\bar{v}^{\ddagger}_{2\bar{\eta}\bar{\eta}}-\alpha_0\left[\frac{\bar{u}_0\bar{u}_1^{\ddagger}}{g_c h_c U_{0\eta_c}}+\left(\frac{\bar{u}_0^3}{6g_c^2h_c^2U_{0\eta_c}^2}\right)_{\bar{\eta}}\right]_{X\bar{\eta}\bar{\eta}}\\ -\left[\frac{\bar{u}_0\bar{w}_1^{\ddagger}+\bar{u}_1^{\ddagger}\bar{w}_0}{g_c h_c U_{0\eta_c}}+\left(\frac{\bar{u}_0^2\bar{w}_0}{2g_c^2h_c^2U_{0\eta_c}^2}\right)_{\bar{\eta}}\right]_{\bar{\eta}\bar{\eta}\zeta}\end{aligned} \tag{3.35}$$

where the linear component $\bar{v}^{\dagger}_{2\bar{\eta}\bar{\eta}}$ is given by (B 36) and the nonlinear component satisfies (B 40). For purposes of obtaining the evolution equation for $A(x_1)$, it is only necessary to determine the quantity

$$\tilde{\bar{v}}^{\ddagger}_{2\bar{\eta}\bar{\eta}}\equiv\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi/\beta}a_{00}\mathrm{e}^{-\mathrm{i}X}\bar{v}^{\ddagger}_{2\bar{\eta}\bar{\eta}}\mathrm{d}\zeta\mathrm{d}X \tag{3.36}$$

which, as shown in appendix B, is determined by

$$\begin{aligned}\mathrm{L}\tilde{\bar{v}}^{\ddagger}_{2\bar{\eta}\bar{\eta}}=&A\mathrm{Re}[(2k_1+k_4+3k_5)G^{(0)}_{\bar{\eta}}-(k_3+2k_4+k_5)H_{\bar{\eta}}-(k_1+k_2)(E^{(1)}E^{(1)*}_{\bar{\eta}\bar{\eta}})_{\bar{\eta}}\\&-(k_1+\tfrac{1}{2}k_3-\tfrac{1}{2}k_4)(E^{(1)}E^{(1)*}_{\bar{\eta}})_{\bar{\eta}\bar{\eta}}]+\mathrm{i}A\mathrm{Re}[-\mathrm{i}(k_3+2k_4+k_5)G^{(0)}_{\bar{\eta}}\\&+\mathrm{i}(k_1+k_2)(E^{(1)}E^{(1)*}_{\bar{\eta}})_{\bar{\eta}\bar{\eta}}]+\tfrac{1}{2}A^*[(2k_1-k_3+k_4+2k_5)G^{(2)}_{\bar{\eta}}\\&-(k_1-k_2)(E^{(1)}E^{(1)}_{\bar{\eta}\bar{\eta}})_{\bar{\eta}}-(k_2+\tfrac{1}{2}k_3+\tfrac{1}{2}k_4)(E^{(1)}E^{(1)}_{\bar{\eta}})_{\bar{\eta}\bar{\eta}}]-\mathrm{L}\tilde{\phi}_{2\bar{\eta}\bar{\eta}}\end{aligned} \tag{3.37}$$

where

$$\mathrm{L}\equiv c_0\frac{\partial}{\partial x_1}+\mathrm{i}(\alpha_0U_{0\eta_c}\bar{\eta}-S_1), \tag{3.38}$$

$$(\,\cdot\,)^{(m)}\equiv\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}mX}(\,\cdot\,)\mathrm{d}X, \tag{3.39}$$

the real constants k_n are given as

$$\{k_1, k_2, k_3, k_4, k_5\} \equiv \int_0^{2\pi/\beta} \frac{h_c}{g_c}\{\gamma_1\gamma_2,\ \gamma_2\gamma_3,\ \gamma_1^2,\ \gamma_1\gamma_3,\ \gamma_3^2\}\mathrm{d}\zeta, \tag{3.40}$$

the function $\tilde{\phi}_2(x_1, \bar{\eta})$ is given as

$$\tilde{\phi}_2 \equiv \frac{1}{\pi}\int_0^{2\pi} \mathrm{i}\mathrm{e}^{-\mathrm{i}X}\mathrm{Re}(E)[k_3\mathrm{Re}(F_X) - (k_3 + k_4)\mathrm{Re}(\mathrm{i}F) + \mathrm{i}(2k_1 + k_3)\mathrm{Re}(\mathrm{i}E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]\mathrm{d}X, \tag{3.41}$$

and the fact that $F_{\bar{\eta}\bar{\eta}}^{(0)} = G_{\bar{\eta}}^{(0)}$ has been used in arriving at (3.37). The solution to (3.37) turns out to be

$$\begin{aligned}\bar{\bar{v}}^{\ddagger}_{2\bar{\eta}\bar{\eta}} = {} & (2k_1 + k_3 + 3k_4 + 4k_5)Q_1 + (2k_1 - k_3 - k_4 + 2k_5)Q_2 \\ & + (2k_1 - k_3 + k_4 + 2k_5)Q_3 - (k_3 + 2k_4 + k_5)Q_4 - (k_1 + k_2)Q_5 \\ & - (k_1 - k_2)Q_6 - (2k_1 + k_2 + \tfrac{1}{2}k_3 - \tfrac{1}{2}k_4)Q_7 + (k_2 - \tfrac{1}{2}k_3 + \tfrac{1}{2}k_4)Q_8 \\ & - (k_2 + \tfrac{1}{2}k_3 + \tfrac{1}{2}k_4)Q_9 - \tilde{\phi}_{2\bar{\eta}\bar{\eta}},\end{aligned} \tag{3.42}$$

where the functions $Q_n(x_1, \bar{\eta})$ are determined by

$$\mathrm{L}\{Q_1, Q_2, Q_3, Q_4\} = \tfrac{1}{2}\{AG_{\bar{\eta}}^{(0)},\ AG_{\bar{\eta}}^{(0)*},\ A^*G_{\bar{\eta}}^{(2)},\ 2A\mathrm{Re}(H_{\bar{\eta}})\}, \tag{3.43}$$

$$\mathrm{L}\{Q_5, Q_6\} = \tfrac{1}{2}\{2A\mathrm{Re}(E^{(1)}E_{\bar{\eta}\bar{\eta}}^{(1)*})_{\bar{\eta}},\ A^*(E^{(1)}E_{\bar{\eta}\bar{\eta}}^{(1)})_{\bar{\eta}}\} \tag{3.44}$$

$$\mathrm{L}\{Q_7, Q_8, Q_9\} = \tfrac{1}{2}\{A(E^{(1)}E_{\bar{\eta}}^{(1)*})_{\bar{\eta}\bar{\eta}},\ A(E^{(1)*}E_{\bar{\eta}}^{(1)})_{\bar{\eta}\bar{\eta}},\ A^*(E^{(1)}E_{\bar{\eta}}^{(1)})_{\bar{\eta}\bar{\eta}}\}, \tag{3.45}$$

together with the condition that the $Q_n \to 0$ as $x_1 \to -\infty$. Explicit expressions for the Q_n are given in appendix C.

4. Amplitude evolution equation

The velocity jump induced by the flow in the critical layer will now be computed and combined with (2.18) and (2.19) in order to determine the functions $b_{m3}^{\pm}$. These results will then be used in (2.14) to obtain the governing equation for $A(x_1)$.

By using the relation

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\bar{Y}(\xi - x)} \mathrm{d}\bar{\eta} = \frac{2\pi c_0}{\alpha_0 U_{0\eta_c}} \delta(\xi - x) \tag{4.1}$$

where δ denotes the Dirac delta function, one can show from (3.10), (3.21), (3.24), (B 18)–(B 20) and (3.29) that

$$\int_{-\infty}^{+\infty} \acute{u}_{\bar{\eta}} \mathrm{d}\bar{\eta} = -\frac{\pi}{\alpha_0} \mathrm{Re}\left(e_{01}^{(L)} A \mathrm{e}^{\mathrm{i}X}\right) - 2\pi \frac{U_{0\eta_c}}{c_0^3} \gamma_{2\zeta} \int_{-\infty}^{x_1} (x_1 - \xi)^2 |A(\xi)|^2 \mathrm{d}\xi + O(\sigma), \tag{4.2}$$

which, when combined with (1.19), (2.18) and (A 36), yields

$$b_{03}^{+} - b_{03}^{-} = \mathrm{i}\,\pi\, a_{03}^{(L)}. \tag{4.3}$$

In order to match with the X-independent term on the right-hand side of (4.2), a mean-flow component must be included in the solution for the perturbation $\{\acute{u}, \acute{p}\}$. The 'steady' Rayleigh problem that governs this component outside the critical layer is given in appendix D where it is shown that the corresponding streamwise velocity is of the same order of magnitude as the instability wave that produced it and further that the slowly varying amplitude of this velocity component is given by

$$B = \int_{-\infty}^{x_1} (x_1 - \xi)^2 |A(\xi)|^2 \mathrm{d}\xi. \tag{4.4}$$

Again using (4.1), one can show from (B 37) together with the definitions of α_1 and c_1

that

$$\frac{1}{\pi A}\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi/\beta} a_{00}\mathrm{e}^{-\mathrm{i}X} q\,\mathrm{d}\zeta\,\mathrm{d}\bar{\eta}\,\mathrm{d}X = 2\mathrm{i}\frac{U_{0\eta_c}}{\alpha_0}\left(I_R\frac{\alpha_1}{\alpha_0}+J_R\frac{c_1}{c_0}\right) + 3\pi\int_0^{2\pi/\beta} a_{00}\Phi_0\left[\left(\frac{\alpha_1}{\alpha_0}-\frac{c_1 U_{0\eta\eta_c}}{U^2_{0\eta_c}}\right)a_{03}^{(L)} - a_{13}^{(L)}\right]\mathrm{d}\zeta, \qquad (4.5)$$

where

$$I_R \equiv -\mathrm{i}\pi\int_0^{2\pi/\beta}\frac{\alpha_0^2 g_c^2}{\bar{g}_c U_{0\eta_c}}\left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c}-2\frac{g_{\eta_c}}{g_c}\right)a_{00}^2\,\mathrm{d}\zeta, \qquad (4.6)$$

$$J_R \equiv -\mathrm{i}\pi\int_0^{2\pi/\beta}\frac{c_0 a_{00}}{\bar{g}_c U^2_{0\eta_c}}\left[3\left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c}+\frac{U_{0\eta\eta_c}}{2U_{0\eta_c}}\right)a_{03}^{(L)} - \frac{\bar{g}_{\eta\eta_c}}{\bar{g}_c}a_{02} + \mathcal{D}_2 a_{00}\right]\mathrm{d}\zeta, \qquad (4.7)$$

$\bar{g} \equiv gU_{0\eta}/h$ and $\mathcal{D}_2$ is defined in appendix A. It now follows from (2.19), (3.11), (3.35), (3.36) and (4.3) that

$$\int_0^{2\pi/\beta} a_{00}\left[\frac{2c_1\Phi_1}{U_{0\eta_c}}(b_{03}^+ - b_{03}^-) - \Phi_0(b_{13}^+ - b_{13}^- - \mathrm{i}\pi a_{13}^{(L)})\right]\mathrm{d}\zeta = -\frac{2U_{0\eta_c}}{3\alpha_0}\left(I_R\frac{\alpha_1}{\alpha_0}+J_R\frac{c_1}{c_0}\right) + \frac{\mathrm{i}}{3A}\int_{-\infty}^{+\infty}\tilde{\bar{v}}^{\ddagger}_{2\bar{\eta}\bar{\eta}}\,\mathrm{d}\bar{\eta}. \qquad (4.8)$$

Using (3.42) and the results of appendix C, one can show that

$$\int_{-\infty}^{+\infty}\tilde{\bar{v}}^{\ddagger}_{2\bar{\eta}\bar{\eta}}\,\mathrm{d}\bar{\eta} = \frac{\mathrm{i}\pi}{4c_0^5}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3} K(x_1\,|\,\xi_3,\xi_2)A(\xi_3)A(\xi_2)A^*(\xi_3+\xi_2-x_1)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3, \qquad (4.9)$$

where

$$K \equiv (x_1-\xi_3)[\nu_1(x_1-\xi_2)(\xi_3-\xi_2) - \nu_2(x_1-\xi_3)^2 - \nu_3(x_1-\xi_2)^2], \qquad (4.10)$$

$$\nu_1 \equiv 4\alpha_0^2 U^2_{0\eta_c}(k_3+2k_4+k_5) = \int_0^{2\pi/\beta}\frac{g_c}{h_c}\left(\frac{a^2_{00\zeta}}{h_c^2}+\alpha_0^2 a_{00}^2\right)^2_{\zeta}\mathrm{d}\zeta, \qquad (4.11)$$

$$\nu_2 \equiv -4\alpha_0^2 U^2_{0\eta_c}(k_3-k_5) = -\int_0^{2\pi/\beta}\frac{g_c}{h_c}\left[\left(\frac{a^2_{00\zeta}}{h_c^2}\right)^2_{\zeta} - \left(\alpha_0^2 a_{00}^2\right)^2_{\zeta}\right]\mathrm{d}\zeta, \qquad (4.12)$$

$$\nu_3 - \nu_1 \equiv -8\alpha_0^2 U^2_{0\eta_c}(k_1+k_2) = -\int_0^{2\pi/\beta}\frac{2}{h_c}\left(\frac{g_c a^2_{00\zeta}}{h_c^2}\right)_{\zeta}\left(\frac{a^2_{00\zeta}}{h_c^2}+\alpha_0^2 a_{00}^2\right)_{\zeta}\mathrm{d}\zeta. \qquad (4.13)$$

Combining (4.8) and (4.9) with the solvability condition (2.14) and using the result

$$\int_0^{2\pi/\beta} \Phi_0(a_{00}a_{13}^{(L)} - a_{03}^{(L)}a_{10})\mathrm{d}\zeta = \left[\frac{\bar{h}_{\eta_c}}{3\alpha_0\bar{h}_c^2}(a_{00}a_{10\zeta} - a_{00\zeta}a_{10})\right]_{\zeta=0}^{2\pi/\beta} = 0 \tag{4.14}$$

leads to the following amplitude-evolution equation

$$A' = \kappa A + \mathrm{i}\mu\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_1} K(x_1\,|\,\xi_1,\xi_2)A(\xi_1)A(\xi_2)A^*(\xi_1+\xi_2-x_1)\mathrm{d}\xi_2\mathrm{d}\xi_1 \tag{4.15}$$

where

$$\kappa \equiv \frac{\mathrm{i}\alpha_0 J S_1}{(J-I)S_0}, \tag{4.16}$$

$$\mu \equiv \frac{\pi\alpha_0^2}{8c_0^5 U_{0\eta_c}(J-I)}, \tag{4.17}$$

$$I \equiv I_P + I_R = \int_0^{2\pi/\beta}\int_{y_0}^{\infty} \frac{\alpha_0^2\hat{p}_0^2}{(U_0-c_0)^2} gh\mathrm{d}\eta\mathrm{d}\zeta, \tag{4.18}$$

$$J \equiv J_P + J_R = \int_0^{2\pi/\beta}\int_{y_0}^{\infty} \frac{c_0}{(U_0-c_0)^3}\left(\nabla_T\hat{p}_0\cdot\nabla_T\hat{p}_0 + \alpha_0^2\hat{p}_0^2\right) gh\mathrm{d}\eta\mathrm{d}\zeta, \tag{4.19}$$

and the η integration in (4.18) and (4.19) is performed along a contour in the complex-η plane that lies below the singularity at $\eta = \eta_c$.

Appendix A. Near-critical-level expansions

In this appendix, the near-critical-level expansions of $\{\hat{u}_m, \hat{p}_m\}$ are determined by first expressing (2.3) and (2.4) in terms of η and ζ. The resulting equations are

$$\hat{p}_{0\eta\eta} - \frac{\Pi}{\eta-\eta_c}\hat{p}_{0\eta} + \mathcal{D}\hat{p}_0 = 0, \tag{A 1}$$

and

$$\hat{p}_{1\eta\eta} - \frac{\Pi}{\eta-\eta_c}\hat{p}_{1\eta} + \mathcal{D}\hat{p}_1 = 2\alpha_1\Lambda\hat{p}_0 + 2c_1\frac{\Omega}{(\eta-\eta_c)^2}\hat{p}_{0\eta}, \tag{A 2}$$

where

$$\Pi \equiv (\eta-\eta_c)\frac{h}{g(U_0-c_0)^2}\left[\frac{g(U_0-c_0)^2}{h}\right]_\eta = \sum_{n=0}^{\infty}\Pi_n(\eta-\eta_c)^n, \tag{A 3}$$

$$\mathcal{D}(\,\cdot\,) \equiv \frac{g}{h}\frac{\partial}{\partial\zeta}\left[\frac{g}{h}\frac{\partial}{\partial\zeta}(\,\cdot\,)\right] - \alpha_0^2 g^2(\,\cdot\,) = \sum_{n=0}^{\infty}\mathcal{D}_n(\,\cdot\,)(\eta-\eta_c)^n, \tag{A 4}$$

$$\Lambda \equiv \alpha_0 g^2 = \sum_{n=0}^{\infty}\Lambda_n(\eta-\eta_c)^n, \tag{A 5}$$

and

$$\Omega \equiv (\eta-\eta_c)^2\frac{U_{0\eta}}{(U_0-c_0)^2} = \sum_{n=0}^{\infty}\Omega_n(\eta-\eta_c)^n. \tag{A 6}$$

Expressions for $\mathcal{D}_n$ and Λ_n are easily obtained from the Taylor series expansions of $\mathcal{D}$ and Λ about $\eta=\eta_c$. The first few coefficients in the near-critical-level expansions of Π and Ω are

$$\Pi_0 = 2, \quad \Pi_1 = \frac{\bar{g}_{\eta_c}}{\bar{g}_c}, \quad \Pi_2 = \left(\frac{\bar{g}_\eta}{\bar{g}}\right)_{\eta_c} - \frac{U_{0\eta\eta\eta_c}}{3U_{0\eta_c}} + \frac{U^2_{0\,\eta\eta_c}}{2U^2_{0\,\eta_c}}, \tag{A 7}$$

and

$$\Omega_0 = \frac{1}{U_{0\eta_c}}, \quad \Omega_1 = 0, \quad \Omega_2 = \frac{U_{0\eta\eta\eta_c}}{6U^2_{0\,\eta_c}} - \frac{U^2_{0\,\eta\eta_c}}{4U^3_{0\,\eta_c}}, \tag{A 8}$$

where $\bar{g} \equiv gU_{0\eta}/h$ and the c subscript denotes evaluation at $\eta=\eta_c$.

Substituting (2.12) and (2.13) into (A 1) and (A 2) and equating like powers of $\eta-\eta_c$ leads to

$$a_{m2} = \tfrac{1}{2}\mathcal{D}_0 a_{m0}, \tag{A 9}$$

$$a^{(L)}_{m3} = -\tfrac{1}{3}(\mathcal{D}_1 a_{m0} - 2\Pi_1 a_{m2}), \tag{A 10}$$

$$a^{(L)}_{m4} = \tfrac{3}{4}\Pi_1 a^{(L)}_{m3}, \tag{A 11}$$

$$a_{m4} = -\tfrac{1}{4}[\mathcal{D}_2 a_{m0} + (\mathcal{D}_0 - 2\Pi_2)a_{m2} - \Pi_1 a^{(L)}_{m3} + 5a^{(L)}_{m4}], \tag{A 12}$$

$$b^{\pm}_{m4} = \tfrac{3}{4}\Pi_1 b^{\pm}_{m3}, \tag{A 13}$$

and

$$d_{11} = -2c_1\Omega_0 a_{02}, \tag{A 14}$$

$$d_{12}^{(L)} = -3c_1\Omega_0 a_{03}^{(L)}, \tag{A 15}$$

$$d_{12}^{\pm} = -\alpha_1\Lambda_0 a_{00} - c_1\Omega_0(a_{03}^{(L)} + 3b_{03}^{\pm}) - \tfrac{1}{2}(\Pi_1 d_{11} - d_{12}^{(L)}), \tag{A 16}$$

$$d_{13}^{(L)} = \tfrac{2}{3}\alpha_1\Lambda_1 a_{00} + \tfrac{2}{3}c_1[2\Omega_2 a_{02} + \Omega_0(a_{04}^{(L)} + 4a_{04} + 4b_{04}^{\pm})]$$
$$-\tfrac{1}{3}[(\mathcal{D}_0 - \Pi_2)d_{11} - \Pi_1(d_{12}^{(L)} + 2d_{12}^{\pm})], \tag{A 17}$$

where $m = 0, 1$.

In view of (2.1), the shape functions corresponding to the normalized velocity components introduced in (2.17) should expand like

$$\{\hat{\hat{u}}, \hat{\hat{v}}, \hat{\hat{w}}\} = \{\hat{\hat{u}}_0, \hat{\hat{v}}_0, \hat{\hat{w}}_0\} + \sigma\{\hat{\hat{u}}_1, \hat{\hat{v}}_1, \hat{\hat{w}}_1\} + \cdots, \tag{A 18}$$

as $\sigma \to 0$. Substituting (A 18) into (2.17) and the result together with (1.24), (1.25), (2.2) and (2.10) into (1.26) and (1.27) and equating like powers of σ leads to

$$\mathrm{i}\alpha_0\hat{\hat{u}}_0 + \hat{\hat{v}}_{0\eta} + \hat{\hat{w}}_{0\zeta} = 0, \tag{A 19}$$

$$\mathrm{i}\alpha_0\hat{\hat{u}}_1 + \mathrm{i}\alpha_1\hat{\hat{u}}_0 + \hat{\hat{v}}_{1\eta} + \hat{\hat{w}}_{1\zeta} = 0, \tag{A 20}$$

and

$$\{\hat{\hat{v}}_0, \hat{\hat{w}}_0\} = \frac{\mathrm{i}}{\eta - \eta_c}\{\Phi\hat{p}_{0\eta}, \Theta\hat{p}_{0\zeta}\}, \tag{A 21}$$

$$\{\hat{\hat{v}}_1, \hat{\hat{w}}_1\} = \frac{\mathrm{i}}{\eta - \eta_c}\{\Phi\hat{p}_{1\eta}, \Theta\hat{p}_{1\zeta}\} - \left(\frac{\alpha_1}{\alpha_0} - c_1\frac{\Psi}{\eta - \eta_c}\right)\{\hat{\hat{v}}_0, \hat{\hat{w}}_0\}, \tag{A 22}$$

where

$$\Phi \equiv (\eta - \eta_c)\frac{h}{\alpha_0 g(U_0 - c_0)} = \sum_{n=0}^{\infty}\Phi_n(\eta - \eta_c)^n, \tag{A 23}$$

$$\Theta \equiv (\eta - \eta_c)\frac{g}{\alpha_0 h(U_0 - c_0)} = \sum_{n=0}^{\infty}\Theta_n(\eta - \eta_c)^n, \tag{A 24}$$

and

$$\Psi \equiv (\eta - \eta_c)\frac{1}{U_0 - c_0} = \sum_{n=0}^{\infty}\Psi_n(\eta - \eta_c)^n. \tag{A 25}$$

The first few coefficients in the near-critical-level expansions of Φ, Θ and Ψ are

$$\Phi_0 = \frac{1}{\alpha_0 \bar{g}_c}, \quad \Phi_1 = \frac{1}{\alpha_0 \bar{g}_c}\left(\frac{U_{0\eta\eta c}}{2U_{0\eta c}} - \frac{\bar{g}_{\eta c}}{\bar{g}_c}\right),$$

$$\Phi_2 = \frac{1}{\alpha_0 \bar{g}_c}\left(\frac{U_{0\eta\eta\eta c}}{3U_{0\eta c}} - \frac{U^2_{0\eta\eta c}}{4U^2_{0\eta c}} - \frac{\bar{g}_{\eta\eta c}}{2\bar{g}_c} - \frac{\bar{g}_{\eta c}U_{0\eta\eta c}}{2\bar{g}_c U_{0\eta c}} + \frac{\bar{g}^2_{\eta c}}{\bar{g}^2_c}\right), \tag{A 26}$$

$$\Theta_0 = \frac{1}{\alpha_0 \bar{h}_c}, \quad \Theta_1 = \frac{1}{\alpha_0 \bar{h}_c}\left(\frac{U_{0\eta\eta c}}{2U_{0\eta c}} - \frac{\bar{h}_{\eta c}}{\bar{h}_c}\right),$$

$$\Theta_2 = \frac{1}{\alpha_0 \bar{h}_c}\left(\frac{U_{0\eta\eta\eta c}}{3U_{0\eta c}} - \frac{U^2_{0\eta\eta c}}{4U^2_{0\eta c}} - \frac{\bar{h}_{\eta\eta c}}{2\bar{h}_c} - \frac{\bar{h}_{\eta c}U_{0\eta\eta c}}{2\bar{h}_c U_{0\eta c}} + \frac{\bar{h}^2_{\eta c}}{\bar{h}^2_c}\right), \tag{A 27}$$

and

$$\Psi_0 = \frac{1}{U_{0\eta c}}, \quad \Psi_1 = -\frac{U_{0\eta\eta c}}{2U^2_{0\eta c}}, \quad \Psi_2 = -\frac{U_{0\eta\eta\eta c}}{6U^2_{0\eta c}} + \frac{U^2_{0\eta\eta c}}{4U^3_{0\eta c}}, \tag{A 28}$$

where $\bar{h} \equiv hU_{0\eta}/g$. It turns out that $\hat{\bar{v}}_m$ and $\hat{\bar{w}}_m$ expand like

$$\hat{\bar{v}}_0 = e_{00} + (e^{(L)}_{01}\ln|\eta-\eta_c| + e^{\pm}_{01})(\eta-\eta_c) + (e^{(L)}_{02}\ln|\eta-\eta_c| + e^{\pm}_{02})(\eta-\eta_c)^2$$
$$+ O[(\eta-\eta_c)^3\ln|\eta-\eta_c|], \tag{A 29}$$

$$\hat{\bar{v}}_1 = e^{(L)}_{10}\ln|\eta-\eta_c| + e^{\pm}_{10} + (e^{(L)}_{11}\ln|\eta-\eta_c| + e^{\pm}_{11})(\eta-\eta_c) + O[(\eta-\eta_c)^2\ln|\eta-\eta_c|], \tag{A 30}$$

and

$$\hat{\bar{w}}_0 = f_{0\text{-}1}(\eta-\eta_c)^{-1} + f_{00} + f_{01}(\eta-\eta_c) + O[(\eta-\eta_c)^2\ln|\eta-\eta_c|], \tag{A 31}$$

$$\hat{\bar{w}}_1 = f_{1\text{-}2}(\eta-\eta_c)^{-2} + f_{1\text{-}1}(\eta-\eta_c)^{-1} + f_{10} + O[(\eta-\eta_c)\ln|\eta-\eta_c|], \tag{A 32}$$

as $\eta \to \eta_c$, where the coefficients e_{mn} and f_{mn} are at most functions of x_1 and ζ. It therefore follows from (A 19) and (A 20) that

$$\hat{\bar{u}}_0 = \mathrm{i}\alpha_0^{-1}[f_{0\text{-}1\zeta}(\eta-\eta_c)^{-1} + e^{(L)}_{01}\ln|\eta-\eta_c| + e^{(L)}_{01} + e^{\pm}_{01} + f_{00\zeta}$$
$$+ (2e^{(L)}_{02}\ln|\eta-\eta_c| + e^{(L)}_{02} + 2e^{\pm}_{02} + f_{01\zeta})(\eta-\eta_c)] + O[(\eta-\eta_c)^2\ln|\eta-\eta_c|], \tag{A 33}$$

$$\hat{u}_1 = \mathrm{i}\alpha_0^{-1}[f_{1\text{-}2\zeta}(\eta-\eta_c)^{-2} + (e_{10}^{(L)} + f_{1\text{-}1\zeta} - \alpha_1\alpha_0^{-1} f_{0\text{-}1\zeta})(\eta-\eta_c)^{-1}$$
$$+ (e_{11}^{(L)} - \alpha_1\alpha_0^{-1} e_{01}^{(L)}) \ln|\eta-\eta_c| + e_{11}^{(L)} + e_{11}^{\pm} + f_{10\zeta} - \alpha_1\alpha_0^{-1}(e_{01}^{(L)} + e_{01}^{\pm} + f_{00\zeta})]$$
$$+ O[(\eta-\eta_c)\ln|\eta-\eta_c|], \qquad \text{(A 34)}$$

as $\eta \to \eta_c$.

Substituting (2.12), (2.13) and (A 29)–(A 32) into (A 21) and (A 22) and equating like powers of $\eta - \eta_c$ leads to

$$e_{00} = \mathrm{i}2\Phi_0 a_{02}, \qquad \text{(A 35)}$$

$$e_{01}^{(L)} = \mathrm{i}3\Phi_0 a_{03}^{(L)}, \qquad \text{(A 36)}$$

$$e_{01}^{\pm} = \mathrm{i}[2\Phi_1 a_{02} + \Phi_0(a_{03}^{(L)} + 3b_{03}^{\pm})], \qquad \text{(A 37)}$$

$$e_{02}^{(L)} = \mathrm{i}(3\Phi_1 a_{03}^{(L)} + 4\Phi_0 a_{04}^{(L)}), \qquad \text{(A 38)}$$

$$e_{02}^{\pm} = \mathrm{i}[2\Phi_2 a_{02} + \Phi_1(a_{03}^{(L)} + 3b_{03}^{\pm}) + \Phi_0(a_{04}^{(L)} + 4a_{04} + 4b_{04}^{\pm})], \qquad \text{(A 39)}$$

$$e_{10}^{(L)} = \mathrm{i}2\Phi_0 d_{12}^{(L)} + c_1 \Psi_0 e_{01}^{(L)}, \qquad \text{(A 40)}$$

$$e_{10}^{\pm} = \mathrm{i}[\Phi_1 d_{11} + \Phi_0(d_{12}^{(L)} + 2a_{12} + 2d_{12}^{\pm})] - \alpha_1\alpha_0^{-1} e_{00} + c_1(\Psi_1 e_{00} + \Psi_0 e_{01}^{\pm}), \qquad \text{(A 41)}$$

$$e_{11}^{(L)} = \mathrm{i}[2\Phi_1 d_{12}^{(L)} + 3\Phi_0(a_{13}^{(L)} + d_{13}^{(L)})] - \alpha_1\alpha_0^{-1} e_{01}^{(L)} + c_1(\Psi_1 e_{01}^{(L)} + \Psi_0 e_{02}^{(L)}), \qquad \text{(A 42)}$$

$$e_{11}^{\pm} = \mathrm{i}[\Phi_2 d_{11} + \Phi_1(d_{12}^{(L)} + 2a_{12} + 2d_{12}^{\pm}) + \Phi_0(a_{13}^{(L)} + d_{13}^{(L)} + +3b_{13}^{\pm})]$$
$$- \alpha_1\alpha_0^{-1} e_{01}^{\pm} + c_1(\Psi_2 e_{00} + \Psi_1 e_{01}^{\pm} + \Psi_0 e_{02}^{\pm}), \qquad \text{(A 43)}$$

$$f_{0\text{-}1} = \mathrm{i}\Theta_0 a_{00\zeta}, \qquad \text{(A 44)}$$

$$f_{00} = \mathrm{i}\Theta_1 a_{00\zeta}, \qquad \text{(A 45)}$$

$$f_{01} = \mathrm{i}(\Theta_2 a_{00\zeta} + \Theta_0 a_{02\zeta}), \qquad \text{(A 46)}$$

and

$$f_{1\text{-}2} = c_1 \Psi_0 f_{0\text{-}1}, \tag{A 47}$$

$$f_{1\text{-}1} = \mathrm{i}\Theta_0 a_{10\zeta} - \alpha_1 \alpha_0^{-1} f_{0\text{-}1} + c_1(\Psi_1 f_{0\text{-}1} + \Psi_0 f_{00}), \tag{A 48}$$

$$f_{10} = \mathrm{i}(\Theta_1 a_{10\zeta} + \Theta_0 d_{11\zeta}) - \alpha_1 \alpha_0^{-1} f_{00} + c_1(\Psi_2 f_{0\text{-}1} + \Psi_1 f_{00} + \Psi_0 f_{01}). \tag{A 49}$$

Appendix B. Higher-order critical-layer problems

In this appendix, the higher-order critical-layer problems obtained by substituting (3.10)–(3.13) into (3.4)–(3.7) and equating like powers of σ are given. The order-σ problem reads

$$\alpha_0 \bar{u}_{1X} + \bar{u}_{0x_1} + \bar{v}_{1\bar{\eta}} + \bar{w}_{1\zeta} = 0, \tag{B 1}$$

$$\mathcal{L}_0 \bar{u}_1 + \mathcal{L}_1 \bar{u}_0 + U_{0\eta_c}\left(\bar{v}_1 + \frac{\bar{f}_{\eta_c}}{\bar{f}_c}\bar{\eta}\bar{v}_0\right) + g_c h_c(\alpha_0 p_{1X} + p_{0x_1}) + 2\alpha_0 (gh)_{\eta_c}\bar{\eta} p_{0X} = -\psi_1, \tag{B 2}$$

$$p_{1\bar{\eta}} = 0, \tag{B 3}$$

$$\mathcal{L}_0 \bar{w}_1 + \mathcal{L}_1 \bar{w}_0 + \frac{g_c}{h_c} p_{1\zeta} + 2\frac{g_{\eta_c}}{h_c}\bar{\eta} p_{0\zeta} = -\theta_1, \tag{B 4}$$

where $\bar{f} \equiv ghU_{0\eta}$,

$$\mathcal{L}_1 \equiv \frac{(gh)_{\eta_c}}{g_c h_c}\bar{\eta}\mathcal{L}_0 + U_{0\eta_c}\bar{\eta}\frac{\partial}{\partial x_1} + \frac{1}{2}\alpha_0 U_{0\eta\eta_c}\bar{\eta}^2\frac{\partial}{\partial X}, \tag{B 5}$$

$$\psi_1 \equiv \alpha_0\left(\frac{\bar{u}_0^2}{g_c h_c}\right)_X + \left(\frac{\bar{u}_0 \bar{v}_0}{g_c h_c}\right)_{\bar{\eta}} + \left(\frac{\bar{u}_0 \bar{w}_0}{g_c h_c}\right)_\zeta, \tag{B 6}$$

$$\theta_1 \equiv \alpha_0\left(\frac{\bar{u}_0 \bar{w}_0}{g_c h_c}\right)_X + \left(\frac{\bar{v}_0 \bar{w}_0}{g_c h_c}\right)_{\bar{\eta}} + \frac{1}{h_c}\left(\frac{\bar{w}_0^2}{g_c}\right)_\zeta. \tag{B 7}$$

It follows directly from (B 3) and matching with the outer linear solution that

$$p_1 = \mathrm{Re}\left(a_{10} A \mathrm{e}^{\mathrm{i}X}\right). \tag{B 8}$$

It turns out that, for purposes of computing the velocity jump $\Delta\bar{u}$ across the critical layer, it is only necessary to know $\bar{u}_{1\bar{\eta}}$, $\bar{v}_{1\bar{\eta}\bar{\eta}}$ and $\bar{w}_1$. Therefore (B 1), (B 2) and (B 4) are rewritten as

$$\alpha_0(\bar{u}_{1\bar{\eta}} - \bar{u}^{\dagger}_{1\bar{\eta}})_X + \bar{v}_{1\bar{\eta}\bar{\eta}} - \bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}} + (\bar{w}_1 - \bar{w}^{\dagger}_1)_{\bar{\eta}\zeta} = 0, \tag{B 9}$$

$$\mathcal{L}_0(\bar{u}_{1\bar{\eta}} - \bar{u}^{\dagger}_{1\bar{\eta}}) = U_{0\eta_c}(\bar{w}_1 - \bar{w}^{\dagger}_1)_\zeta - \psi_{1\bar{\eta}}, \tag{B 10}$$

$$\mathcal{L}_0(\bar{w}_1 - \bar{w}^{\dagger}_1) = -\theta_1, \tag{B 11}$$

where $\bar{u}^{\dagger}_{1\bar{\eta}}$, $\bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}}$ and $\bar{w}^{\dagger}_1$ satisfy the linear equations

$$\alpha_0\bar{u}^{\dagger}_{1\bar{\eta}X} + \bar{u}_{0x_1\bar{\eta}} + \bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}} + \bar{w}^{\dagger}_{1\bar{\eta}\zeta} = 0, \tag{B 12}$$

$$\mathcal{L}_0\bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}} + \left(\frac{g_c}{h_c}\frac{\bar{h}_{\eta_c}}{\bar{h}_c}p_{0\zeta}\right)_\zeta + \alpha_0^2 g_c h_c\left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c} - 2\frac{g_{\eta_c}}{g_c}\right)p_{0XX} = 0, \tag{B 13}$$

$$\mathcal{L}_0\bar{w}^{\dagger}_1 + \mathcal{L}_1\bar{w}_0 + \frac{g_c}{h_c}p_{1\zeta} + 2\frac{g_{\eta_c}}{h_c}\bar{\eta}p_{0\zeta} = 0, \tag{B 14}$$

and have the following large-$\bar{\eta}$ behavior

$$\{\bar{u}^{\dagger}_{1\bar{\eta}}, \bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}}, \bar{w}^{\dagger}_1\} \sim \mathrm{Re}\left(\{\mathrm{i}e^{(L)}_{01}/\alpha_0\bar{\eta},\ e^{(L)}_{01}/\bar{\eta},\ f_{00}\}A e^{\mathrm{i}X}\right) \tag{B 15}$$

which ensures that the solutions to (B 9)–(B 11) match with the outer linear solution as $\bar{\eta} \to \pm\infty$. By using (3.21) and (3.22) together with the relation

$$\mathcal{L}_1(\,\cdot\,) = \mathcal{L}_0[\mathcal{M}_1(\,\cdot\,)] - \left[\left(\frac{U_{0\eta_c}}{c_0} - \frac{U_{0\eta\eta_c}}{2U_{0\eta_c}}\right)\bar{\eta} - \frac{S_1}{S_0}\right]\bar{\eta}\frac{\partial}{\partial\bar{\eta}}\mathcal{L}_0(\,\cdot\,), \tag{B 16}$$

where

$$\mathcal{M}_1(\,\cdot\,) \equiv \left[\left(\frac{U_{0\eta_c}}{c_0} - \frac{U_{0\eta\eta_c}}{2U_{0\eta_c}}\right)\bar{\eta} - \frac{S_1}{S_0}\right]\bar{\eta}\frac{\partial}{\partial\bar{\eta}}(\,\cdot\,) + \frac{(ghU_0)_{\eta_c}}{g_c h_c c_0}\bar{\eta}(\,\cdot\,), \tag{B 17}$$

it can be shown that

$$\alpha_0\bar{u}^{\dagger}_{1\bar{\eta}} = (\bar{u}_{0x_1\bar{\eta}} + \bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}} + \bar{w}^{\dagger}_{1\bar{\eta}\zeta})_X, \tag{B 18}$$

$$\bar{v}^{\dagger}_{1\bar{\eta}\bar{\eta}} = \mathrm{Re}(\mathrm{i}\alpha_0 U_{0\eta_c} e^{(L)}_{01} E), \tag{B 19}$$

$$\bar{w}^{\dagger}_1 = -\mathrm{Re}\left[\left(\mathrm{i}\alpha_0 U_{0\eta_c} f_{0\text{-}1}\mathcal{M}_1 + 2\frac{g_{\eta c}}{h_c} a_{00\zeta}\bar{\eta} + \frac{g_c}{h_c} a_{10\zeta}\right) E\right]. \tag{B 20}$$

It follows from (B 6), (B 7) and (3.15)–(3.18) that

$$\psi_{1\bar{\eta}} = -\frac{\alpha_0}{U_{0\eta_c}} p_{0X}\bar{u}_{0\bar{\eta}\bar{\eta}} + \left(\frac{\bar{u}_0\bar{w}_0}{g_c h_c}\right)_{\bar{\eta}\zeta} - \mathcal{L}_0\left[\left(\frac{\bar{u}_0\bar{u}_{0\bar{\eta}}}{\bar{f}_c}\right)_{\bar{\eta}}\right], \tag{B 21}$$

$$\theta_1 = \frac{1}{h_c^2 U_{0\eta_c}} p_{0\zeta}\bar{u}_{0\bar{\eta}} - \frac{\alpha_0}{U_{0\eta_c}} p_{0X}\bar{w}_{0\bar{\eta}} + \frac{1}{h_c}\left(\frac{\bar{w}_0^2}{g_c}\right)_{\zeta} - \mathcal{L}_0\left[\left(\frac{\bar{u}_0\bar{w}_0}{\bar{f}_c}\right)_{\bar{\eta}}\right]. \tag{B 22}$$

Combining these expressions with (B 10), (B 11), (3.21), (A 35) and (A 44) then leads to (3.26) and (3.27).

The order-σ^2 critical-layer problem reads

$$\alpha_0\bar{u}_{2X} + \bar{u}_{1x_1} + \bar{v}_{2\bar{\eta}} + \bar{w}_{2\zeta} = 0, \tag{B 23}$$

$$\begin{aligned}\mathcal{L}_0\bar{u}_2 + \mathcal{L}_1\bar{u}_1 + \mathcal{L}_2\bar{u}_0 + U_{0\eta_c}\left(\bar{v}_2 + \frac{\bar{f}_{\eta c}}{\bar{f}_c}\bar{\eta}\bar{v}_1 + \frac{\bar{f}_{\eta\eta c}}{2\bar{f}_c}\bar{\eta}^2\bar{v}_0\right) + g_c h_c(\alpha_0 p_{2X} + p_{1x_1})\\ + 2(gh)_{\eta c}\bar{\eta}(\alpha_0 p_{1X} + p_{0x_1}) + \frac{(g^2h^2)_{\eta\eta c}}{2g_c h_c}\bar{\eta}^2\alpha_0 p_{0X} = -\psi_2,\end{aligned} \tag{B 24}$$

$$\frac{g_c}{h_c}\mathcal{L}_0\bar{v}_0 + p_{2\bar{\eta}} = \frac{h_{\eta c}}{g_c^2 h_c}\bar{w}_0^2, \tag{B 25}$$

$$\mathcal{L}_0\bar{w}_2 + \mathcal{L}_1\bar{w}_1 + \mathcal{L}_2\bar{w}_0 + \frac{g_c}{h_c}p_{2\zeta} + 2\frac{g_{\eta c}}{h_c}\bar{\eta}p_{1\zeta} + \frac{(g^2)_{\eta\eta c}}{2g_c h_c}\bar{\eta}^2 p_{0\zeta} = -\theta_2, \tag{B 26}$$

where

$$\mathcal{L}_2 \equiv \left[\frac{(gh)_{\eta\eta c}}{2g_c h_c} - \frac{(gh)^2_{\eta c}}{g_c^2 h_c^2}\right]\bar{\eta}^2\mathcal{L}_0 + \frac{(gh)_{\eta c}}{g_c h_c}\bar{\eta}\mathcal{L}_1 + \frac{1}{2}U_{0\eta\eta_c}\bar{\eta}^2\frac{\partial}{\partial x_1} + \frac{1}{6}\alpha_0 U_{0\eta\eta\eta_c}\bar{\eta}^3\frac{\partial}{\partial X}, \tag{B 27}$$

$$\begin{aligned}\psi_2 \equiv 2\alpha_0\left(\frac{\bar{u}_0\bar{u}_1}{g_c h_c}\right)_X + \left(\frac{\bar{u}_1\bar{v}_0 + \bar{u}_0\bar{v}_1}{g_c h_c}\right)_{\bar{\eta}} + \left(\frac{\bar{u}_1\bar{w}_0 + \bar{u}_0\bar{w}_1}{g_c h_c}\right)_{\zeta}\\ + \left(\frac{\bar{u}_0^2}{g_c h_c}\right)_{x_1} - \frac{(gh)_{\eta c}}{g_c^2 h_c^2}\bar{u}_0\bar{v}_0 - \frac{1}{g_c h_c}\left[\frac{(gh)_{\eta c}}{g_c h_c}\right]_{\zeta}\bar{\eta}\bar{u}_0\bar{w}_0,\end{aligned} \tag{B 28}$$

$$\theta_2 \equiv \alpha_0\left(\frac{\bar{u}_0\bar{w}_1+\bar{u}_1\bar{w}_0}{g_ch_c}\right)_X+\left(\frac{\bar{v}_0\bar{w}_1+\bar{v}_1\bar{w}_0}{g_ch_c}\right)_{\bar{\eta}}+\frac{2}{h_c}\left(\frac{\bar{w}_0\bar{w}_1}{g_c}\right)_\zeta$$
$$+\left(\frac{\bar{u}_0\bar{w}_0}{g_ch_c}\right)_{x_1}-\frac{1}{g_c^2}\left(\frac{g}{h}\right)_{\eta_c}\bar{v}_0\bar{w}_0-\frac{1}{g_ch_c}\left(\frac{g_{\eta_c}}{g_c}\right)_\zeta\bar{\eta}\bar{w}_0^2. \qquad \text{(B 29)}$$

Fortunately, only the solution for $\bar{v}_{2\bar{\eta}\bar{\eta}}$ is needed in determining the governing equation for $A(x_1)$. Therefore the above equations are combined to give

$$\mathcal{L}_0(\bar{v}_{2\bar{\eta}\bar{\eta}}-\bar{v}^\dagger_{2\bar{\eta}\bar{\eta}})=[\alpha_0\psi_{2X}+\psi_{1x_1}+\theta_{2\zeta}+\mathcal{L}_{1\zeta}(\bar{w}_1-\bar{w}_1^\dagger)]_{\bar{\eta}}$$
$$-\left[\mathcal{L}_1\frac{\partial^2}{\partial\bar{\eta}^2}+\frac{(gh)_{\eta_c}}{g_ch_c}\mathcal{L}_0\frac{\partial}{\partial\bar{\eta}}-\alpha_0\frac{\bar{f}_{\eta_c}}{\bar{f}_c}U_{0\eta_c}\frac{\partial}{\partial X}\right](\bar{v}_1-\bar{v}_1^\dagger)+\bar{\mathcal{D}}\left(\frac{h_{\eta_c}\bar{w}_0^2}{g_c^2h_c}\right) \qquad \text{(B 30)}$$

where

$$\bar{\mathcal{D}}\equiv\frac{\partial}{\partial\zeta}\left(\frac{g_c}{h_c}\frac{\partial}{\partial\zeta}\right)+\alpha_0^2g_ch_c\frac{\partial^2}{\partial X^2}, \qquad \text{(B 31)}$$

and $\bar{v}^\dagger_{2\bar{\eta}\bar{\eta}}$ is determined by the linear equation

$$\mathcal{L}_0\bar{v}^\dagger_{2\bar{\eta}\bar{\eta}}+\mathcal{L}_1\bar{v}^\dagger_{1\bar{\eta}\bar{\eta}}+\frac{(gh)_{\eta_c}}{g_ch_c}\mathcal{L}_0\bar{v}^\dagger_{1\bar{\eta}}-\frac{\bar{f}_{\eta_c}}{\bar{f}_c}U_{0\eta_c}(\alpha_0\bar{v}^\dagger_{1X}+\bar{v}_{0x_1})-(\mathcal{L}_{1\zeta}\bar{w}_1^\dagger+\mathcal{L}_{2\zeta}\bar{w}_0)_{\bar{\eta}}$$
$$+\bar{\mathcal{D}}\left(\frac{g_c}{h_c}\mathcal{L}_0\bar{v}_0\right)-\bar{\eta}\left\{\alpha_0\frac{\bar{f}_{\eta\eta_c}}{\bar{f}_c}U_{0\eta_c}\bar{v}_{0X}+\left[\frac{(g^2)_{\eta\eta_c}}{g_ch_c}p_{0\zeta}\right]_\zeta+\alpha_0^2\frac{(g^2h^2)_{\eta\eta_c}}{g_ch_c}p_{0XX}\right\}$$
$$-4\alpha_0(gh)_{\eta_c}p_{0x_1X}-2\left[\left(\frac{g_{\eta_c}}{h_c}p_{1\zeta}\right)_\zeta+\alpha_0^2(gh)_{\eta_c}p_{1XX}\right]=0, \qquad \text{(B 32)}$$

together with the boundary condition

$$\bar{v}^\dagger_{2\bar{\eta}\bar{\eta}}\to\mathrm{Re}[(2e_{02}^{(L)}\ln|\sigma\bar{\eta}|+3e_{02}^{(L)}+2e_{02}^{\pm}+e_{11}^{(L)}/\bar{\eta})A\mathrm{e}^{\mathrm{i}X}] \quad \text{as} \quad \bar{\eta}\to\pm\infty \qquad \text{(B 33)}$$

which ensures that the solution to (B 30) matches with the outer linear solution. By manipulating (3.15)–(3.18) and (B 1)–(B 4), one can show that

$$U_{0\eta_c}(\alpha_0\bar{v}^\dagger_{1X}+\bar{v}_{0x_1})=\mathcal{L}_0\bar{v}^\dagger_{1\bar{\eta}}-\bar{\eta}\left(\alpha_0\frac{\bar{f}_{\eta_c}}{\bar{f}_c}U_{0\eta_c}\bar{v}_{0X}+\frac{h_c}{g_c}\mathcal{D}_1p_0+2\frac{h_{\eta_c}}{g_c}\mathcal{D}_0p_0\right)$$
$$-2\alpha_0g_ch_cp_{0x_1X}-\frac{h_c}{g_c}\mathcal{D}_0p_1 \qquad \text{(B 34)}$$

and

$$(\mathcal{L}_{1\zeta}\bar{w}_1^\dagger + \mathcal{L}_{2\zeta}\bar{w}_0)_{\bar{\eta}} = -\left[\frac{(gh)_{\eta_c}}{g_c h_c}\right]_\zeta \left[2\bar{\eta}\left(\frac{g}{h}\right)_{\eta_c} p_{0\zeta} + \frac{g_c}{h_c}p_{1\zeta}\right] - \bar{\eta}\left[\frac{(gh)_{\eta\eta_c}}{g_c h_c}\right]_\zeta \frac{g_c}{h_c}p_{0\zeta} \qquad \text{(B 35)}$$

where the $\mathcal{D}_n$ are defined in appendix A. Combining these results with (3.21), (B 8), (B 16) and (B 19) then leads to

$$\begin{aligned}\bar{v}^\dagger_{2\bar{\eta}\bar{\eta}} &= -\left[\mathcal{M}_1 + \left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c} - 2\frac{h_{\eta_c}}{h_c}\right)\bar{\eta}\right]\bar{v}^\dagger_{1\bar{\eta}\bar{\eta}} + \frac{U_{0\eta\eta_c}}{U_{0\eta_c}}\bar{v}^\dagger_{1\bar{\eta}} - \operatorname{Re}\left(\mathrm{i}2\Phi_0\mathcal{D}_0 a_{02}A\mathrm{e}^{\mathrm{i}X}\right)\\ &\quad - \operatorname{Re}\left\{2\frac{h_c}{g_c}\left[\mathrm{i}\alpha_0\left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c} - 2\frac{g_{\eta_c}}{g_c}\right)g_c^2 a_{00}\frac{\partial}{\partial x_1} + \frac{\bar{g}_{\eta\eta_c}}{\bar{g}_c}a_{02}\bar{\eta} - \mathcal{D}_2 a_{00}\bar{\eta} + \frac{3}{2}a_{13}^{(L)}\right]E\right\}. \qquad \text{(B 36)}\end{aligned}$$

For purposes of computing the induced velocity jump, it is convenient to express (B 36) as

$$\begin{aligned} q \equiv \bar{v}^\dagger_{2\bar{\eta}\bar{\eta}} - \frac{U_{0\eta\eta_c}}{U_{0\eta_c}}\bar{v}^\dagger_{1\bar{\eta}} - \operatorname{Re}\left[\left(e_{02}^{(L)} + 2e_{02}^{\pm} - \frac{U_{0\eta\eta_c}}{U_{0\eta_c}}e_{01}^{\pm}\right)A\mathrm{e}^{\mathrm{i}X}\right]\\ = -\operatorname{Re}\left\{3\frac{h_c}{g_c}\left[a_{13}^{(L)} - \frac{S_1}{S_0}a_{03}^{(L)} + \mathrm{i}\frac{2}{3}\alpha_0\left(\frac{\bar{g}_{\eta_c}}{\bar{g}_c} - 2\frac{g_{\eta_c}}{g_c}\right)g_c^2 a_{00}\frac{\partial}{\partial x_1}\right]E\right\}\\ - \operatorname{Re}\left\{2\frac{h_c}{g_c}\left[3\left(\frac{U_{0\eta_c}}{2c_0} - \frac{\bar{g}_{\bar{\eta}_c}}{\bar{g}_c}\right)a_{03}^{(L)} + \frac{\bar{g}_{\bar{\eta}\bar{\eta}_c}}{\bar{g}_c}a_{02} - \mathcal{D}_2 a_{00}\right]\bar{\mathrm{L}}E\right\}\\ + \operatorname{Re}\left\{3\frac{h_c}{g_c}a_{03}^{(L)}\left[\left(\frac{U_{0\eta_c}}{c_0} - \frac{U_{0\eta\eta_c}}{2U_{0\eta_c}}\right)\bar{\mathrm{L}} - \frac{S_1}{S_0}\right]\bar{\mathrm{L}}E_{\bar{\eta}}\right\} \qquad \text{(B 37)}\end{aligned}$$

where

$$\bar{\mathrm{L}} \equiv \frac{1}{\mathrm{i}\alpha_0 U_{0\eta_c}}\left(\mathrm{i}S_1 - c_0\frac{\partial}{\partial x_1}\right), \qquad \text{(B 38)}$$

and (3.22) and (B 19) were used in arriving at (B 37).

It follows from (B 28), (B 29), (3.15)–(3.18) and (B 9)–(B 11) that

$$\begin{aligned}(\alpha_0\psi_{2X} + \theta_{2\zeta})_{\bar{\eta}} &= -\frac{\alpha_0^2}{U_{0\eta_c}}(p_{0X}\bar{u}^\ddagger_{1\bar{\eta}\bar{\eta}})_X - \frac{1}{U_{0\eta_c}}\left(\frac{p_{0\zeta}\bar{u}^\ddagger_{1\bar{\eta}\bar{\eta}}}{h_c^2}\right)_\zeta - \frac{\alpha_0}{U_{0\eta_c}}\left(p_{0X}\bar{w}^\ddagger_{1\bar{\eta}\bar{\eta}}\right)_\zeta\\ &\quad + \left[\frac{1}{h_c}\left(2\frac{\bar{w}_0\bar{w}_1^\ddagger}{g_c} + \frac{\bar{u}_{0\bar{\eta}}\bar{w}_0^2}{g_c\bar{f}_c}\right)_{\bar{\eta}\zeta}\right]_\zeta + (\alpha_0\psi^\dagger_{2X} + \theta^\dagger_{2\zeta})_{\bar{\eta}}\\ &\quad - \mathcal{L}_0\left\{\alpha_0\left[\frac{\bar{u}_0\bar{u}_1^\ddagger}{\bar{f}_c} + \left(\frac{\bar{u}_0^3}{6\bar{f}_c^2}\right)_{\bar{\eta}}\right]_X + \left[\frac{\bar{u}_0\bar{w}_1^\ddagger + \bar{u}_1^\ddagger\bar{w}_0}{\bar{f}_c} + \left(\frac{\bar{u}_0^2\bar{w}_0}{2\bar{f}_c^2}\right)_{\bar{\eta}}\right]_\zeta\right\}_{\bar{\eta}\bar{\eta}} \qquad \text{(B 39)}\end{aligned}$$

where $\psi_2^\dagger$ and $\theta_2^\dagger$ are given by the right-hand sides of (B 28) and (B 29), respectively, but with $\{\bar{u}_1, \bar{v}_1, \bar{w}_1\}$ replaced by $\{\bar{u}_1^\dagger, \bar{v}_1^\dagger, \bar{w}_1^\dagger\}$. Introducing the above relation into (B 30) leads to

$$
\begin{aligned}
\mathcal{L}_0 \bar{v}^\ddagger_{2\bar{\eta}\bar{\eta}} = &-\frac{\alpha_0^2}{U_{0\eta_c}}(p_{0X}\bar{u}^\ddagger_{1\bar{\eta}\bar{\eta}})_X - \frac{1}{U_{0\eta_c}}\left(\frac{p_{0\zeta}\bar{u}^\ddagger_{1\bar{\eta}\bar{\eta}}}{h_c^2}\right)_\zeta - \frac{\alpha_0}{U_{0\eta_c}}(p_{0X}\bar{w}^\ddagger_{1\bar{\eta}\bar{\eta}})_\zeta \\
&+ \left[\frac{1}{h_c}\left(2\frac{\bar{w}_0\bar{w}_1^\dagger}{g_c} + \frac{\bar{u}_{0\bar{\eta}}\bar{w}_0^2}{g_c\bar{f}_c}\right)_{\bar{\eta}\zeta}\right]_\zeta + \left[\alpha_0\psi^\dagger_{2X} + \psi_{1x_1} + \theta^\dagger_{2\zeta} + \mathcal{L}_{1\zeta}(\bar{w}_1 - \bar{w}_1^\dagger)\right]_{\bar{\eta}} \\
&- \left[\mathcal{L}_1\frac{\partial^2}{\partial\bar{\eta}^2} + \frac{(gh)_{\eta_c}}{g_c h_c}\mathcal{L}_0\frac{\partial}{\partial\bar{\eta}} - \alpha_0\frac{\bar{f}_{\eta_c}}{\bar{f}_c}U_{0\eta_c}\frac{\partial}{\partial X}\right](\bar{v}_1 - \bar{v}_1^\dagger) + \bar{\mathcal{D}}\left(\frac{h_{\eta_c}\bar{w}_0^2}{g_c^2 h_c}\right)
\end{aligned}
\qquad \text{(B 40)}
$$

where $\bar{v}^\ddagger_{2\bar{\eta}\bar{\eta}}$ is given by (3.35). Substituting (3.21), (3.29) and (3.30) into (B 40), multiplying the result by $a_{00}\mathrm{e}^{-\mathrm{i}X}/\pi$, and integrating from $\zeta = 0$ to $2\pi/\beta$ then using the relations

$$
\begin{aligned}
[2\alpha_0 U_{0\eta_c}\mathrm{Re}(E)\mathrm{Re}(F_X)]_{X\bar{\eta}} = &\ \mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(F_X) + \mathrm{Re}(E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}} \\
&+ \mathrm{Re}\left(\mathrm{i}A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(E)\mathrm{Re}(E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}} - \mathcal{L}_0[\mathrm{Re}(E)\mathrm{Re}(F_X)]_{\bar{\eta}\bar{\eta}},
\end{aligned}
\qquad \text{(B 41)}
$$

$$
\begin{aligned}
[2\alpha_0 U_{0\eta_c}\mathrm{Re}(E)\mathrm{Re}(\mathrm{i}F)]_{X\bar{\eta}} = &\ \mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(\mathrm{i}F)]_{\bar{\eta}\bar{\eta}} \\
&+ \mathrm{Re}\left(\mathrm{i}A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(E)\mathrm{Re}(E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}} - \mathcal{L}_0[\mathrm{Re}(E)\mathrm{Re}(\mathrm{i}F)]_{\bar{\eta}\bar{\eta}},
\end{aligned}
\qquad \text{(B 42)}
$$

$$
\begin{aligned}
[\alpha_0 U_{0\eta_c}\mathrm{Re}(E)^2\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]_{XX\bar{\eta}} = &\ \mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(\mathrm{i}E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}} \\
&+ \mathrm{Re}\left(\mathrm{i}A\mathrm{e}^{\mathrm{i}X}\right)[\mathrm{Re}(E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}} - \mathcal{L}_0[\mathrm{Re}(E)\mathrm{Re}(\mathrm{i}E)\mathrm{Re}(\mathrm{i}E_{\bar{\eta}})]_{\bar{\eta}\bar{\eta}},
\end{aligned}
\qquad \text{(B 43)}
$$

and integrating from $X = 0$ to 2π leads to (3.37).

Appendix C. Expressions for the Q_n

The solutions to (3.43)–(3.45) are

$$
Q_1 = -\mathrm{i}\frac{M}{2}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_2}(\xi_2 - \xi_1)^3 C_1(x_1, \bar{\eta}\,|\,\xi_3, \xi_2, \xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \qquad \text{(C 1)}
$$

$$Q_2 = \mathrm{i}\frac{M}{2}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_2}(\xi_2-\xi_1)^3 C_2(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 2}$$

$$Q_3 = \mathrm{i}\frac{M}{2}\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_2}(\xi_2-\xi_1)^2(2\xi_3-\xi_1-\xi_2) C_3(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 3}$$

$$Q_4 = \mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_2}(\xi_3-\xi_2)(\xi_2-\xi_1)^2[C_1(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1) - C_2(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 4}$$

$$Q_5 = -\mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_3}(\xi_2-\xi_1)[(\xi_3-\xi_2)^2 + (\xi_3-\xi_1)^2] C_1(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 5}$$

$$Q_6 = \mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_3}(\xi_3-\xi_1)^2(2\xi_3-\xi_2-\xi_1) C_3(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 6}$$

$$Q_7 = -\mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_3}(\xi_3-\xi_1)(\xi_2-\xi_1)^2 C_1(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 7}$$

$$Q_8 = \mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_3}(\xi_3-\xi_2)(\xi_2-\xi_1)^2 C_1(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 8}$$

and

$$Q_9 = \mathrm{i}M\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\int_{-\infty}^{\xi_3}(\xi_3-\xi_1)(2\xi_3-\xi_2-\xi_1)^2 C_3(x_1,\bar{\eta}\,|\,\xi_3,\xi_2,\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 9}$$

where

$$C_1 \equiv A(\xi_3)A(\xi_2)A^*(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(\xi_3+\xi_2-\xi_1-x_1)}, \tag{C 10}$$

$$C_2 \equiv A(\xi_3)A^*(\xi_2)A(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(\xi_3-\xi_2+\xi_1-x_1)}, \tag{C 11}$$

$$C_3 \equiv A^*(\xi_3)A(\xi_2)A(\xi_1)\mathrm{e}^{\mathrm{i}\bar{Y}(-\xi_3+\xi_2+\xi_1-x_1)}, \tag{C 12}$$

and $M \equiv \alpha_0^3 U_{0\,\eta_c}^3/2c_0^6$.

By using (4.1), one can show that

$$\int_{-\infty}^{+\infty} Q_1 \mathrm{d}\bar{\eta} = -\mathrm{i}N\int_{-\infty}^{x_1}\int_{-\infty}^{\xi_3}\tfrac{1}{2}(x_1-\xi_3)^3 D(x_1\,|\,\xi_3,\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{C 13}$$

$$\int_{-\infty}^{+\infty} Q_4 \mathrm{d}\bar{\eta} = \int_{-\infty}^{+\infty} Q_8 \mathrm{d}\bar{\eta} = \mathrm{i}N \int_{-\infty}^{x_1} \int_{-\infty}^{\xi_3} (x_1 - \xi_3)^2 (\xi_3 - \xi_2) D(x_1 \,|\, \xi_3, \xi_2) \mathrm{d}\xi_2 \mathrm{d}\xi_3, \quad \text{(C 14)}$$

$$\int_{-\infty}^{+\infty} Q_5 \mathrm{d}\bar{\eta} = -\mathrm{i}N \int_{-\infty}^{x_1} \int_{-\infty}^{\xi_3} (x_1 - \xi_3)[(x_1 - \xi_2)^2 + (\xi_3 - \xi_2)^2] D(x_1 \,|\, \xi_3, \xi_2) \mathrm{d}\xi_2 \mathrm{d}\xi_3, \quad \text{(C 15)}$$

$$\int_{-\infty}^{+\infty} Q_7 \mathrm{d}\bar{\eta} = -\mathrm{i}N \int_{-\infty}^{x_1} \int_{-\infty}^{\xi_3} (x_1 - \xi_3)^2 (x_1 - \xi_2) D(x_1 \,|\, \xi_3, \xi_2) \mathrm{d}\xi_2 \mathrm{d}\xi_3, \quad \text{(C 16)}$$

and

$$\int_{-\infty}^{+\infty} Q_2 \mathrm{d}\bar{\eta} = \int_{-\infty}^{+\infty} Q_3 \mathrm{d}\bar{\eta} = \int_{-\infty}^{+\infty} Q_6 \mathrm{d}\bar{\eta} = \int_{-\infty}^{+\infty} Q_9 \mathrm{d}\bar{\eta} = 0, \quad \text{(C 17)}$$

where

$$D \equiv A(\xi_3) A(\xi_2) A^*(\xi_3 + \xi_2 - x_1), \quad \text{(C 18)}$$

and $N \equiv \pi \alpha_0^2 U_{0\eta_c}^2 / c_0^5$.

Appendix D. Mean-flow distortion

In this appendix, the solution for the mean-flow distortion generated by the critical-layer nonlinearity is analyzed. When the mean-flow distortion terms are made explicit in (1.19) and (1.20), these equations become

$$\acute{u} = \mathrm{Re}\left(A\hat{u}\mathrm{e}^{\mathrm{i}X}\right) + \mathrm{Re}\left(iB\frac{\check{\bar{u}}}{gh} + l\sigma B'\frac{\check{\bar{v}}}{h} + m\sigma B'\frac{\check{\bar{w}}}{g}\right) + \ldots, \quad \text{(D 1)}$$

$$\acute{p} = \mathrm{Re}\left(A\hat{p}\mathrm{e}^{\mathrm{i}X}\right) + \mathrm{Re}(\sigma^2 B''\check{p}) + \ldots, \quad \text{(D 2)}$$

where $B(x_1)$ is a slowly varying amplitude function and the functions $\check{\bar{u}}$, $\check{\bar{v}}$, $\check{\bar{w}}$ and $\check{p}$ of x_1, y and z expand like

$$\{\check{\bar{u}}, \check{\bar{v}}, \check{\bar{w}}, \check{p}\} = \{\check{\bar{u}}_0, \check{\bar{v}}_0, \check{\bar{w}}_0, \check{p}_0\}(y, z) + \cdots, \quad \text{(D 3)}$$

as $\sigma \to 0$. Substituting (D 1)–(D 3) into (1.15)–(1.17) shows that $\check{p}_0$ satisfies the 'steady' Rayleigh equation

$$\nabla_T \cdot \left(\frac{\nabla_T \check{p}_0}{U_0^2}\right) = 0, \quad \text{(D 4)}$$

while the velocity fluctuations are determined in terms of $\check{p}_0$ by

$$\{\check{\tilde{u}}_0, \check{\tilde{v}}_0, \check{\tilde{w}}_0\} = \frac{1}{U_0}\left\{\frac{hU_{0\eta}}{gU_0}\check{p}_{0\eta}, \; -\frac{h}{g}\check{p}_{0\eta}, \; -\frac{g}{h}\check{p}_{0\zeta}\right\}. \tag{D 5}$$

Near the critical level, $\check{p}_0$ expands like

$$\check{p}_0 = r_{00} + r_{01}^{\pm}(\eta - \eta_c) + \cdots \tag{D 6}$$

where (3.21), (B 8) and (B 25) have been used to conclude that the mean pressure fluctuation is continuous across the critical layer to $O(\sigma^2\epsilon)$. It follows from (D 5) that the discontinuity in (D 6) leads to a jump in the streamwise velocity component

$$\Delta\check{\tilde{u}}_0 = \frac{h_c U_{0\eta_c}}{g_c c_0^2}(r_{01}^{+} - r_{01}^{-}) \tag{D 7}$$

across the critical layer. Matching this jump with (4.2) yields

$$r_{01}^{+} - r_{01}^{-} = -2\pi\frac{g_c\gamma_{2\zeta}}{h_c c_0}, \tag{D 8}$$

and the amplitude equation (4.4).

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It is well known that the presence of a weak cross flow in an otherwise two-dimensional shear flow results in a spanwise variation in the mean streamwise velocity profile that can lead to an amplification of certain three-dimensional disturbances through a kind of resonant-interaction mechanism (Goldstein and Wundrow 1994). The spatial evolution of an initially linear, finite-growth-rate, instability wave in such a spanwise-varying shear flow is considered, The base flow, which is governed by the three-dimensional parabolized Navier-Stokes equations, is initiated by imposing a spanwise-periodic cross-flow velocity on an otherwise two-dimensional shear flow at some fixed streamwise location. The resulting mean-flow distortion initially grows with increasing streamwise distance, reaches a maximum and eventually decays through the action of viscosity. This decay, which coincides with the viscous spread of of the shear layer, means that the local growth rate of the instability wave will eventually decrease as the wave propagates downstream. Nonlinear effects can then become important within a thin spanwise-modulated critical later once the local instability-wave amplitude and growth rate become sufficiently large and small, respectively. The amplitude equation that describes this stage of evolution is shown to be a generalization of the one obtained by Goldstein and Choi (1989) who considered the related problem of the interaction of two oblique modes in a two-dimensional shear layer.

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