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On the transition towards slow manifold in shallow-water and 3D Euler equations in a rotating frame

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The long-time, asymptotic state of rotating homogeneous shallow-water equations is investigated. Our analysis is based on long-time averaged rotating shallow-water equations describing interactions of large-scale, horizontal, two-dimensional motions with surface inertial-gravity waves field for a shallow, uniformly rotating fluid layer. These equations are obtained in two steps: first by introducing a Poincaré/Kelvin linear propagator directly into classical shallow-water equations, then by averaging. The averaged equations describe interaction of wave fields with large-scale motions on time scales long compared to the time scale $1/f_0$ introduced by rotation (f_0/2-angular velocity of background rotation). The present analysis is similar to the one presented by Waleffe (1991) for 3D Euler equations in a rotating frame. However, since three-wave interactions in rotating shallow-water equations are forbidden, the final equations describing the asymptotic state are simplified considerably. Special emphasis is given to a new conservation law found in the asymptotic state and de-coupling of the dynamics of the divergence free part of the velocity field. The possible rising of a de-coupled dynamics in the asymptotic state is also investigated for homogeneous turbulence subjected to a background rotation. In our analysis we use long-time expansion, where the velocity field is decomposed into the 'slow manifold' part (the manifold which is unaffected by the linear 'rapid' effects of rotation or the inertial waves) and a formal 3D disturbance. We derive the physical space version of the long-time averaged equations and consider an invariant, basis-free derivation. This formulation can be used to generalize Waleffe's (1991) helical decomposition to viscous inhomogeneous flows (e.g. problems in cylindrical geometry with no-slip boundary conditions on the cylinder surface and homogeneous in the vertical direction).

1. Introduction

There are many important engineering (e.g. turbomachinery, ship propellers) and geophysical (geophysical turbulence) problems in which rotation significantly modify the turbulence properties of fluid flows. The interest in the effects of rotation is reflected in the large body of theoretical, experimental, and numerical work documenting them (Hopfinger 1989). As the effects are both multi-fold and subtle, the development of models which account for the effects of rotation require an understanding of the processes occurring in these flows. The objective of this work

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is to examine the long-time evolution of solutions of shallow-water and 3D Euler equations in a rotating frame. We are especially interested in possible de-coupling of dynamics on slow manifold and in developing new conservation laws valid in the asymptotic state.

The paper is organized as follows. The long-time averaged rotating shallow-water equations are presented in Section 2. Special emphasis is given to a new conservation law found in the asymptotic state and to the de-coupling of dynamics of the divergence free part of the velocity field. Section 3 is devoted to homogeneous turbulence subjected to uniform rotation. In our analysis we use long-time expansion, where the velocity field is decomposed into the 'slow manifold' part (the manifold which is unaffected by the linear 'rapid' effects of rotation or the inertial waves) and a formal 3D disturbance. We derive the physical space version of the long-time averaged equations and consider an invariant, basis-free derivation. This can be useful in generalizations of Waleffe (1991) helical decomposition to viscous inhomogeneous flows (e.g. problems in cylindrical geometry with no-slip boundary conditions on the cylinder surface and homogeneous in the vertical direction). Finally, the possible rising of a de-coupled dynamics of the slow manifold is investigated. This study was motivated by the previous studies of Squires *et al.* (1993) and Cambon *et al.* (1994).

2. Rotating shallow-water equations

It has been long realized that geophysical flows admit motions varying on different time scales. In many situations, the low frequency class contains the majority of energy and is therefore the class of main interest. However, long-time computation of this class of motions using unmodified Eulerian equations is prohibitive due to severe accuracy and time step restrictions (Browning *et al.* 1990). The disparity of time scales leads to problems in the numerical solution of the equations because the Courant number is determined by the fastest time scale and therefore limits the time step which makes explicit solution impractical. In our approach collective contribution to the dynamics made by fast wave motions is accounted for by an averaging procedure. Elimination of "fast" time scales through the averaging procedure leads to equations that more accurately and efficiently describe dynamics on long time scales. These equations are obtained in two steps: first by introducing a Poincaré/Kelvin linear propagator directly into classical shallow-water equations, then by averaging. Averaged equations describe interaction of wave fields with large-scale motions on time scales long compared to a time scale $1/f_0$ introduced by rotation ($f_0/2$-angular velocity of background rotation). These are fully nonlinear equations containing classical quasigeostrophic equation as a completely decoupled subsystem. Two other equations describing departure from quasigeostrophy are coupled to the latter equation. Special emphasis is given to a new conservation law found in the asymptotic state and de-coupling of dynamics of the divergence free part of the velocity field.

The equations for shallow-water are:

$$[\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla]\mathbf{U} = -g\nabla H + f_0 \mathbf{U} \times e_3, \tag{1}$$

$$\frac{\partial}{\partial t}H = -\nabla \cdot [\mathbf{U}H] \tag{2}$$

where ∇ and $\mathbf{U}$ are two-dimensional and in the (x_1, x_2)-plane, $\mathbf{U}$ is the velocity, g is the reduced acceleration of gravity (a constant), H is a function of x_1, x_2, and t and is the free-surface height of the shallow layer of fluid, $\hat{H}_0$ is a constant and is the mean depth of the layer, e_3 is the unit vector in x_3, and f_0 is the Coriolis parameter. Equations (1) and (2) with boundary conditions describe the full system. The system (1)-(2) could also be thought of as describing a two-dimensional compressible gas, with H being the density. Note that (1) and (2) imply

$$[\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla]Q = 0 \tag{3}$$

where Q is the potential vorticity

$$Q \equiv \frac{f_0 + \Omega}{H} \tag{4}$$

where Ω is the vorticity. Note that Eqs. (1), (3), and (4) are also a complete description of shallow-water motion. The fast time scale that we average over is $1/f_0$. Here we treat f_0 as constant, though later we will modify it to be a function of x_2 or in polar coordinates a function of r.

2.1 Linear propagator

We have for the linear propagator

$$\frac{\partial \mathbf{U}}{\partial t} = -g\nabla H + f_0 \mathbf{U} \times e_3, \quad \frac{\partial H}{\partial t} = -\hat{H}_0 \nabla \cdot \mathbf{U}. \tag{5}$$

Introducing $D = \nabla \cdot \mathbf{U}$ and $\Omega = curl\ \mathbf{U} \cdot e_3$, we obtain

$$\frac{\partial D}{\partial t} = -g\Delta H + f_0\Omega, \quad \frac{\partial \Omega}{\partial t} = -f_0 D, \quad \frac{\partial H}{\partial t} = -\hat{H}_0 D. \tag{6}$$

In the case of homogeneous flows we use Fourier representation (for simplicity we assume 2π periodicity in x_1 and x_2)

$$D = \sum_k e^{ik\cdot x} D_k, \quad \Omega = \sum_k e^{ik\cdot x}\Omega_k, \quad H = \sum_k e^{ik\cdot x} H_k$$

where $k = (k_1, k_2)$, $x = (x_1, x_2)$. Then in Fourier space Eq. (6) becomes

$$\frac{\partial D_k}{\partial t} = g|k|^2 H_k + f_0\Omega_k, \quad \frac{\partial \Omega_k}{\partial t} = -f_0 D_k, \quad \frac{\partial H_k}{\partial t} = -\hat{H}_0 D_k. \tag{7}$$

We supplement (7) with initial conditions

$$D_k(0) = D_k^0, \quad \Omega_k(0) = \Omega_k^0, \quad H_k(0) = H_k^0. \tag{8}$$

Solutions of the linear propagator Eqs. (7)-(8) are well-known (e.g. Pedlosky 1979); those corresponding to the free surface gravity waves represent the fast time scale. Defining $\phi_k = f_0\sqrt{1+L_R^2|k|^2}$ where $L_R^2 = g\hat{H}_0/f_0^2$ is the Rossby deformation radius, we have

$$\begin{pmatrix} D_k(t) \\ \Omega_k(t) \\ H_k(t) \end{pmatrix} = (\mathbf{A}(k) + \cos(\phi_k t)\mathbf{A}_c(k) + \sin(\phi_k t)\mathbf{A}_s(k)) \begin{pmatrix} D_k^0 \\ \Omega_k^0 \\ H_k^0 \end{pmatrix} \equiv \exp(\mathbf{L}t) \begin{pmatrix} D_k^0 \\ \Omega_k^0 \\ H_k^0 \end{pmatrix}. \tag{9}$$

Here $\mathbf{A}(k)$, $\mathbf{A}_c(k)$ and $\mathbf{A}_s(k)$ are 3×3 matrices defined as follows

$$\mathbf{A}(k) = \frac{1}{\phi_k^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & g\hat{H}_0|k|^2 & -f_0 g|k|^2 \\ 0 & -\hat{H}_0 f_0 & f_0^2 \end{pmatrix}, \quad \mathbf{A}_c(k) = \frac{1}{\phi_k^2}\begin{pmatrix} \phi_k^2 & 0 & 0 \\ 0 & f_0^2 & f_0 g|k|^2 \\ 0 & \hat{H}_0 f_0 & g\hat{H}_0|k|^2 \end{pmatrix},$$
$$\mathbf{A}_s(k) = \frac{1}{\phi_k}\begin{pmatrix} 0 & f_0 & g|k|^2 \\ -f_0 & 0 & 0 \\ -\hat{H}_0 & 0 & 0 \end{pmatrix}. \tag{10}$$

2.2 The asymptotic limit of rotating homogeneous shallow-water equations

We write classical rotating shallow-water equations (1)-(2) using the variables $D = \nabla\cdot\mathbf{U}$, $\Omega = curl\mathbf{U}\cdot e_3$ and H. Letting $\mathbf{V}^{tr} = (D, \Omega, H)$ (tr- transpose), these equations can be written symbolically in the form

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{L}\mathbf{V} + \mathbf{B}(\mathbf{V},\mathbf{V}). \tag{11}$$

Here $\mathbf{L}$ is the linear propagator operator corresponding to Poincaré/Kelvin waves. We introduce this linear propagator directly into nonlinearity using the change of variables $\mathbf{V} = \exp(\mathbf{L}t)\mathbf{v}$ where $\mathbf{v}^{tr} = (d, \omega, h)$ and $\exp(\mathbf{L}t)$ is defined by (9). Equation (11) written in $\mathbf{v}$ variables has the form

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{B}(t, \mathbf{v}, \mathbf{v}) \tag{12}$$

where

$$\mathbf{B}(t,\mathbf{v},\mathbf{v}) = \exp(-\mathbf{L}t)\mathbf{B}(\exp(\mathbf{L}t)\mathbf{v}, \exp(\mathbf{L}t)\mathbf{v}).$$

Equation (12) is explicitly time-dependent with rapidly varying coefficients. Using the standard averaging methods (e.g. Hale 1980), we average over the fast time scale of the gravity waves and obtain

$$\frac{\partial \mathbf{v}}{\partial t} = \tilde{\mathbf{B}}(\mathbf{v},\mathbf{v}), \ \tilde{\mathbf{B}}(\mathbf{v},\mathbf{v}) = \frac{1}{2T}\int_{-T}^{T} \mathbf{B}(s,\mathbf{v},\ \mathbf{v})ds. \tag{13}$$

These equations are exact in the limit $\mu = f_0 T \gg 1$ where μ is a non-dimensional parameter. We present explicit final form of the regularized (long-time averaged) Eqs. (13). The calculation leading to these equations is lengthy but straightforward. Here we present only the final result for periodic case. We use the following notations: m and k are wavevectors so that $m = (m_1, m_2)$, $k = (k_1, k_2)$; $(mke_3) = (m \times k) \cdot e_3 = m_1 k_2 - k_1 m_2$; $k \cdot x = k_1 x_1 + k_2 x_2$ and $m \cdot k = m_1 k_1 + m_2 k_2$; $\phi_k^2 = f_0^2 + g\hat{H}_0 |k|^2$. The regularized (long-time averaged) equations are conveniently written in new variables

$$A = d, \; B = f_0 \omega - g\Delta h, \; C = \hat{H}_0 \omega - f_0 h \tag{14}$$

where Δ is the Laplace operator. Physical interpretation of these variables is as follows. $C/\hat{H}_0$ is quasigeostrophic potential vorticity; $g\Delta h / f_0$ is quasigeostrophic component of vorticity and, therefore, B/f_0 is ageostrophic component of vorticity. In the quasigeostrophic case we have $A = B = 0$. Let

$$A = \sum_n e^{in\cdot x} A_n, \quad B = \sum_n e^{in\cdot x} B_n, \quad C = \sum_n e^{in\cdot x} C_n. \tag{15}$$

We note that three-wave interactions are forbidden since the equation $\pm\phi_k \pm \phi_m = \phi_n$ $(n = k+m)$ has no solutions. Then the final equations describing the asymptotic state are much simpler than in the case of 3D Euler equations considered by Waleffe (1991). We obtain after averaging (Eq. (13) in A, B, C variables)

$$\begin{aligned}
\frac{dA_n}{dt} &= \sum_{k:|k|=|n|} C_{n-k}(\alpha(k, n-k)A_k + \beta(k, n-k)B_k), \\
\frac{dB_n}{dt} &= \sum_{k:|k|=|n|} C_{n-k}\left(-\phi_n^2 \beta(k, n-k)A_k + \frac{\phi_n^2}{\phi_k^2}\alpha(k, n-k)B_k\right), \\
\frac{dC_n}{dt} &= \sum_{k:\text{no restriction}} \phi_n^2 \gamma(k, n-k) C_{n-k} C_k.
\end{aligned} \tag{16}$$

Here

$$\alpha(k,m) = \frac{g(mke_3)(2\phi_k^2(m\cdot k) + f_0^2(|k|^2 - |m|^2))}{2|k|^2\phi_k^2\phi_m^2},$$

$$\beta(k,m) = \frac{f_0 g(|m|^4 - |k|^4 - \frac{5}{2}|m|^2|k|^2)}{2|k|^2\phi_k^2\phi_m^2}, \quad \gamma(k,m) = \frac{g(mke_3)}{\phi_m^2 \phi_{k+m}^2} \tag{17}$$

are geometric parameters. Equations (16) are exact in the limit $\mu = f_0 T \gg 1$. They have property that the third equation is completely decoupled from the first two equations. After the third equation is solved, the first two equations become linear equations in A and B. From the third equation in (16), it can be easily shown that $C(t,x)$ obeys in physical space

$$\frac{\tilde{D}}{\tilde{D}t} C = 0 \tag{18}$$

where $\frac{\tilde{D}}{Dt}$ is the advective derivative, based on the velocity $\tilde{\mathbf{v}} = e_3 \times \nabla\tilde{\psi}$ where $\tilde{\psi} \equiv (1 - L_R^2\Delta)^{-1}C/\hat{H}_0$. Note that in the quasigeostrophic limit, $\tilde{\psi}$ is the stream function associated with the quasigeostrophic potential vorticity $C/\hat{H}_0$, and $\tilde{\mathbf{v}}$ is the quasigeostrophic velocity. The above Eq. (18) implies that C and all of its moments are conserved because $\tilde{\mathbf{v}}$ is divergence free (a well-known fact within the context of two-dimensional incompressible turbulence). For a given geostrophic component $C(t, x)$ found from (18), equations for ageostrophic components A and B have the form

$$\begin{aligned} \frac{dA_n}{dt} &= \sum_{k:|k|=|n|} C_{n-k}(\alpha(k, n-k)A_k + \beta(k, n-k)B_k), \\ \frac{dB_n}{dt} &= \sum_{k:|k|=|n|} C_{n-k}(-\phi_n^2\beta(k, n-k)A_k + \frac{\phi_n^2}{\phi_k^2}\alpha(k, n-k)B_k). \end{aligned} \tag{19}$$

It can be easily shown that these equations have a new conservation law

$$M = \sum_n (\phi_n^2|A_n|^2 + |B_n|^2). \tag{20}$$

The conservation of M is proved by direct differentiation of (20), substitution of expressions for dA_n/dt and dB_n/dt from (19) and using properties of the geometric coefficients $\alpha(k, m)$ and $\beta(k, m)$ given by (17). Another approach for determining this conservation law would be to derive the long-time averaged equations from a variational principle. Existence of conserved quantity M allows us to control the size of ageostrophic component in terms of its initial value. One could use this conservation law to examine the Lyapunov stability conditions for the equilibrium solutions of the long-time averaged equations. We note that M is not conserved by the full rotating shallow-water Eqs. (1)-(2).

3. On the asymptotic limit of 3D Euler equations in a rotating frame

In a frame of reference rotating with constant angular velocity $f_0/2$ about the x_3 axis, the inviscid Euler equations have the form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} + f_0\mathbf{J}\mathbf{U} = -\nabla p, \quad \nabla\cdot\mathbf{U} = 0. \tag{21}$$

Here $\mathbf{U} = (U, V, W)$ is the velocity field, p is modified pressure and $\mathbf{J}$ is the rotation matrix ($\mathbf{J}\mathbf{U} = e_3 \times \mathbf{U}$). It is convenient to write (21) in the form

$$\frac{\partial \mathbf{U}}{\partial t} + f_0\mathbf{J}\mathbf{U} = \mathbf{U} \times curl\mathbf{U} - \nabla(p + |\mathbf{U}|^2/2), \quad \nabla\cdot\mathbf{U} = 0. \tag{22}$$

We present a coordinate-free derivation of long-time averaged Euler equations describing dynamics on time scales long compared to the time scale $1/f_0$ introduced by rotation. This can be useful in generalizations of Waleffe (1991) helical decomposition to viscous inhomogeneous flows (e.g. problems in cylindrical geometry with no-slip boundary conditions on the cylinder surface and homogeneous in the vertical direction).

We define the Leray projection $\mathbf{P}$ (projection on divergence free vector fields). Then Eq. (22) can be written in operator form as follows

$$\frac{d\mathbf{U}}{dt} + f_0(\mathbf{PJ})\mathbf{U} = \mathbf{P}\{\mathbf{U} \times curl\mathbf{U}\}. \tag{23}$$

We introduce a change of variables (canonical, preserving both energy and helicity)

$$\mathbf{U} = \exp(-f_0\mathbf{PJ}t)\mathbf{u}. \tag{24}$$

This transformation is analogous to the van der Pol transformation widely used in the theory of nonlinear oscillators (e.g. Wiggins 1990). Equation (23) written in $\mathbf{u}$-variables now becomes

$$\frac{d\mathbf{u}}{dt} = \exp(f_0\mathbf{PJ}t)\mathbf{P}\{\exp(-f_0\mathbf{PJ}t)\mathbf{u} \times curl(\exp(-f_0\mathbf{PJ}t)\mathbf{u})\}. \tag{25}$$

We note that the operator $\mathbf{PJ}$ has the following representation ($\mathbf{A} = curl^2$ is the Stokes operator)

$$\mathbf{PJ} = -\mathbf{A}^{-1}\frac{\partial}{\partial x_3}curl. \tag{26}$$

This representation in the periodic (homogeneous) case reduces to $\mathbf{PJ}(e^{ik\cdot x}\mathbf{u}_k) = e^{ik\cdot x}\frac{k_3}{|k|^2}k \times \mathbf{u}_k \equiv e^{ik\cdot x}\frac{k_3}{|k|^2}\mathbf{R}(k)\mathbf{u}_k$ where $\mathbf{u}_k$ is the Fourier coefficient corresponding to wavevector $k = (k_1, k_2, k_3)$ and $\mathbf{R}(k)\mathbf{u}_k = k \times \mathbf{u}_k$. The operator $\mathbf{R}(k)$ becomes diagonal in the helical basis described in Waleffe (1991) (Eq. (3) in his paper). In the case of horizontally inhomogeneous flows a more general representation (26) should be used. From (26) we obtain

$$\exp(f_0\mathbf{PJ}t) = \mathbf{C}(t) - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl \tag{27}$$

and

$$curl\exp(f_0\mathbf{PJ}t) = \mathbf{C}(t)curl - \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t) \tag{28}$$

where

$$\mathbf{C}(t) = \cosh(f_0 t\mathbf{A}^{-\frac{1}{2}}\frac{\partial}{\partial x_3}), \quad \mathbf{S}(t) = \sinh(f_0 t\mathbf{A}^{-\frac{1}{2}}\frac{\partial}{\partial x_3}).$$

We note that $\mathbf{C}(t)$ is an even and $\mathbf{S}(t)$ is an odd operator-valued function. Using (27), we have for the nonlinear term in (25)

$$\begin{aligned}\exp(f_0\mathbf{PJ}t)\mathbf{P}\{\mathbf{U} \times curl\mathbf{U}\} &= (\mathbf{C}(t) - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl)\mathbf{P}\{\mathbf{U} \times curl\mathbf{U}\} = \\ &\mathbf{C}(t)\{\mathbf{U} \times curl\mathbf{U}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{U}, curl\mathbf{U}] - \nabla\mathbf{q}\end{aligned} \tag{29}$$

since $curl(\mathbf{U} \times curl\mathbf{U}) = [\mathbf{U}, curl\mathbf{U}]$. Here $[\ ,\ ]$ is a commutator of two vector fields: $[\mathbf{U}, curl\mathbf{U}] \equiv (\mathbf{U} \cdot \nabla)curl\mathbf{U} - (curl\mathbf{U} \cdot \nabla)\mathbf{U}$; q is a modified pressure. From (24), (27) and (28) we find that

$$\mathbf{U} = \mathbf{C}(t)\mathbf{u} + \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u}, \quad curl\mathbf{U} = \mathbf{C}(t)curl\mathbf{u} + \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}.$$

We substitute these expressions in (29). Then equation (25) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{B}(t, \mathbf{u}, \mathbf{u}) - \nabla q, \quad \nabla \cdot \mathbf{u} = 0. \tag{30}$$

Here $\mathbf{B}(t, \mathbf{u}, \mathbf{u})$ is given by

$$\mathbf{B}(t, \mathbf{u}, \mathbf{u}) = \mathbf{C}(t)\{\mathbf{U} \times curl\mathbf{U}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{U}, curl\mathbf{U}] =$$

$$\mathbf{C}(t)\{\mathbf{C}(t)\mathbf{u} \times \mathbf{C}(t)curl\mathbf{u}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{C}(t)\mathbf{u}, \mathbf{C}(t)curl\mathbf{u}] +$$

$$\mathbf{C}(t)\{\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u} \times \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u}, \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}] + \tag{31}$$

$$\mathbf{C}(t)\{\mathbf{C}(t)\mathbf{u} \times \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{C}(t)\mathbf{u}, \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}] +$$

$$\mathbf{C}(t)\{\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u} \times \mathbf{C}(t)curl\mathbf{u}\} - \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u}, \mathbf{C}(t)curl\mathbf{u}].$$

Separation of "fast" and "slow" oscillations in (31) is equivalent to a search for resonances which is a geometry dependent problem. One immediate simplification is that odd terms make no contribution to the time-averaged equations. Thus averaging of (30)-(31) reduces to averaging of four even terms

$$\begin{aligned} &\mathbf{C}(t)\{\mathbf{C}(t)\mathbf{u} \times \mathbf{C}(t)curl\mathbf{u}\}, \quad \mathbf{C}(t)\{\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u} \times \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}\}, \\ &\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{C}(t)\mathbf{u}, \mathbf{A}^{\frac{1}{2}}\mathbf{S}(t)\mathbf{u}], \quad \mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)[\mathbf{A}^{-\frac{1}{2}}\mathbf{S}(t)curl\mathbf{u}, \mathbf{C}(t)curl\mathbf{u}]. \end{aligned} \tag{32}$$

It is convenient to work in the basis of the Stokes operator $\mathbf{A}$. The operators $\mathbf{C}(t)$ and $\mathbf{S}(t)$ have a block-diagonal form in this basis. Since $\mathbf{C}(t) = \cosh(f_0 t \mathbf{A}^{-\frac{1}{2}} \frac{\partial}{\partial x_3})$ and $\mathbf{S}(t) = \sinh(f_0 t \mathbf{A}^{-\frac{1}{2}} \frac{\partial}{\partial x_3})$, they become simple oscillations in the basis of $\mathbf{A}$. These oscillations have frequencies $f_0 \frac{j_3}{\sqrt{\lambda_j}}$ where λ_j are the eigenvalues of $\mathbf{A}$ and j_3 is axial wavenumber. Then resonances in (32) are given by

$$\pm \frac{k_3}{\sqrt{\lambda_k}} \pm \frac{m_3}{\sqrt{\lambda_m}} = \frac{n_3}{\sqrt{\lambda_n}}.$$

In the case of periodic (homogeneous) flows the helical basis formed by eigenfunctions of the *curl* operator acting in the space of divergence free vector fields is very useful. The operator $\mathbf{PJ}$ ($\frac{k_3}{|k|^2}\mathbf{R}(k)$ in Fourier space) becomes diagonal in this basis. In this case we have $\lambda_j = |j|$ where $j = (j_1, j_2, j_3)$ is the wavevector. Then averaging of (30) leads to equations studied by Waleffe (1991). In particular, he argued that in the asymptotic state of rotating homogeneous turbulence interactions are restricted to pairs of wavevectors satisfying resonance conditions ($n = k + m$)

$$\pm\frac{k_3}{|k|} \pm \frac{m_3}{|m|} = \frac{n_3}{|n|}. \tag{33}$$

Now we consider homogeneous (periodic in three directions) flows and discuss equations describing the asymptotic limit of 3D Euler equations in a rotating frame. Both computations and experiments have noted an increase in integral length scales along the rotation axis relative to those in non-rotating turbulence. Increase in the integral length scales has been thought to be a prelude to a Taylor-Proudman reorganization to two-dimensional turbulence. It has been also suggested that this process is self-similar (Squires *et al.* 1993). The 'slow manifold' interactions (interactions which are unaffected by the linear 'rapid' effects of rotation or the inertial waves) are obtained in (33) with $k_3 = m_3 = 0$. These interactions are invariant under scalings of k_3, m_3 in a trivial way. It is reasonable to conjecture that at zero Rossby number the only acting triads in the asymptotic state are those which are invariant under the scalings $k_3 \to \xi k_3$, $m_3 \to \eta m_3$ where ξ and η are arbitrary real numbers. This is called the ***selection principle for triad interactions in the asymptotic state.*** It is inspired by the observations of Squires *et al.* (1993) on self-similar asymptotic states of rotating homogeneous turbulence. One can easily find solutions of the resonant Eq. (33) obeying the selection principle. These are $\{m_3 = 0, |n| = |k|\}$, $\{k_3 = 0, |n| = |m|\}$ and $\{n_3 = 0, |k| = |m|\}$ with the $+, -$ signs chosen appropriately. Since these interactions are unaffected by scalings of the vertical wavenumber, they remain active even at zero Rossby number. It is possible to show that the interactions $\{n_3 = 0, |k| = |m|\}$ make no contribution to the averaged equations. Defining $k_\perp = (k_1, k_2)$ to be the horizontal wavevector perpendicular to the rotation axis, interactions active in the asymptotic state can be written in the form $\{k_3 = m_3 = 0\}$, $\{m_3 = 0, |n_\perp| = |k_\perp|\}$ and $\{k_3 = 0, |n_\perp| = |m_\perp|\}$. The form of these interaction manifolds suggests splitting the velocity field into two parts, each separately related to the horizontal or vertical velocities. We proceed to define splitting by first defining ω to be vertical (in the direction of rotation) component of the vorticity vector. Then we define splitting of the total velocity $\mathbf{u}$ as follows

$$\mathbf{u} = \tilde{\mathbf{v}} + \mathbf{v} \quad \textit{where} \quad \tilde{\mathbf{v}} = e_3 \times \nabla\tilde{\psi}$$

and $\tilde{\psi}$ is related to ω by the equation $-\Delta\tilde{\psi} = \omega$. Applying averaging to the transformed Euler Eq. (30) (we note that only averaging of the even terms given by (32) is required) and using the selection principle, we obtain equations valid in the asymptotic state. Our analysis is similar to the one presented in Section 2.2 where de-coupling of the divergence free part of the total shallow-water velocity field was rigorously proved (see Eq. (18)). We find that in the asymptotic limit $\omega(t,x)$ obeys in physical space

$$\frac{\tilde{D}}{\tilde{D}t}\omega = 0 \qquad (34)$$

where $\frac{\tilde{D}}{\tilde{D}t}$ is the advective derivative, based on the velocity $\tilde{\mathbf{v}} = e_3 \times \nabla\tilde{\psi}$ where $-\Delta\tilde{\psi} = \omega$. This equation should be compared with Eq. (18). Thus we obtain de-coupling (in the asymptotic state) of the dynamics of $\tilde{\mathbf{v}}$ (two-component and horizontally divergence free). It satisfies classical 2D Euler equations (34). The de-coupling which is linked to the $\tilde{\mathbf{v}}$ part of the total velocity can reflect the rise of an inverse energy cascade. Our future plans include confirmation of inverse cascade using the CTR database. Another avenue is to search for new conservation laws in the asymptotic limit of rotating homogeneous 3D Euler equations similar to the one we found for the long-time averaged shallow-water equations (Section 2.2, Eq. (20)). One possible approach for determining new conservation laws would be to derive the long-time averaged Euler equations from a variational principle.

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