N95- 22968

GEOMETRICAL INTERPRETATION
FOR THE
$SU(3)$ OUTER MULTIPLICITY LABEL

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Abstract

A geometrical interpretation for the outer multiplicity ρ that occurs in a reduction of the product of two $SU(3)$ representations, $(\lambda_\pi, \mu_\pi) \times (\lambda_\nu, \mu_\nu) \to \sum_\rho (\lambda, \mu)_\rho$, is introduced. This coupling of proton (π) and neutron (ν) representations arises, for example, in both boson and fermion descriptions of heavy deformed nuclei. Attributing a geometry to the coupling raises the possibility of introducing a simple interaction that provides a physically meaningful way for distinguishing multiple occurrences of (λ, μ) values that can arise in such products.

1   Introduction

The objective of our program in nuclear structure physics has been to bridge the gap that exists between collective and shell-model descriptions of observed nuclear phenomena. Progress has been slow because of the difficulty in making realistic shell-model calculations, at least when measured against the background of the success of simpler collective models. Algebraic shell-model theories come closest to realizing this objective. Regarding the latter, there are two basic types of algebraic theories: those based on a boson description of the dynamics, such as the Interacting Boson Model (IBM) [1], and those which treat the nucleons as fermions.

The first and most familiar algebraic fermion model is the Elliott $SU(3)$ scheme. It is known to work well for light ($A \leq 28$) nuclei [2]. Another is the $Sp(3, R)$ (denoted $Sp(6, R)$ sometimes) or symplectic model which is a natural multi-$\hbar\omega$ extension of the Elliott scheme [3]. For heavier systems ($A \geq 150$) there are currently two algebraic models being employed: the so-called Fermion Dynamical Symmetry Model (FDSM) which identifies s and d fermion pair operators that form an algebra which closes under commutation (the $SO(8)$ group for the $n = 4$ shell and $Sp(6)$ for $n = 5$ and $n = 6$, which has $SU(3)$ as a subgroup) and gives a possible microscopic interpretation of the IBM [4], and the pseudo-$SU(3)$ model and its pseudo-symplectic extension which builds on the concept of good pseudo-spin symmetry in heavy nuclei [5, 6, 7].

The common algebraic structure in these theories is the $SU(3)$ group. This is understandable because the angular momentum L and the deformation generating quadrupole operator Q – when restricted to a single major oscillator shell – are generators of $SU(3)$. In particular, large irreducible representations (irreps) of $SU(3)$ correspond to configurations of constant deformation. In

the next section we expand on the $SU(3)$ - rotor connection, and in so doing establish a basis for the geometrical picture of the $SU(3)$ outer multiplicity that is presented in the subsequent section. While no proofs are given, it should be clear from the discussion that the proposed scheme has potentially far reaching consequences regarding a physically motivated interpretation of the outer multiplicity whenever there is an applicable group contraction-expansion procedure, which is $SU(3) \leftrightarrow T_5 \wedge SO(3)$ in the present case. Here $T_5 \wedge SO(3)$ is the symmetry group of the rotor.

2 $SU(3)$ - Rotor Connection

A geometrical interpretation for $SU(3)$ can be achieved by looking at a shell-model interpretation of collective quadrupole motion as depicted in terms of a triaxial quantum rotor. The trick that we apply is to first express the Hamiltonian for the rotor in a frame-independent form because that expression can then be rewritten in terms of its corresponding microscopic operators. The rotor is a particularly elegant example because this prescription is easy to apply and leads immediately to the sought after shell-model representation. Furthermore, the operators that enter into the expression have historical significance, dating back to Racah's pioneering work on the $SU(3) \supset SO(3)$ symmetry group [8]. Since the argument is illustrative it bears repeating, but in an abbreviated form. A more complete description can be found in the book by Casten [9].

The triaxial rotor Hamiltonian is given by

$$H_{ROT} = A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2 \tag{1}$$

where I_α ($\alpha = 1, 2, 3$) is the projection of the total angular momentum on the α-th body-fixed symmetry axis and A_α is the corresponding inertia parameter: $A_\alpha = 1/(2\mathcal{J}_\alpha)$ where $\mathcal{J}_\alpha$ is the moment of inertia about the α-th principal axis. This familiar principal-axis form can be rewritten in a frame-independent representation by introducing three special scalar operators:

$$L^2 = \sum_\alpha L_\alpha L_\alpha = \sum_\alpha I_\alpha^2,$$

$$X_3^c = \sum_{\alpha,\beta} L_\alpha Q_{\alpha\beta}^c L_\beta = \sum_\alpha \lambda_\alpha I_\alpha^2,$$

$$X_4^c = \sum_{\alpha,\beta,\gamma} L_\alpha Q_{\alpha\beta}^c Q_{\beta\gamma}^c L_\gamma = \sum_\alpha \lambda_\alpha^2 I_\alpha^2. \tag{2}$$

The L_α and $Q_{\alpha\beta}^c$ in this equation are Cartesian forms for the total angular momentum and collective quadrupole operators, respectively. (The superscript c appended to the Q denotes the *collective* quadrupole operator which has non-vanishing matrix elements between major shells ($n' = n$, $n \pm 2$), in contrast with the *algebraic* quadrupole operators, $Q_{\alpha\beta}^a$, which have non-vanishing matrix elements only within a major shell, $n' = n$.) The last expression given for each scalar in eq.(2) is the form these operators take in the body-fixed, principal-axis system where the eigenvalues of the $Q_{\alpha\beta}^c$ are presumed to be sharp: $\langle Q_{\alpha\beta}^c \rangle = \lambda_\alpha \delta_{\alpha,\beta}$. These equations can be inverted to yield the I_α^2 in terms of L^2, X_3^c, and X_4^c:

$$I_\alpha^2 = \left[(\lambda_1 \lambda_2 \lambda_3) L^2 + (\lambda_\alpha^2) X_3^c + (\lambda_\alpha) X_4^c \right] / D_\alpha \text{ where } D_\alpha \equiv 2\lambda_\alpha^3 + \lambda_1 \lambda_2 \lambda_3. \tag{3}$$

Substituting this result for the I_α^2 into eq.(1) yields

$$H_{ROT} = aL^2 + bX_3^c + cX_4^c \; ; \tag{4}$$

where a, b and c depend on the inertia parameters and the eigenvalues of $Q_{\alpha\beta}^c$:

$$a = \sum_\alpha a_\alpha A_\alpha , \quad b = \sum_\alpha b_\alpha A_\alpha , \quad c = \sum_\alpha c_\alpha A_\alpha , \tag{5}$$

$$a_\alpha = \frac{\lambda_\beta \lambda_\gamma}{2\lambda_\alpha^2 + \lambda_\beta \lambda_\gamma} , \quad b_\alpha = \frac{\lambda_\alpha}{2\lambda_\alpha^2 + \lambda_\beta \lambda_\gamma} , \quad c_\alpha = \frac{1}{2\lambda_\alpha^2 + \lambda_\beta \lambda_\gamma}$$

where $\alpha \neq \beta \neq \gamma \neq \alpha$.

A shell-model image of the rotor Hamiltonian can be obtained by substituting single-particle forms for the collective L_α and $Q_{\alpha\beta}^c$ operators: $L_\alpha = \sum_i l_\alpha(i)$ and $Q_{\alpha\beta}^c = \sum_i q_{\alpha\beta}^c(i)$. However, this ignores the shell structure and the fermion character of the many-body system. It is important to remember that while the L_α have non-vanishing matrix elements only within a major oscillator shell, the $Q_{\alpha\beta}^c$ couple shells differing by two quanta ($n' = n, n \pm 2$). Indeed, the off-diagonal ($n' = n \pm 2$) couplings are about equal in magnitude to the diagonal ($n' = n$) ones. It follows from this that operators like $Q^c \cdot Q^c$ and the X_3^c and X_4^c (even if used only as residual interactions) can destroy the shell structure. This catastrophe can be avoided easily by simply setting all off-diagonal couplings between major shells to zero, an action which corresponds to replacing the $Q_{\alpha\beta}^c$ operators by their algebraic counterparts, $Q_{\alpha\beta}^a$. Elliott was the first person to recognize that the $Q_{\alpha\beta}^a$ operators, along with the L_α, generate $SU(3)$, the symmetry algebra of the isotropic harmonic oscillator Hamiltonian. The appropriate shell-model image of the rotor Hamiltonian, eqs.(1) and (4), is thus given by

$$H_{SU3} = H_0 + aL^2 + bX_3^c + cX_4^c, \tag{6}$$

where H_0 is the harmonic oscillator Hamiltonian.

Shell-model values for the λ_α are required to complete the mapping. This follows by equating invariants of the two theories, a very natural thing to do since constants of the motion relate to the important physics, which in turn should be independent of the particular description. Because $SU(3)$ is a rank two group it has two invariants: C_2 with eigenvalue $[\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu)]$, and C_3 with eigenvalue $[(\lambda - \mu)(\lambda + 2\mu + 3)(2\lambda + \mu + 3)/2]$, where λ and μ are $SU(3)$ representation labels with $(\lambda + \mu)$ and μ, respectively, specifying the number of boxes in the first and second rows in a standard Young diagram labeling of irreps of the $SU(3)$ group. Note that C_2 is of degree two in the generators of $SU(3)$ while C_3 is of degree three. The symmetry group of the rotor $[T_5 \wedge SO(3)]$ also has two invariants: traces of the square $\{\text{Trace}[(Q_2)^2]\}$ and cube $\{\text{Trace}[(Q_2)^3]\}$ of the collective quadrupole matrix. The eigenvalues of these two invariant operator forms are $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \rightarrow (k\beta)^2$ and $\lambda_1 \lambda_2 \lambda_3 \rightarrow (k\beta)^3 \cos(3\gamma)$, respectively, where (β, γ) are the shape variables of the collective model and $k^2 = \frac{5}{9\pi}(A\bar{r}^2)^2$. The requirement of a linear correspondence between these two sets of invariants leads to the following relations,

$$\lambda_1 = -(\lambda - \mu)/3, \quad \lambda_2 = -(\lambda + 2\mu + 3)/3, \quad \lambda_3 = +(2\lambda + \mu + 3)/3. \tag{7}$$

This correspondence, in turn, sets up a direct relationship between the (β, γ) shape variables of the collective model and the (λ, μ) irrep labels of $SU(3)$,

$$\beta^2 = \frac{4\pi}{5(A\bar{r}^2)^2} \left[\lambda^2 + \lambda\mu + \mu^2 + 3(\lambda + \mu) + 3 \right], \quad \gamma = \tan^{-1}\left(\frac{\sqrt{3}(\mu + 1)}{2\lambda + \mu + 3} \right) . \tag{8}$$

Since λ and μ are positive integers, this translates into a regular grid when superimposed on a traditional (β, γ) plot, with β the radius vector and γ the azimuthal angle:

$$k\beta_x = k\beta\cos(\gamma) = \frac{2\lambda + \mu + 3}{3}, \quad k\beta_y = k\beta\sin(\gamma) = \frac{\mu + 1}{\sqrt{3}}. \tag{9}$$

Each (λ, μ)-irrep corresponds to a unique value for the (β, γ)-pair. In the limit of large (λ, μ) values the constant $+3$ factor in λ_2 and λ_3 can be dropped and in so doing one arrives at the asymptotic results [3]. The $+3$ and $+1$ factors in β^2 and γ as well as those in β_x and β_y also disappear in this limit.

3 $SU(3)$ - Outer Mulitiplicity

Having established the $SU(3)$ - rotor connection, it is instructive to push the $(\beta, \gamma) \leftrightarrow (\lambda, \mu)$ connection to a consideration of a coupled double-rotor picture which is commonly used to describe heavy nuclei in a collective model framework, see Figure 1 ahead, with one rotor representing the protons (π) and another the neutrons (ν). This associates physics with the $SU(3)$ coupling picture and, as we will see in greater detail later, it also leads naturally to a geometrical interpretation for the $SU(3)$ outer multiplicity label. This picture also suggests a natural way for parameterizing the proton-neutron interaction in terms of the geometry of this simple scheme, for example, one with final states of the same (λ, μ) but different multiplicity energetically separated from one another due to a simple interaction that senses the relative orientation of the parent proton-neutron configurations. We will return to this matter after making the geometrical picture quantitative for the special case of prolate proton-neutron ($\pi - \nu$) parent configurations.

To get a feeling for the proposed scheme, consider the special case of prolate $\pi - \nu$ factors ($\gamma_\pi = 0$ and $\gamma_\nu = 0$) in the parent configuration. In this case it is sufficient to introduce a single angle θ which measures the relative orientation of the principal axes of the two distributions; rotations about either the proton or neutron symmetry axis effect no change, only rotations about an axis that is perpendicular to the plane defined by the principal axes are distinguishable. (The scissors mode used to describe $B(M1)$ strengths gets its name from this simple picture ... θ measures the angle between the two blades of the scissors. Also note that the Exclusion Principle, which applies because the nucleons are considered to be fermions, is not violated by the coupling because the two distributions are made up of different particle types.) For $\theta = 0^o$ the two axially symmetric ellipsoids overlap maximally (aligned principal axes) whereas when $\theta = 90^o$ the principal axes are perpendicular to one another and the resulting overlap is a minimum.

The (β, γ) value of the product can be determined once β_π, β_ν and θ are specified. Recall that β and γ are determined respectively by the trace of the square and cube of the quadrupole matrix, see eq.(8), and that the quadrupole matrix of the joint distribution is just the sum of the separate proton and neutron distributions, with the second (Q_ν) rotated by an angle θ relative to the first (Q_π): $Q = Q_\pi + RQ_\nu R^{-1}$ where $R = \exp(i\theta \cdot \hat{n})$ and $\hat{n}$ points in the direction of $\hat{n}_\pi \times \hat{n}_\nu$ with $\hat{n}_\pi$ and $\hat{n}_\nu$ defined to be unit vectors that point respectively along the proton and neutron symmetry axes. Or vice-versa, given β_ν, β_π and (β, γ) one can clearly deduce the relative orientation angle θ. This construction corresponds to the $(\lambda_\pi, \mu_\pi = 0) \otimes (\lambda_\nu, \mu_\nu = 0) \rightarrow \sum \oplus (\lambda, \mu)$ coupling in the $SU(3)$ case which is known to be simply reducible, that is, each of the allowed (λ, μ) irreps in the product $[(\lambda, \mu) = (\lambda_\pi + \lambda_\nu, 0), (\lambda_\pi + \lambda_\nu - 2, 1), (\lambda_\pi + \lambda_\nu - 4, 2), \ldots, (\lambda_> - \lambda_<, \lambda_<)$, where

$\lambda_> = \max(\lambda_\pi, \lambda_\nu)$ and $\lambda_< = \min(\lambda_\pi, \lambda_\nu)$], occurs once and only once. Arguing by analogy with the collective model picture, it is relatively easy to see that a discrete orientation angle θ_n can be associated with the $(\lambda_\pi + \lambda_\nu - 2n, n)$ irrep in the product $(\lambda_\pi, \mu_\pi = 0) \otimes (\lambda_\nu, \mu_\nu = 0)$ where n is an integer given by $n = 0$ ($\theta = 0^o \to \parallel$), $1, \ldots, \min(\lambda_\pi, \lambda_\nu)$ ($\theta = 90^o \to \perp$).

$$|(\beta,\gamma)\rangle \times |(\beta',\gamma')\rangle = \int f_\Omega |(\beta'',\gamma'')\rangle d(\beta'',\gamma'')$$

relative orientation

$\Omega = (\psi, \theta, \phi)$

$$|(\lambda,\mu)\rangle \times |(\lambda',\mu')\rangle = \sum_{\rho_\Omega(\lambda'',\mu'')} |(\lambda'',\mu'')\rangle_{\rho_\Omega}$$

Figure 1: Schematic representation for the expansion of a product of two quadrupole mass distributions in terms of other quadrupole mass distributions. The upper product is for triaxial quantum rotors, which are characterized by the (β, γ) shape variables of the collective model and have a [$T_5 \wedge SO(3)$] symmetry; the lower coupling is for (λ, μ) irreps of $SU(3)$. The overlap function f_Ω is the inner product $\langle (\beta'', \gamma'') | (\beta, \gamma); (\beta', \gamma') \rangle_\Omega$ where $\Omega = (\psi, \theta, \phi)$ specifies the Euler angles giving the relative orientation of the principal axes of the unprimed ($|(\beta, \gamma)\rangle$) and primed ($|(\beta', \gamma')\rangle$) systems. In the $SU(3)$ case, the decomposition is a sum of $SU(3)$ irreps with integer multiplicity ρ which can be determined by the Littlewood rules for coupling Young diagrams. The multiplicity ρ_Ω, like f_Ω, can be related to the number of distinguishable orientations of the two initial distributions that yield the final one.

Finding an expression for θ_n in terms of $(\lambda_\pi, \mu_\pi = 0)$, $(\lambda_\nu, \mu_\nu = 0)$, and the final (λ, μ) illustrates a prescription that can also be applied to the case of general shapes when the μ values of the factors ($\mu_\alpha \neq 0; \alpha = \pi, \nu$) are non-zero. First of all note that the various (λ, μ) values that enter determine the eigenvalues of the corresponding quadrupole matrix, see eq.(7). It follows from this that an analytic form for θ_n can be derived by requiring that the roots of the characteristic

equations for $Q_\pi + RQ_\nu R^{-1}$ and Q coincide: $|Q_\pi + RQ_\nu R^{-1}| \Leftrightarrow |Q|$. The solution to the set of equations that this condition generates, yields the following general result for θ_n as a function of λ_π and λ_ν:

$$\theta_n = \sin^{-1}([n(\lambda_\pi + \lambda_\nu - n)/(\lambda_\pi \lambda_\nu)]^{1/2}), \tag{10}$$

where the integer index $n = 0, 1, \ldots, \min(\lambda_\pi, \lambda_\nu) = \lambda_<$. Although this expression is symmetric in λ_π and λ_ν and goes respectively to 0° and 90° for $n = 0$ and $n = \lambda_<$ as required, it has no other obvious symmetry properties, and in particular, note that the allowed θ values are not distributed symmetrically about the $\theta = 45^o$ plane, a result that is related to the occurrence of the square root in the argument of the inverse sine function.

When one of the two factor distributions is triaxial ($\gamma_\pi \neq 0$ and $\gamma_\nu = 0$ or $\gamma_\pi = 0$ and $\gamma_\nu \neq 0$) the situation is only slightly more complicated. In this case two angles rather than one are required to specify the relative orientation of the two distributions: θ as introduced above to specify the relative orientation of the major axes, and another angle φ that specifies the rotation of the minor axes of the triaxial shape relative to an axis that is perpendicular to the plane defined by the principal axes of the two factor distributions. Only values of θ and φ that lie between 0° and 90° lead to distinguishable configurations. In the $SU(3)$ case this construction corresponds to the $(\lambda_\pi, \mu_\pi) \otimes (\lambda_\nu, \mu_\nu) \rightarrow \sum \oplus (\lambda, \mu)$ coupling, where $\mu_\pi \neq 0$ and $\mu_\nu = 0$ or $\mu_\pi = 0$ and $\mu_\nu \neq 0$, respectively. While this $SU(3)$ coupling is more complicated than the previous case, it remains simply reducible, that is, each of the allowed (λ, μ) irreps in the product occurs just one time. However, because one of the two μ values is now nonzero, the pattern of allowed final (λ, μ) values is considerably richer than in the previous case: $(\lambda, \mu) = (\lambda_\pi + \lambda_\nu, \mu_>), (\lambda_\pi + \lambda_\nu - 2, \mu_> + 1), \ldots, (\lambda_\pi + \lambda_\nu - 1, \mu_> - 1), (\lambda_\pi + \lambda_\nu - 3, \mu_>), \ldots, (\lambda_\pi + \lambda_\nu - 2, \mu_> - 2), (\lambda_\pi + \lambda_\nu - 4, \mu_> - 1), \ldots$, where $\mu_> = \max(\mu_\pi, \mu_\nu)$. The general result, $(\lambda_\pi, \mu_\pi) \otimes (\lambda_\nu, \mu_\nu) \rightarrow \sum_{m,n} \oplus (\lambda_\pi + \lambda_\nu - 2n - m, \mu_> + n - m)$, requires one additional non-negative integer (m) that specifies the number of completed (three box) columns in the final Young diagram.

In general one must deal with two triaxial shapes ($\gamma_\pi \neq 0$ and $\gamma_\nu \neq 0$) and the corresponding product distribution: $(\beta_\pi, \gamma_\pi) \times (\beta_\nu, \gamma_\nu) \rightarrow (\beta, \gamma)$. The geometrical interpretation is considerably more complicated in this case because three Euler angles (φ, θ, ϕ) are required to specify the relative orientation of the factor distributions. For $(\varphi, \theta, \phi) = (0^o, 0^o, 0^o)$ the major and minor axes of the sub-distributions coincide (maximum alignment) whereas if $(\varphi, \theta, \phi) = (0^o, 90^o, 0^o)$ the semi-axes (y) remain aligned but the major (z) and minor (x) axes of the two systems are perpendicular to one another, etc. In the corresponding $SU(3)$ case the allowed product configurations are again determined by the Littlewood Rules but now for the coupling of two two-rowed Young diagrams. There is a need for three $(\varphi, \theta, \phi) \leftrightarrow (m, n, \rho)$ rather than one [prolate shapes: $(\theta) \leftrightarrow (n)$] or two [one prolate and one triaxial shape: $(\theta, \phi) \leftrightarrow (m, n)$] quantum labels in this general case: $(\lambda_\pi, \mu_\pi) \otimes (\lambda_\nu, \mu_\nu) \rightarrow (\lambda_\pi + \lambda_\mu + m, \mu_\pi + \mu_\nu + n)_\rho$, where ρ is a non-negative integer index ($\rho = 1, 2, \ldots, \rho_{max}$) labeling distinct occurrences of the same (λ, μ) in the $(\lambda_\pi, \mu_\pi) \otimes (\lambda_\nu, \mu_\nu)$ product. Working backwards, it should also be clear that the $(\beta, \gamma) \leftrightarrow (\lambda, \mu)$ correspondence can be used to give a geometrical interpretation to the abstract group theoretical concept of the outer multiplicity – at least for the $SU(3)$ case – which has up until now escaped a simple physical interpretation. Specifically, the multiplicity ρ, together with m and n, can be considered to be a measure of the relative orientation of the two factor distributions. In this way the first ($\rho = 1$) occurrence of (λ, μ) corresponds to a parent configuration oriented with one set of angles $(\varphi_1, \theta_1, \phi_1)$ while the second ($\rho = 2$) solution corresponds to another set $(\varphi_2, \theta_2, \phi_2)$, and so on.

If $\rho_{max} = 1$, the corresponding (λ, μ) distribution can only be realized in one way. With this interpretation in hand the evaluation of reduced matrix elements and especially $SU(3)$ coupling and recoupling coefficients should be revisited, looking for asymptotic solutions that exploit the geometrical concept of overlapping ellipsoidal mass distributions.

It is instructive to view the relationship between the rotor and $SU(3)$ theories at a more fundamental level. This can be achieved by comparing the algebras of their symmetry groups. The symmetry group of the quantum rotor is the semi-direct product $T_5 \wedge SO(3)$ where T_5 is generated by the five independent components of the (spherical) collective quadrupole operator (Q_μ^c) and $SO(3)$ is generated by the angular momentum operators (L_μ). The generators of $SU(3)$, on the other hand, are the Q_μ^a [see the discussion following eq.(1)] and the L_μ operators. If Q^x denotes a generic quadrupole operator, the commutation relations of the L_μ and the Q_μ^x are

$$
\begin{aligned}
[L_\mu, L_\nu] &= -\sqrt{2} < 1\mu, 1\nu | 1, \mu + \nu > L_{\mu+\nu} \ , \\
[L_\mu, Q_\nu^x] &= -\sqrt{6} < 1\mu, 2\nu | 2, \mu + \nu > Q_{\mu+\nu}^x \ , \\
\left[Q_\mu^x, Q_\nu^x \right] &= c < 2\mu, 2\nu | 1, \mu + \nu > L_{\mu+\nu} \ ,
\end{aligned}
\tag{11}
$$

where $c = 0$ for $T_5 \wedge SO(3)$, $(Q^x = Q^c)$, $c = +3\sqrt{10}$ for $SU(3)$ $(Q^x = Q^a)$, and $c = -3\sqrt{10}$ for a heretofore not mentioned group $Sl(3,R)$ $[Q^x = Q^b \sim (x_i p_j + p_j x_i)]$ which is associated with shear degrees of freedom. In eq.(11) the $< -, -|- >$ symbol denotes an ordinary $SO(3)$ Clebsch-Gordan coefficient. [All three of these groups, $T_5 \wedge SO(3)$, $SU(3)$, and $Sl(3, R)$, are subgroups of the symplectic group $Sp(3,R)$.] From these commutation relations it is easy to see how the $SU(3)$ algebra reduces to that of $T_5 \wedge SO(3)$: if Q^a is divided by the square root of the second order invariant of $SU(3)$ $(Q^a \leftarrow Q^a/\sqrt{C_2}$ where by definition the invariant $C_2 = (Q^a \cdot Q^a + 3L^2)/4$ commutes with the Q^a and L_μ operators), the first and second commutators in eq.(11) remain unchanged, while the $L_{\mu+\nu}$ on the right-hand-side of the third goes over into $L_{\mu+\nu}/C_2$ and for low L values in large $SU(3)$ irreps, $L_{\mu+\nu}/C_2 \to 0$. This renormalization of the Q^a operator is a group contraction process and the arguments presented show the $SU(3)$ algebra reduces to the algebra of $T_5 \wedge SO(3)$ in the contraction limit, and consequently, the $SU(3)$ theory reduces to that of the quantum rotor. Differences between observables of the two theories occur because $SU(3)$ is a compact group with finite dimensional irreps while $T_5 \wedge SO(3)$ is non-compact with infinite dimensional representations. Band termination and a fall-off in B(E2) strengths are examples.

4   Conclusion

A geometrical interpretation for the outer multiplicity ρ that occurs in a reduction of the product of two $SU(3)$ representations, $(\lambda_\pi, \mu_\pi) \times (\lambda_\nu, \mu_\nu) \to \sum_\rho (\lambda, \mu)_\rho$, has been introduced. This structure arises, for example, in the coupling of proton (π) and neutron (ν) representations that occur in both boson and fermion descriptions of heavy deformed (rare earth and actinide) nuclei. Attributing a geometry to the proton-neutron coupling, raises the possibility of introducing a simple phenomenological interaction that provides a physically meaningful way for distinguishing among different (λ, μ) and multiple occurrences of the same (λ, μ) values that arise, for example, when coupling deformed proton and neutron configurations in heavy deformed nuclei.

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