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A RAMANUJAN-TYPE MEASURE FOR THE ASKEY-WILSON POLYNOMIALS

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Abstract

A Ramanujan-type representation for the Askey-Wilson q-beta integral, admitting the transformation $q \to q^{-1}$, is obtained. Orthogonality of the Askey-Wilson polynomials with respect to a measure, entering into this representation, is proved. A simple way of evaluating the Askey-Wilson q-beta integral is also given.

1 Introduction.

The Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$ [1], which have already become classical, represent a five-parameter system of polynomials. They satisfy the orthogonality relation

$$\int_{-1}^{1} p_m(x; a, b, c, d|q)\, p_n(x; a, b, c, d|q)\, w(x; a, b, c, d|q)\, dx = \delta_{mn} I_n(a, b, c, d|q) \qquad (1.1)$$

with respect to the absolutely continuous measure $d\alpha(x) = w(x)dx$, with the weight function

$$w(x; a, b, c, d|q) = \frac{1}{\sin\theta} \frac{h(\cos 2\theta, 1; q)}{\prod_{v=a,b,c,d} h(\cos\theta, v; q)}, \qquad x = \cos\theta,$$

$$h(a, b; q) = \prod_{j=0}^{\infty} (1 - 2abq^j + b^2 q^{2j}). \qquad (1.2)$$

As special and limiting cases, the Askey-Wilson polynomials contain many known systems of polynomials (see, for example, [2]). In particular, the choice of the parameters $a = -b = \sqrt{\beta}$, $c = -d = \sqrt{q\beta}$, leads to the continuous q-ultraspherical polynomials $C_n(x; \beta|q)$ [3], *i.e.*,

$$p_n(x; \sqrt{\beta}, -\sqrt{\beta}, \sqrt{q\beta}, -\sqrt{q\beta}|q) = \frac{(\beta^2; q)_{2n} (q; q)_n}{(\beta, \beta^2; q)_n} C_n(x; \beta|q), \qquad (1.3)$$

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where we have used the standard notation of the theory of q-special functions

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \qquad (a_1,...,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n. \tag{1.4}$$

In turn, from $C_n(x;\beta|q)$ one can obtain the continuous q-Hermite polynomials $H_n(x|q) = (q;q)_n C_n(x;0|q)$, the Gegenbauer (ultraspherical) polynomials $C_n^\lambda(x) = \lim_{q\to 1} C_n(x;q^\lambda|q)$, and also the Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, by taking the limit $\beta \to 1$ or by putting $\beta = q$ in $C_n(x;\beta|q)$, respectively.

The key ingredient of the original proof of the orthogonality (1.1), which led to the discovery of the Askey-Wilson system of polynomials (see the discussion of this point in [4]), was the evaluation of the Askey-Wilson q-beta integral:

$$I_0(a,b,c,d|q) \equiv \int_{-1}^{1} w(x;a,b,c,d|q)dx = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{1.5}$$

$$max_{v=a,b,c,d}|v| < 1, \quad |q| < 1.$$

The integral (1.5) has acquired its name because in a special case, when the parameters $a = q^{\alpha+1/2}$, $b = -q^{\beta+1/2}$, and $c = -d = q^{1/2}$, the $q \to 1^-$ limit of $I_0(a,b,c,d|q)$ is the beta function (or Euler's integral of the first kind)

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta dx = 2^{\alpha+\beta+1}B(\alpha+1,\beta+1) = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{1.6}$$

A nonstandard form of the orthogonality on the full real line for the continuous q-Hermite polynomials $H_n(\sin\kappa x|q)$, $q = \exp(-2\kappa^2)$, was considered in [5]. It turned out that if one uses the modular transformation and the periodicity property of the ϑ-function appearing in the weight function for these polynomials, the finite interval of orthogonality can be transformed into an infinite one. This technique is of interest both from a mathematical point of view and from the point of view of possible applications in theoretical physics, beginning with a number of problems, related with q-oscillators (see the review [6]).

The purpose of this article is to discuss the applicability of this idea to the more general case, i.e. to the Askey-Wilson q-beta integral (1.5) [7, 8]. To simplify consideration it will be assumed in Sections 2-4 that $|v| < 1$, $v = a,b,c,d$, and that the parameter $q = \exp(-2\kappa^2)$ satisfies the requirement $0 < q < 1$. The possibility of extending these results to other values of the parameters is discussed in Section 5.

2 A Ramanujan-type representation for the q-beta integral.

From the point of view of symmetry the parametrization $x = \sin\varphi$ is most convenient; it corresponds to the change of variable $\theta = \frac{\pi}{2} - \varphi$, $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ in formula (1.2). Consequently, the left side of (1.5) is equal to

$$I_0(a,b,c,d|q) = \int_{-\pi/2}^{\pi/2} \frac{h(-\cos 2\varphi, 1; q)}{\prod_{v=a,d,c,d} h(\sin\varphi, v; q)}\, d\varphi\,. \tag{2.1}$$

Comparison of the numerator

$$h(-\cos 2\varphi, 1; q) = \prod_{j=0}^{\infty} (1 + 2q^j \cos 2\varphi + q^{2j})$$

of the integral (2.1) with Jacobi's expression for the theta-function $\vartheta_2(z,q) \equiv \vartheta_2(z|\tau)$, $q = \exp(\pi i \tau)$ as an infinite product [9]

$$\vartheta_2(z,q) = 2q^{1/4}(q^2;q^2)_\infty \cos z \prod_{j=1}^{\infty} (1 + 2q^{2j}\cos 2z + q^{4j}), \tag{2.2}$$

shows that

$$h(-\cos 2\varphi, 1; q) = \frac{2\cos\varphi}{q^{1/8}(q;q)_\infty}\, \vartheta_2(\varphi, q^{1/2}) \tag{2.3}$$

and therefore

$$I_0(a,b,c,d|q) = \frac{2}{q^{1/8}(q;q)_\infty} \int_{-\pi/2}^{\pi/2} \frac{\vartheta_2(\varphi, q^{1/2})\cos\varphi}{\prod_{v=a,b,c,d} h(\sin\varphi, v; q)}\, d\varphi\,. \tag{2.4}$$

With the aid of the modular transformation [9]

$$\vartheta_2(z|\tau) = \frac{\exp\left(-\frac{iz^2}{\pi\tau}\right)}{(-i\tau)^{\frac{1}{2}}}\, \vartheta_4(z\tau^{-1}|-\tau^{-1}), \qquad \tau = \frac{i\kappa^2}{\pi}\,, \tag{2.5}$$

and the change of variable $\varphi = \kappa x$, the integral (2.4) can be written as

$$I_0(a,b,c,d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q;q)_\infty} \int_{-\pi/2\kappa}^{\pi/2\kappa} \frac{\vartheta_4(\frac{\pi i}{\kappa}x, e^{-\pi^2/\kappa^2})\, e^{-x^2}\cos\kappa x}{\prod_{v=a,b,c,d} h(\sin\kappa x, v; q)}\, dx. \tag{2.6}$$

Using the expansion

$$\vartheta_4(z,q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2ikz} \tag{2.7}$$

and taking into account the uniform convergence of the series (2.7) in any bounded domain of values of z [9], we substitute (2.7) into (2.6) and integrate this series termwise, *i.e.*,

$$I_0(a,b,c,d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k \int_{-\pi/2\kappa}^{\pi/2\kappa} \frac{e^{-(x+\pi/\kappa k)^2}\cos\kappa x dx}{\prod_{v=a,b,c,d} h(\sin\kappa x, v; q)}. \tag{2.8}$$

The change of variable $x_k = x + \frac{\pi}{\kappa}k$, $x_k^{min} = \frac{\pi}{\kappa}(k - \frac{1}{2}) \leq x_k \leq \frac{\pi}{\kappa}(k + \frac{1}{2}) = x_k^{max}$ and an account for the relation $x_{k-1}^{max} = x_k^{min}$ allows to sum the right-hand side of (2.8) with respect to k and represent (2.8) in the form

$$I_0(a,b,c,d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q;q)_\infty}\tilde{I}_0(a,b,c,d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q;q)_\infty}\int_{-\infty}^{\infty} \frac{e^{-x^2}\cos\kappa x dx}{\prod_{v=a,b,c,d} h(\sin\kappa x, v; q)}. \tag{2.9}$$

Thus, combining formulas (1.5) and (2.9) yields the following representation for the Askey-Wilson q-beta integral [7]

$$\tilde{I}_0(a,b,c,d|q) \equiv \int_{-\infty}^{\infty} \rho(\kappa x; a,b,c,d|q)\, e^{-x^2}\cos\kappa x dx = \frac{\sqrt{\pi} q^{\frac{1}{8}}(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{2.10}$$

where, in accordance with the definition (1.2),

$$\rho(x; a,b,c,d|q) = \prod_{v=a,b,c,d} h^{-1}(\sin x, v; q) = \prod_{v=a,b,c,d} e_q(ive^{-ix})e_q(-ive^{ix}), \tag{2.11}$$

and $e_q(z) = (z;q)_\infty^{-1}$ is the q-exponential function [2].

We note that each factor $h^{-1}(\sin\kappa x, v; q)$, $v = a,b,c,d$, in the integrand (2.10) is represented as

$$h^{-1}(\sin\kappa x, v; q) = \sum_{n=0}^{\infty}(iv)^n \sum_{k=0}^{n} \frac{(-1)^k \exp[-i(n-2k)\kappa x]}{(q;q)_k(q;q)_{n-k}}, \tag{2.12}$$

if one uses the generating function for the continuous q-Hermite polynomials $H_n(x|q)$

$$(te^{i\theta}, te^{-i\theta}; q)_\infty^{-1} = \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)}{(q;q)_n} t^n \qquad |t| < 1, \tag{2.13}$$

and their explicit representation [2]

$$H_n(\cos\theta|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta}, \tag{2.14}$$

where the symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient [2]. Therefore the integration over x in (2.10) is reduced to the Fourier transformation formula for the ground state of the linear harmonic oscillator

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-x^2/2 + ixy)dx = \exp(-y^2/2). \tag{2.15}$$

An explicit evaluation of the nonstandard form of the Askey-Wilson q-beta integral (2.10) will be discussed in greater detail in Section 4.

As mentioned above, the weight function (1.2) with the parameters $a = -b = \beta^{1/2}$, $c = -d = aq^{1/2}$, corresponds to the continuous q-ultraspherical polynomials $C_n(x;\beta|q)$. The relations [2]

$$(a;q)_\infty = (a, aq; q^2)_\infty, \qquad (a, -a; q)_\infty = (a^2; q^2)_\infty,$$

enable the representation (2.10) for this particular case to be simplified to

$$\int_{-\infty}^{\infty} \frac{\exp(-x^2 + i\kappa x)dx}{(-\beta \exp(2i\kappa x), -\beta \exp(-2i\kappa x); q)_\infty} = \frac{\sqrt{\pi} q^{1/8} (\beta, q\beta; q)_\infty}{(\beta^2; q)_\infty}. \tag{2.16}$$

If one compares (2.16) with the Ramanujan integral ($q = \exp(-2k^2)$, $|q| < 1$) [10, 11]

$$\int_{-\infty}^{\infty} e^{-x^2+2mx} e_q(aq^{1/2}e^{2ikx}) e_q(bq^{1/2}e^{-2ikx})\, dx = \frac{\sqrt{\pi} e^{m^2}}{(qab; q)_\infty} E_q(aqe^{2imk}) E_q(bqe^{-2imk}), \tag{2.17}$$

it is easy to verify that (2.16) agrees with (2.17) if one sets $2m = ik = i\kappa$ and $a = b = -\beta q^{1/2}$.

3 Orthogonality of the Askey-Wilson polynomials with respect to the measure $\rho(\kappa x; a, b, c, d|q)$.

A direct proof of the orthogonality for the Askey-Wilson polynomials

$$\int_{-\infty}^{\infty} p_m(\sin \kappa x; a, b, c, d|q) p_n(\sin \kappa x; a, b, c, d|q) \rho(\kappa x; a, b, c, d|q) \exp(-x^2) \cos \kappa x dx =$$

$$= \delta_{mn} \tilde{I}_n(a, b, c, d|q) \tag{3.1}$$

with respect to the weight function appearing in the nonstandard integral representation (2.10), is analogous to the proof of eigenfunctions orthogonality for the Sturm-Liouville differential equation [12] . Indeed, the difference differentiation formula for the Askey-Wilson polynomials [1]

$$\sin \kappa \partial_x \, p_n(\sin \kappa x; a, b, c, d|q) = \tag{3.2}$$

$$= q^{-n/2}(1 - q^n)(1 - abcdq^{n-1}) \cos \kappa x \, p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)$$

provides a lowering operator for these polynomials. To find a raising operator one can use the relation

$$w(\sin \varphi; a, b, c, d|q) = \frac{2\vartheta_2(\varphi, q^{1/2})}{q^{1/8}(q; q)_\infty} \rho(\varphi; a, b, c, d|q), \tag{3.3}$$

which follows from (1.2), (2.3) and (2.11), and write the difference equation for the Askey-Wilson polynomials [1] in the form

$$\sin \kappa \partial_x \left[\frac{\vartheta_2(\kappa x, q^{1/2})}{\cos \kappa x} \rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) \sin \kappa \partial_x p_n(\sin \kappa x; a, b, c, d|q) \right] =$$

$$= (1 - q^{-n})(1 - abcdq^{n-1}) \cos \kappa x \, \vartheta_2(\kappa x, q^{1/2}) \rho(\kappa x; a, b, c, d|q) \, p_n(\sin \kappa x; a, b, c, d|q). \tag{3.4}$$

Now, using the difference differentiation formula (3.2) in the left-hand side of (3.4) and the periodicity property of the ϑ_2-function [9],

$$\vartheta_2(z \pm \pi\tau, q) = q^{-1} \exp(\mp 2iz)\vartheta_2(z, q), \qquad q = \exp(\pi i\tau), \tag{3.5}$$

we arrive at the raising operator

$$(\sin 2\kappa x \cos \kappa\partial_x - \cos 2\kappa x \sin \kappa\partial_x)\rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)$$

$$p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = q^{\frac{1-n}{2}} \cos \kappa x \, \rho(\kappa x; a, b, c, d|q) p_n(\sin \kappa x; a, b, c, d|q). \tag{3.6}$$

We are now in a position to give a direct proof of the orthogonality relation (3.1). We multiply both sides of the equality (3.6) by $p_m(\sin \kappa x; a, b, c, d|q)\ \exp(-x^2)$ and integrate in x over the full real line. As a result we obtain in the right-hand side ,

$$q^{\frac{1-n}{2}} \int_{-\infty}^{\infty} p_m(\sin \kappa x; a, b, c, d|q)\, p_n(\sin \kappa x; a, b, c, d, |q)\, \rho(\kappa x; a, b, c, d|q)\, e^{-x^2} \cos \kappa x \, dx \equiv$$

$$q^{\frac{1-n}{2}} I_{mn}(a, b, c, d|q). \tag{3.7}$$

The left-hand side

$$\int_{-\infty}^{\infty} dx p_m(\sin \kappa x; a, b, c, d|q) e^{-x^2} (\sin 2\kappa x \cos \kappa\partial_x - \cos 2\kappa x \sin \kappa\partial_x)$$

$$\rho(\kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)\, p_{n-1}(\sin \kappa x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q), \tag{3.8}$$

can be integrated by parts with the aid of (3.2) and the evident relations

$$\int_{-\infty}^{\infty} dx f(x) \cos \kappa\partial_x \, \varphi(x) = \int_{-\infty}^{\infty} dx \, \varphi(x) \cos \kappa\partial_x f(x),$$

$$\int_{-\infty}^{\infty} dx f(x) \sin \kappa\partial_x \, \varphi(x) = - \int_{-\infty}^{\infty} dx \, \varphi(x) \sin \kappa\partial_x f(x), \tag{3.9}$$

which apply to (3.8) because the function $\rho(\kappa z; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)$ has no singularities inside of the strip $-\kappa \le y \le \kappa, \quad -\infty < x < \infty$ in the complex plane $z = x + iy$. This leads to

$$q^{\frac{1-m}{2}}(1 - q^m)(1 - abcdq^{m-1}) I_{m-1n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q). \tag{3.10}$$

Equating the right-hand (3.7) and left-hand (3.10) sides thus yields

$$q^{\frac{m-n}{2}} I_{mn}(a, b, c, d|q) = (1 - q^m)(1 - abcdq^{m-1}) I_{m-1n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q). \tag{3.11}$$

We now interchange m and n in (3.11) and take into account that the integral $I_{mn}(a,b,c,d|q)$ is symmetric in m and n due to the definition (3.7), *i.e.*,

$$q^{\frac{n-m}{2}} I_{mn}(a,b,c,d|q) = (1-q^n)(1-abcdq^{n-1}) I_{m-1n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q). \qquad (3.11')$$

Finally, multiplying both sides of (3.11) by $(1-q^n)(1-abcdq^{n-1})$ and of (3.11′) by $(1-q^m)(1-abcdq^{m-1})$ and subtracting the second expression from the first, we get

$$(q^{\frac{m-n}{2}} - q^{\frac{n-m}{2}})(1-abcdq^{m+n-1}) I_{mn}(a,b,c,d|q) = 0. \qquad (3.12)$$

From (3.12) it follows that $I_{mn}(a,b,c,d|q) = \delta_{mn} \tilde{I}_n(a,b,c,d|q)$, confirming the orthogonality (3.1) of the Askey-Wilson polynomials for $m \neq n$ [8].

We note that as special and limiting cases, (3.1) contains the orthogonality relations for other known sets of polynomials, such as the continuous q-ultraspherical polynomials $C_n(x;\beta|q)$, the continuous q-Hermite polynomials $H_n(x;q) = (q;q)_n\ C_n(x;0|q)$ (the corresponding special case of (3.1), when the all parameters a,b,c,d are equal to zero, is considered in [5]), the Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$, and so on.

4 Evaluation of the integrals $\tilde{I}_n(a,b,c,d|q)$.

Iterating the recurrence relation

$$\tilde{I}_n(a,b,c,d|q) = (1-q^n)(1-abcdq^{n-1}) \tilde{I}_{n-1}(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q), \qquad (4.1)$$

which follows from (3.11) or (3.11′) when $m = n$, allows to express the normalization integrals $\tilde{I}_n(a,b,c,d|q)$, $n = 1,2,...$, through a known value of the Askey-Wilson q-beta integral $\tilde{I}_0(a,b,c,d|q)$, i.e.

$$\tilde{I}_n(a,b,c,d|q) = \frac{(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-abcdq^{2n-1})(abcd;q)_{n-1}} \tilde{I}_0(a,b,c,d|q). \qquad (4.2)$$

It only remains to evaluate the integral $\tilde{I}_0(a,b,c,d|q)$ itself. To this end, having defined the symmetrical $\rho_+(x)$ and antisymmetrical $\rho_-(x)$ combinations with respect to the inversion $x \to -x$,

$$\rho_\pm(x;a,b,c,d|q) = \tfrac{1}{2}[\rho(x;a,b,c,d|q) \pm \rho(-x;a,b,c,d|q)], \qquad (4.3)$$

it is convenient to rewrite (2.10) as

$$\tilde{I}_0(a,b,c,d|q) = \int_{-\infty}^{\infty} dx \exp(-x^2 + i\kappa x) \rho_+(\kappa x;a,b,c,d|q). \qquad (4.4)$$

Let us carry out the replacements $v \to v\sqrt{q}$, $v = a,b,c,d$, and the subsequent shift of the variable of integration $x \to x + i\kappa$ in (4.4). (We remind that the function $\rho(\kappa z; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q)$ does not have singularities in the strip $-\kappa \leq y \leq \kappa$, $\quad -\infty < x < \infty$ of the complex plane $z = x + iy$). Then, taking into account that in accordance with the definitions (1.2) and (2.11)

$$\rho(\kappa(x+i\kappa); aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}|q) = \rho(\kappa x;a,b,c,d|q) \prod_{v=a,b,c,d} (1 + iv\exp(i\kappa x)), \qquad (4.5)$$

we obtain

$$\tilde{I}_0(aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}|q) = (1-s_2)\tilde{I}_0(a,b,c,d|q)+$$

$$+s_4\int_{-\infty}^{\infty} dx\exp(-x^2+3i\kappa x)\rho_+(\kappa x;a,b,c,d|q) - is_3\int_{-\infty}^{\infty} dx\exp(-x^2+2i\kappa x)\rho_-(\kappa x;a,b,c,d|q), \tag{4.6}$$

where

$$s_2 = ab+ac+ad+bc+bd+cd,$$

$$s_3 = abc+abd+acd+bcd, \quad s_4 = abcd. \tag{4.7}$$

It remains only to express the second and third integrals in the right-hand side of (4.6) in terms of $\tilde{I}_0(a,b,c,d|q)$. To that end one can use the $n=1$ case of (3.6)

$$(\sin 2\kappa x\cos\kappa\partial_x - \cos 2\kappa x\sin\kappa\partial_x)\rho(\kappa x;aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}|q) =$$

$$= [(1-s_4)\sin 2\kappa x + (s_3-s_1)\cos\kappa x]\rho(\kappa x;a,b,c,d|q), \tag{4.8}$$

taking into account that $p_0(x;a,b,c,d|q) = 1$, $p_1(x;a,b,c,d|q) = 2(1-s_4)x+s_3-s_1$ and $s_1 = a+b+c+d$. The symmetrization of (4.8) leads to the relations

$$(\sin 2\kappa x\cos\kappa\partial_x - \cos 2\kappa x\sin\kappa\partial_x)\rho_\pm(\kappa x;aq^{1/2},bq^{1/2},c\,q^{1/2},d\,q^{1/2}|q) =$$

$$= (1-s_4)\sin 2\kappa x\,\rho_\pm(\kappa x;a,b,c,d|q) + (s_3-s_1)\cos\kappa x\,\rho_\mp(\kappa x;a,b,c,d|q). \tag{4.9}$$

Multiplying both sides of the equality (4.9) for the antisymmetrical combination $\rho_-(\kappa x)$ by $\exp(-x^2)$ and integrating over the variable x yields

$$(1-s_4)\int_{-\infty}^{\infty} dx\exp(-x^2+2i\kappa x)\rho_-(\kappa x;a,b,c,d|q) = i(s_1-s_3)\tilde{I}_0(a,b,c,d|q). \tag{4.10}$$

Now we multiply both sides of (4.9) for $\rho_+(\kappa x;aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}|q)$ by $\exp(-x^2+i\kappa x)$ and integrate over x. Using (4.10), the result can be written as

$$\int_{-\infty}^{\infty} dx\exp(-x^2+3i\kappa x)\rho_+(\kappa x;a,b,c,d|q) =$$

$$= \left[1-\frac{(s_3-s_1)^2}{(1-s_4)^2}\right]\tilde{I}_0(a,b,c,d|q) - \frac{1-q}{1-s_4}\tilde{I}_0(aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}|q). \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.6), we find

$$(1-abcd)(1-qabcd)\,\tilde{I}_0(aq^{1/2},bq^{1/2},cq^{1/2},dq^{1/2}|q) = \tag{4.12}$$

$$= (1-ab)(1-ac)(1-ad)(1-bc)(1-bd)(1-cd)\,\tilde{I}_0(a,b,c,d|q).$$

Since $0 < q < 1$, by iterating equation (4.12) one can express the Askey-Wilson q-beta integral (2.10) with arbitrary parameters in terms of its value for vanishing parameters a, b, c, d, *i.e.*,

$$\tilde{I}_0(a,b,c,d|q) = \frac{(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}\,\tilde{I}_0(0,0,0,0|q). \tag{4.13}$$

The value of $\tilde{I}_0(0,0,0,0|q)$ is easily found from (2.10) and (3.1) with the aid of the Fourier transformation formula (2.15) for the quadratically decreasing exponential function, *i.e.*,

$$\tilde{I}_0(0,0,0,0|q) = \int_{-\infty}^{\infty} dx \exp\left(-x^2 + i\kappa x\right) = \sqrt{\pi}q^{1/8}. \tag{4.14}$$

Combining formulas (4.13) and (4.14) leads to

$$\tilde{I}_0(a,b,c,d|q) = \frac{\sqrt{\pi}q^{1/8}(abcd;q)_\infty}{(ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{4.15}$$

which is the known value of the Askey-Wilson q-beta integral [1]

$$I_0(a,b,c,d|q) = \frac{2\sqrt{\pi}}{q^{1/8}(q;q)_\infty}\tilde{I}_0(a,b,c,d|q) = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}. \tag{4.15$'$}$$

Substituting (4.15) into (4.2), we finally obtain the explicit form for the normalization integral

$$\tilde{I}_n(a,d,c,d|q) = \frac{\sqrt{\pi}q^{1/8}(q;q)_n(abcdq^{n-1};q)_\infty}{(1-abcdq^{2n-1})(abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}. \tag{4.16}$$

The complications arising in the evaluation of the standard form of the Askey-Wilson q-beta integral (1.5) can be illustrated by the following short quotation from reference [4]: "This was surprisingly hard, and it has taken over five years before relatively simple ways of evaluating this integral were found".

5 The transformation $q \to q^{-1}$.

It is necessary to emphasize that the nonstandard orthogonality relation (3.1) admits the transformation $q \to q^{-1}$ [7, 8]. The standard form of the Askey-Willson integral (1.5) does not in general have this property. Even in the simplest case of vanishing parameters a, b, c and d, which corresponds to the continuous q-Hermite polynomials $H_n(x|q)$, the definition of a weight function for the system of polynomials $h_n(x;q) = i^{-n}H_n(ix|q^{-1})$ requires a special analysis [13, 14].

Since

$$(z;q^{-1})_\infty = (qz;q)_\infty^{-1}, \tag{5.1}$$

the change $q \to q^{-1}$ (i.e. $\kappa \to i\kappa$) in the function $\rho(\kappa x; a,b,c,d|q)$ appearing in (2.10) and (3.1), transforms it into

$$\rho(i\kappa x; a,b,c,d|q^{-1}) = \prod_{v=a,b,c,d} (ivqe^{\kappa x}, -ivqe^{-\kappa x}; q)_\infty = \prod_{v=a,b,c,d} E_q(ivqe^{-\kappa x})E_q(-ivqe^{\kappa x}), \tag{5.2}$$

where $E_q(z) = e_q^{-1}(-z) = (-z;q)_\infty$ [2]. Therefore, under the transformation $q \to q^{-1}$, the orthogonality relation (3.1) for the Askey-Wilson polynomials with the parameter $q < 1$ converts into the following orthogonality relation for the Askey-Wilson polynomials with $q > 1$:

$$\int_{-\infty}^{\infty} p_m(i\sinh\kappa x; a,b,c,d|q^{-1})\, p_n(i\sinh\kappa x; a,b,c,d|q^{-1})\, \rho(i\kappa x; a,b,c,d|q^{-1})\, e^{-x^2}\cosh\kappa x dx =$$

$$\delta_{mn}\tilde{I}_n(a,b,c,d|q^{-1}) \tag{5.3}$$

The explicit form of $\tilde{I}_n(a,b,c,d|q^{-1})$ is readily obtained from (4.16), upon making use of the formulas (5.1) and $(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-n(n-1)/2}$ [2].

On the other hand, with the aid of the explicit representation for the Askey-Wilson polynomials [1, 2]

$$p_n(\sin\varphi; a,b,c,d|q) = (ab,ac,ad;q)_n\, a^{-n}\, {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, iae^{i\varphi}, -iae^{-i\varphi} \\ ab, ac, ad \end{matrix} ; q, q \right] \tag{5.4}$$

and the inversion formula (with respect to the transformation $q \to q^{-1}$) for the basic hypergeometric series ${}_4\phi_3$ (see [2], p.21, exercise 1.4(i)), it is easy to show that

$$p_n(x; a,b,c,d|q^{-1}) = (-1)^n (abcd)^n\, q^{-\frac{3}{2}n(n-1)}\, p_n(x; a^{-1},b^{-1},c^{-1},d^{-1}|q). \tag{5.5}$$

Consequently, from (5.3) and (5.5) it follows the orthogonality relation

$$\int_{-\infty}^{\infty} p_m(i\sinh\kappa x; a^{-1},b^{-1},c^{-1},d^{-1}|q) p_n(i\sinh\kappa x; a^{-1},b^{-1},c^{-1},d^{-1}|q) \rho(i\kappa x; a,b,c,d|q^{-1}) *$$

$$e^{-x^2}\cosh\kappa x dx = \frac{(q, 1/ab, 1/ac, 1/ad, 1/bc, 1/bd, 1/cd; q)_n}{(1-q^{2n-1}/abcd)(1/abcd;q)_{n-1}}\, \tilde{I}_0(a,b,c,d|q^{-1})\, \delta_{mn} \tag{5.6}$$

for the Askey-Wilson polynomials with the parameters $|v| > 1, v = a,b,c,d$ and $0 < q < 1$. The value of the integral $\tilde{I}_0(a,b,c,d|q^{-1})$ is simple to obtain from (4.15) by means of the formula (5.1).

6 Concluding remarks.

The orthogonality relations (3.1) and (5.6) are bound to be related by the Fourier transformation for the Askey-Wilson functions, analogous to the well-known transformation for the harmonic oscillator wave functions $H_n(x)\exp(-x^2/2)$ (or Hermite functions in the terminology of mathematicians [15, 16]) connecting the coordinate and momentum realizations in quantum mechanics. It should be interesting to compare this Fourier transformation with the q-transformations, that reproduce the Askey-Wilson polynomials [17, 18]. For the q-Hermite functions $H_n(\sin \kappa x|q)\exp(-x^2/2)$, $q = \exp(-2\kappa^2)$, which are the simplest case of the Askey-Wilson functions with vanishing parameters a, b, c, and d, such Fourier transformation has the form [5]

$$\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \exp(ixy - x^2/2) H_n(\sin \kappa x|q) dx = i^n q^{n^2/4} h_n(\sinh \kappa y|q) \exp(-y^2/2).$$

The general case needs to be analyzed in greater detail.

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