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COHERENT STATES FOR A GENERALIZATION OF THE HARMONIC OSCILLATOR

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Abstract

Coherent states for a family of isospectral oscillator Hamiltonians are derived from a suitable choice of annihilation and creation operators. The Fock-Bargmann representation is also obtained.

1 Generalized Oscillator

Let us consider the harmonic oscillator Hamiltonian and its annihilation and creation operators

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2, \quad a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right), \quad [a, a^\dagger] = 1. \tag{1}$$

We obviously have $a^\dagger a = H - \frac{1}{2}$, $aa^\dagger = H + \frac{1}{2}$, $Ha^\dagger = a^\dagger(H+1)$ and $Ha = a(H-1)$. The eigenstates verify

$$|\psi_n\rangle = \frac{(a^\dagger)^n|\psi_0\rangle}{\sqrt{n!}}; \qquad a^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle, \qquad a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle. \tag{2}$$

In his paper of 1984, Mielnik [1] (see also [2]) looked for operators b and $b^\dagger$ such that $bb^\dagger = H + \frac{1}{2}$ and taking the following form:

$$b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta(x)\right), \qquad b^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta(x)\right). \tag{3}$$

Hence, $\beta(x)$ must verify the Riccati equation

$$\beta' + \beta^2 = 1 + x^2, \quad \text{whose general solution is} \quad \beta(x) = x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}, \quad \lambda \in \mathbf{R}. \tag{4}$$

The inverted product of the new operators is not related to the oscillator Hamiltonian, but gives a one-parametric family of operators:

$$H_\lambda = b^\dagger b + \frac{1}{2} = -\frac{1}{2}\frac{d^2}{dx^2} + V_\lambda(x) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} - \frac{d}{dx}\left[\frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2}dy}\right]. \tag{5}$$

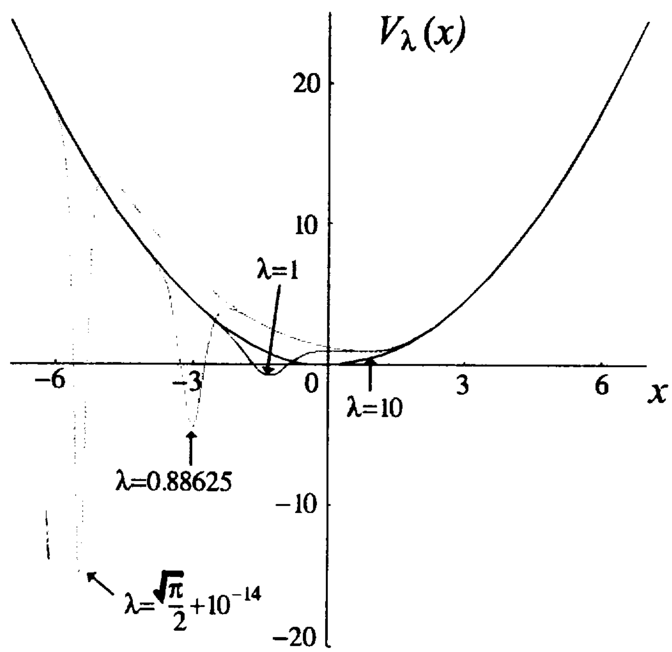

FIG. 1. The potentials $V_\lambda(x)$ associated to H_λ.

The operator $b^\dagger$ connects H and H_λ: $H_\lambda b^\dagger = b^\dagger(H+1)$. Therefore, the normalized eigenstates and eigenvalues of H_λ are

$$|\theta_n\rangle = \frac{b^\dagger|\psi_{n-1}\rangle}{\sqrt{n}}, \qquad E_n = n + \frac{1}{2}, \qquad n = 1, 2, \ldots. \tag{6}$$

They do not generate all $L^2(\mathbf{R})$. There is a missing vector $|\theta_0\rangle$ verifying $b|\theta_0\rangle = 0$ and given by

$$\theta_0(x) = \frac{C_0 e^{-x^2/2}}{\lambda + \int_0^x e^{-y^2}dy}. \tag{7}$$

It is an eigenvector of H_λ with eigenvalue $1/2$; then H_λ is a Hamiltonian with spectrum equal to that of the harmonic oscillator. The annihilation and creation operators for H_λ can be chosen

$$A = b^\dagger a b, \qquad A^\dagger = b^\dagger a^\dagger b. \tag{8}$$

2 New Coherent States

It is well-known that there are several non-equivalent definitions of coherent states [3, 4]. One of the possibilities is to look for eigenstates of an annihilation operator. We have seen that A is such an operator. Hence, the states $|z\rangle$ we are looking for must verify

$$A|z\rangle = z|z\rangle, \qquad |z\rangle = \sum_{n=0}^{\infty} a_n|\theta_n\rangle. \tag{9}$$

After normalizing, we get

$$|z\rangle = \frac{1}{\sqrt{{}_0F_2(1,2;|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!\sqrt{(n+1)!}}|\theta_{n+1}\rangle, \tag{10}$$

where the generalized hypergeometric function is defined as [5]

$${}_0F_2(\alpha,\beta;x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+n)\Gamma(\beta+n)}\frac{x^n}{n!}. \tag{11}$$

We see that $z = 0$ is a doubly degenerated eigenvalue for A, with eigenvectors $|0\rangle \equiv |\theta_1\rangle$ and $|\theta_0\rangle$. We analyze now the overcompleteness. The resolution of the identity must take the form

$$I_{\mathcal{H}} = |\theta_0\rangle\langle\theta_0| + \int |z\rangle\langle z|d\mu(z), \tag{12}$$

where the measure $d\mu(z)$ can be determined as in [6] (see [7] for details). This measure is positive and non-singular. Some other interesting results are the form of the reproducing kernel $\langle z|z'\rangle$

$$\langle z|z'\rangle = \frac{{}_0F_2(1,2;\bar{z}z')}{\sqrt{{}_0F_2(1,2;|z|^2)\ {}_0F_2(1,2;|z'|^2)}}, \tag{13}$$

the dynamical evolution of the coherent states

$$U(t)|z\rangle = \frac{1}{\sqrt{{}_0F_2(1,2,|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!\sqrt{(n+1)!}} e^{-itH_\lambda}|\theta_{n+1}\rangle = e^{-i3t/2}|e^{-it}z\rangle, \tag{14}$$

and the expected value of the Hamiltonian H_λ in a coherent state

$$\langle z|H_\lambda|z\rangle = \frac{{}_0F_2(1,1;|z|^2)}{{}_0F_2(1,2;|z|^2)} + \frac{1}{2}. \tag{15}$$

3 The harmonic oscillator limit

Notice that H_λ tends to the harmonic oscillator Hamiltonian when $|\lambda| \to \infty$. Let us consider this limit to see if there is a relationship between the coherent states we have computed and the harmonic oscillator ones. In the limit, $\beta(x) \to x$; therefore, $b \to a$ and $b^\dagger \to a^\dagger$. Then, we get $|\theta_n\rangle \to |\psi_n\rangle$. Nevertheless, $A \to A_o = a^\dagger a^2$; as a consequence, the coherent states (10) become

$$|z\rangle_o \equiv \lim_{|\lambda|\to\infty} |z\rangle = \frac{1}{\sqrt{{}_0F_2(1,2;|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!\sqrt{(n+1)!}} |\psi_{n+1}\rangle, \tag{16}$$

which are not the usual coherent states. For $|z\rangle$ it is difficult to compute the expectation values of the position and momentum operators, but for $|z\rangle_o$ the problem can be easily solved using

$$\hat{x} = \frac{1}{\sqrt{2}}(a^\dagger + a), \qquad \hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a). \tag{17}$$

For the position operator we get

$${}_o\langle z|\hat{x}|z\rangle_o = \frac{z+\bar{z}}{\sqrt{2}} \frac{{}_0F_2(2,2;|z|^2)}{{}_0F_2(1,2;|z|^2)}; \tag{18}$$

$${}_o\langle z|\hat{x}^2|z\rangle_o = \frac{1}{2\,{}_0F_2(1,2;|z|^2)} \left(3\,{}_0F_2(1,2;|z|^2) + \frac{(z+\bar{z})^2}{2} {}_0F_2(2,3;|z|^2) \right). \tag{19}$$

For the momentum operator we obtain similar results. The uncertainty product is then

$$(\Delta\hat{x})(\Delta\hat{p}) = \sqrt{\left(\frac{3}{2}\right)^2 + \frac{3}{2}|z|^2\varrho(|z|) + [\mathrm{Re}(z)\mathrm{Im}(z)\varrho(|z|)]^2}, \tag{20}$$

where

$$\varrho(|z|) = \frac{{}_0F_2(1,2;|z|^2)\,{}_0F_2(2,3;|z|^2) - 2\,[{}_0F_2(2,2;|z|^2)]^2}{[{}_0F_2(1,2;|z|^2)]^2}. \tag{21}$$

A plot of $(\Delta\hat{x})(\Delta\hat{p})$ is shown in Figure 2. It can be rigorously proved that $1/2 \le (\Delta\hat{x})(\Delta\hat{p}) \le 3/2$.

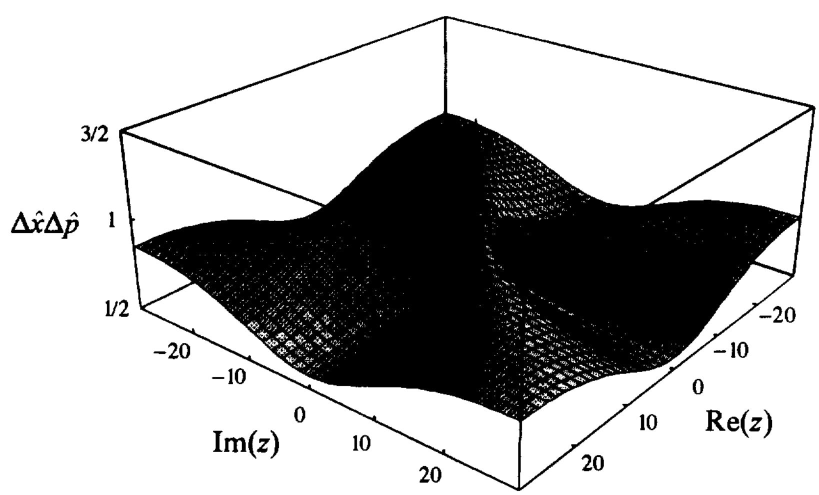

FIG. 2. The uncertainty product $(\Delta\hat{x})(\Delta\hat{p})$ as a function of z.

4 The Fock-Bargmann representation

For the harmonic oscillator it is possible to find a realization of the Hilbert space in terms of entire functions [4, 8]. The same is true for the coherent states of the Lie algebra su(1, 1) [6, 9]. We will show next that we can construct a similar realization for the problem under study. The Hilbert space $\mathcal{H}$ is generated by the basis vectors $\{|\theta_0\rangle, |\theta_1\rangle, |\theta_2\rangle, \dots\}$; the state $|\theta_0\rangle$ is isolated from the others, in the sense that it is an atypical coherent state. Let us call $\mathcal{H}_0$ the one-dimensional subspace generated by $|\theta_0\rangle$ and $\mathcal{H}_1$ the Hilbert space generated by $\{|\theta_1\rangle, |\theta_2\rangle, \dots\}$, so that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. From now on, we are going to concentrate on $\mathcal{H}_1$. A vector $|g\rangle \in \mathcal{H}_1$, is

$$|g\rangle = \sum_{m=1}^{\infty} c_m|\theta_m\rangle \in \mathcal{H}_1; \qquad c_m = \langle\theta_m|g\rangle; \qquad \langle g|g\rangle = \sum_{m=1}^{\infty} |c_m|^2 < \infty. \tag{22}$$

Using (10)

$$\langle z|g\rangle = \frac{1}{\sqrt{{}_0F_2(1,2;|z|^2)}} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!\sqrt{(n+1)!}} \langle\theta_{n+1}|g\rangle. \tag{23}$$

A realization of $\mathcal{H}_1$ as a space $\mathcal{F}$ of entire analytic functions is obtained by associating to every $|g\rangle \in \mathcal{H}_1$ the entire function

$$g(z) = \sum_{n=0}^{\infty} \frac{\langle\theta_{n+1}|g\rangle}{n!\sqrt{(n+1)!}} z^n; \qquad \langle z|g\rangle = \frac{g(\bar{z})}{\sqrt{{}_0F_2(1,2;|z|^2)}}. \tag{24}$$

From the relation $|g(z)| \leq \|g\|\sqrt{{}_0F_2(1,2;|z|^2)}$, $\forall\, g(z) \in \mathcal{F}$ (issued from the Schwarz inequality), we can show that $g(z)$ is an entire function of order 2/3 and type 3/2 (see [7]). This characterizes completely the space $\mathcal{F}$ (the usual coherent states are related to the Segal-Bargmann space of entire functions of growth (1/2, 2)). In particular, the entire function corresponding to a coherent state $|\alpha\rangle$ is

$$\alpha(z) = \frac{{}_0F_2(1,2;\alpha z)}{\sqrt{{}_0F_2(1,2;|\alpha|^2)}}. \tag{25}$$

The functions

$$\theta_{n+1}(z) = \frac{z^n}{n!\sqrt{(n+1)!}}, \quad n = 0, 1, 2, \dots, \tag{26}$$

form an orthonormal basis of $\mathcal{F}$ so that $g(z)$ may be written

$$g(z) = \sum_{n=0}^{\infty} c_{n+1}\theta_{n+1}(z). \tag{27}$$

Notice that the function $\delta(z, z') = {}_0F_2(1, 2; z\bar{z}')$ plays the role of the delta function in $\mathcal{F}$.

Finally, we want to know what is the abstract realization of the operators acting on $\mathcal{F}$ as a multiplication by z and as a derivation $\partial/\partial z$. Let us consider the function

$$zg(z) = \sum_{n=0}^{\infty} c_{n+1}\frac{z^{n+1}}{n!\sqrt{(n+1)!}} = \sum_{m=1}^{\infty} m\sqrt{m+1}\, c_m\theta_{m+1}(z). \tag{28}$$

On the other hand, the action of the operator $A^\dagger$ on $|g\rangle$ is

$$A^\dagger|g\rangle| = b^\dagger a^\dagger b \sum_{m=0}^{\infty} c_{m+1}|\theta_{m+1}\rangle = \sum_{n=1}^{\infty} c_n n\sqrt{n+1}\,|\theta_{n+1}\rangle. \tag{29}$$

Then, $A^\dagger$ is the operator whose realization in $\mathcal{F}$ is a multiplication by z. Let us consider now the function

$$\frac{\partial g(z)}{\partial z} = \sum_{m=1}^{\infty} c_{m+1}\frac{z^{m-1}}{(m-1)!\sqrt{(m+1)!}} = \sum_{m=1}^{\infty} \frac{c_{m+1}}{\sqrt{m+1}}\theta_m(z). \tag{30}$$

As $[A, A^\dagger] \neq I$, the abstract operator corresponding to the derivative is not A. Therefore, we have to find an operator B such that

$$B|g\rangle = \sum_{m=0}^{\infty} c_{m+1}B|\theta_{m+1}\rangle = \sum_{m=1}^{\infty} \frac{c_{m+1}}{\sqrt{m+1}}|\theta_m\rangle. \tag{31}$$

We suppose it has the form

$$B = b^\dagger a f(N) b, \quad N = a^\dagger a, \tag{32}$$

and the function f becomes

$$f(N) = \frac{1}{N(1+N)}. \tag{33}$$

It is easy to see that

$$[B, A^\dagger] = I, \qquad [A, B^\dagger] = I, \tag{34}$$

and therefore, up to normalization,

$$|z\rangle = \exp(zB^\dagger)|\theta_1\rangle. \tag{35}$$

However, it is not possible to obtain $|z\rangle$ as the action of a unitary representation of the algebras in (34).

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