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LYAPUNOV EXPONENTS FROM CHUA'S CIRCUIT TIME SERIES USING ARTIFICIAL NEURAL NETWORKS

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Abstract

In this paper we present the general problem of identifying if a nonlinear dynamic system has a chaotic behavior. If the answer is positive the system will be sensitive to small perturbations in the initial conditions which will imply that there is a chaotic attractor in its state space. A particular problem would be that of identifying a chaotic oscillator. We present an example of three well known different chaotic oscillators where we have knowledge of the equations that govern the dynamical systems and from there we can obtain the corresponding time series. In a similar example we assume that we only know the time series and, finally, in another example we have to take measurements in the Chua's circuit to obtain sample points of the time series. With the knowledge about the time series the phase plane portraits are plotted and from them, by visual inspection, it is concluded whether or not the system is chaotic. This method has the problem of uncertainty and subjectivity and for that reason a different approach is needed. A quantitative approach is the computation of the Lyapunov exponents. We describe several methods for obtaining them and apply a little known method of artificial neural networks to the different examples mentioned above. We end the paper discussing the importance of the Lyapunov exponents in the interpretation of the dynamic behavior of biological neurons and biological neural networks.

1 Introduction

In the companion paper [1] we described some findings about biological oscillators that have been presented in the recent literature. We also showed some examples of oscillator models. Here we want to review some models of chaotic oscillators with the goal of extending the analysis to time series (trains of action potentials) coming from biological oscillators where there are some hints that they are chaotic.

There are many experimental situations where there is no idea of what the mathematical model of a system could be or where the form of the equations is known but the parameters are unknown. There is an extensive literature about methods for systems identification but they are usually limited to linear models. Since the conditions for chaotic behavior arise from the presence of nonlinear elements, the use of linear methods is limited.

In recent years there have been many advances in the understanding of nonlinear dynamic systems and that has produced many methods for identifying whether or not a given system is chaotic. For testing these methods various simple chaotic systems have been discovered or invented. In some of them the equations that govern the system are well known but in others only some sort of approximation is known. In the former case, it is possible to generate the corresponding time series with very high approximation, in the latter case, the measurements yield a sampled version of the corresponding time series. Having at hand the equations and the time series, or at least the time series, it is possible, in very different ways, to compute the asymptotic properties of the system. Two measures are used for this: the Lyapunov exponents in which a positive one indicates chaotic dynamics, and the attractor's topological dimension which indicates topological characteristics and is directly related to the number of non-negative Lyapunov exponents [2].

It is usual to ascertain the existence of chaotic dynamic by means of visual inspection of the phase plane portrait. However, such method presents a considerable amount of uncertainty and subjectiveness. Taking that into account, it is important to have a quantitative method like the one provided by the computation of the Lyapunov exponents.

2 Lyapunov Exponents

To determine if a system possess chaotic dynamics it is necessary to know if it is sensitive to small perturbations on the initial conditions. When this occurs it is then impossible to predict the final state of the system after a finite time. To be able of characterizing a chaotic attractor it is necessary to establish quantitative measures concerning the sensitivity to initial conditions. The spectrum of Lyapunov exponents gives a method of quantifying the dynamics. The Lyapunov exponents describe the average rate of growing or shrinking of small perturbations in different directions in the state space. When the attractor has at least one positive exponent then it has the property of being sensitive to the initial conditions and it is called a chaotic attractor.

There are several methods for computing the Lyapunov exponents. Wolf *et al.* [3] were the first in suggesting a method to compute them directly from the time series, without knowing the equations that govern the system's dynamics. Kurths and Herzel [4] proposed another algorithm. However, in these algorithms the estimations are sensitive to the number of observations, to the sampling frequency and to the noise in the observations [3]. Trying to avoid these problems, Gencay and Dechert [5] designed an algorithm that computes the m Lyapunov exponents from an unknown m-dimensional dynamic system directly from a few observations on the attractor, in such a way that the estimation is robust even for certain amount of noise. This algorithm is based on a result by Hornik *et al.* [6] in which they show that the m Lyapunov exponents of a diffeomorphism that is topologically conjugate to the process that generates the data, are also the m Lyapunov exponents of that process. To obtain a robust estimation considering both few observations on the attractor and the presence of noise, Gencay and Dechert [5] applied artificial neural networks with a cascade architecture. Such procedure is a non-parametric estimation that Hornik *et al.* [6] [7] have shown to be universal approximators, that is, they can asymptotically approximate a function and its derivatives.

3 Computation of the Lyapunov Exponents

The Lyapunov exponents are constants, except for a zero-measure set, and describe the direction of nearby paths that converge or diverge in the state space of a dynamic system. The Lyapunov exponents λ_i are defined as the logarithm of the eigenvalues $\mu_i \; \forall \; i = 1, 2, \cdots, m$ of the symmetric positive matrix

$$\Lambda_x = \lim_{t \to \infty} \left[Y(\mathbf{x}; t)^{tr} Y(\mathbf{x}; t) \right]^{1/2t}, \tag{1}$$

where the matrix Y is dependent on the differential equation that characterizes the dynamical system. A direct application of the above definition is not practical since the Y matrix grows exponentially due to the fast convergence of the columns in the direction of greater expansion. Using topological properties and an appropriate QR decomposition, the Lyapunov exponents are found by computing

$$\lambda_i = \frac{1}{\Delta t} \lim_{t \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left(R_{ii}^j \right), \tag{2}$$

where R_{ii} are the diagonal elements of the triangular matrix R.

An alternative to the aforementioned algorithm is the use of neural networks which are capable of recovering a nonlinear map from a time series of iterates [8]. Here an unknown function is estimated and then it is possible to compute the Lyapunov exponents using the properties of the dynamic system [5].

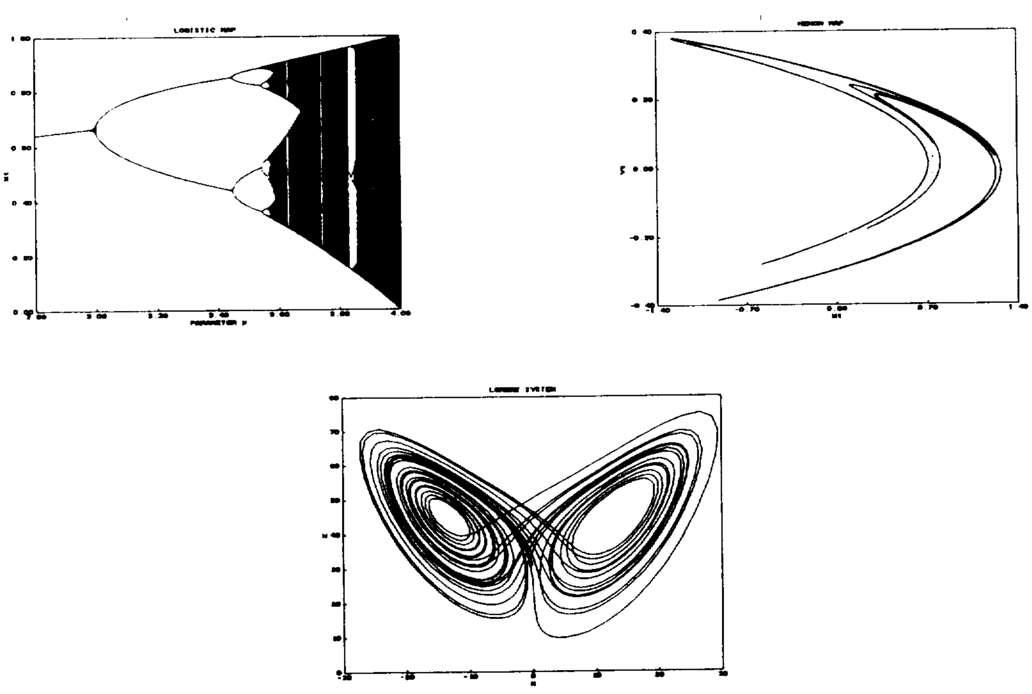

FIG. 1. Phase Plane Portraits for the Logistic Map, the Hénon Map, and the Lorenz System.

4 Examples of chaotic oscillators

To be able of testing the effectiveness of the Lyapunov exponents one has to have at hand dynamical systems with proved chaotic behavior. Many mathematical model systems are known, for instance: Hénon, Rossler-chaos, Lorenz, Rossler- hyperchaos, Mackey-Glass, and others [3]. Physical model systems are more difficult to produce, however we have the Belousov-Zhabotinsky chemical reaction [9] and Chua's nonlinear circuit family [10]. Obviously, there are real physical chaotic systems but they are extremely complex as it is the case of atmospheric turbulence.

TABLE I. True Lyapunov Exponents, equations representing the chaotic systems, initial conditions and parameters.

True Lyapunov Exponents		
Logistic map	Hénon map	Lorenz system
0.673	**0.440** **-1.620**	**1.51** **0.00** **-22.5**
$x_{t+1} = \mu x_t(1-x_t)$	$x_t = 1 - ax_t^2 + y_t$ $y_t = bx_t$	$\dot{x} = a(x-y)$ $\dot{y} = x(b-z) - y$ $\dot{z} = xy - cz$
$x_0 = 0.3, \mu = 4.0$	$x_0 = 0.1, y_0 = 0.0$ $a = 1.4, b = 0.3$	$x_0 = 0.0, y_0 = 1.1, z_0 = 0.0$ $a = 16.0, b = 45.92, c = 4.0$

TABLE II. Estimated Lyapunov Exponents. Logistic Map ($q = 5$, $T = 100$). Hénon Map ($q = 10$, $T = 200$). Lorenz System ($q = 15$, $T = 1000$). The error is less than $1\text{x}10^{-3}$. The non-spurious Lyapunov exponents are shown in boldface.

Estimated Lyapunov Exponents			
p	Logistic map	Hénon map	Lorenz system
1	**0.6794**		
2	**0.6401** -6.7823	**0.3670** **-1.5673**	
3	**0.6350** -2.4378 -2.4930	**0.4502** **-1.7331** -2.8164	**1.7285** **0.0411** **-23.72**
4	**0.6434** -1.61106 -1.7342 -5.0200	**0.4119** **-1.4803** -3.3658 -5.2263	**1.5910** **-0.0710** **-20.973** -80.325
5	**0.6790** -0.9073 -1.3544 -1.8468 -3.2313	**0.4385** **-1.5473** -1.4859 -1.7651 -2.5605	**1.4799** **0.0067** **-20.977** -60.702 -92.584

We divided this section of examples in three parts. In the first part the ordinary differential equations for the Lorenz model were integrated using the SDRIV2 subroutines form Kahaner *et al.* [12]. For the QR decomposition use used the subroutines in [13]. In the second and third parts we computed the Lyapunov exponents by means of the neural networks approach [5].

For the first part of the section we show the results of computing the Lyapunov exponents for mathematical model systems with well known chaotic dynamics [3] [5] [8]. We computed the Lyapunov exponents for the Logistic Map, for the Hénon map and for the Lorenz system [3] [5]. The corresponding phase plane portraits are shown in FIG. 1, and the equations, parameters and computed true values of the Lyapunov exponents are shown in TABLE I above.

For the second part of the section we show the results when we assume that for the same chaotic systems than above, the equations are not known, only the time series. In TABLE II we show the computed Lyapunov exponents where the presence of one positive exponent indicates that the system is chaotic.

According to the established notation for neural network architectures [11], p represents the number of nodes in the input layer and q represents the number of nodes in the hidden layer. The output layer has one node. The error is the quadratic average summation of the differences between the real and the estimated values for the time series, being T the total number of sample points in the sequence.

For the third part of the section we show the results obtained when we used a nonlinear circuit of the Chua's family with the parameters, components and initial conditions shown in FIG. 2. The temporal series was acquired by means of a digital storage scope, the x-coordinate is the voltage in the linear capacitor C1 and the y-coordinate is the voltage in the linear capacitor C2. A part of the phase plane portrait is also shown in FIG. 2, from it the temporal series was obtained using a sampling frequency of 500 Hz.

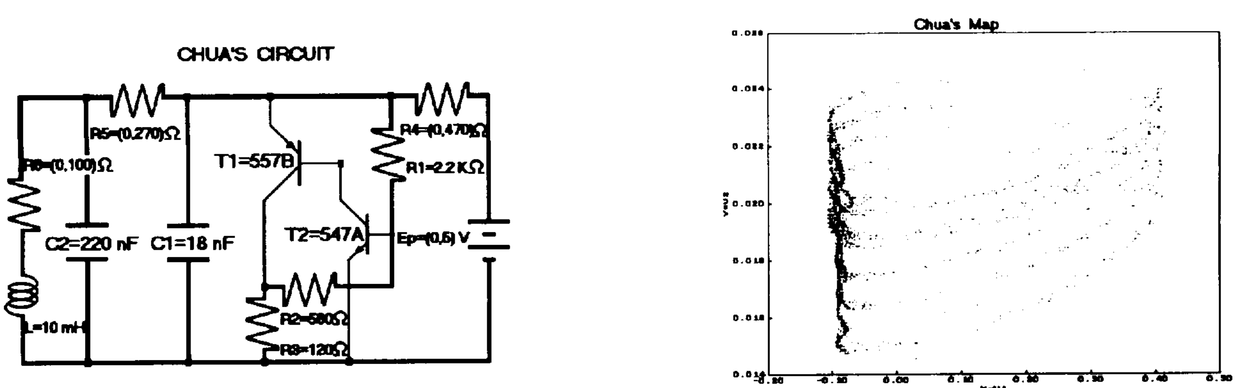

FIG. 2. Nonlinear Circuit of the Chua's Family and a part of its phase plane portrait that was plotted using as state variables the voltages in the capacitors C1 and C2. The phase plane portrait changes when the parameters in the circuit are varied between the limits denoted in the diagram.

The estimated Lyapunov exponents for the Chua's circuit are shown in TABLE III, when using the estimation for $q = 15$ and $T = 2500$. The error was less than 5×10^{-2}. Notice that λ_1 is a positive exponent which means that the dynamic behavior of the circuit is chaotic, as expected.

TABLE III. Estimated Lyapunov Exponents for the Chua's Circuit.

Estimated Lyapunov Exponents	
p	Chua's Circuit
3	λ_1=**+4.07216**
	λ_2=**−0.32160**
	λ_3=**−3.58462**

5 Concluding Remarks

We have shown well known examples of mathematical models of chaotic attractors: Logistic, Hénon, and Lorenz. We also showed an electronic model of a chaotic attractor: a circuit of the Chua's family. In all these examples we computed the Lyapunov exponents as a measure of the system's sensitivity to small perturbations in the initial conditions. For the example where we know the equations we used the standard method; for the other two examples, we used the method of Gencay and Dechert [5] that applies a neuronal network algorithm. We went from the easier examples to the difficult one, that is, the Chua's circuit where the time series is obtained from direct measurements in the circuit. In the former examples the true values of the Lyapunov exponents are known whereas in the Chua's circuit the Lyapunov exponents are estimated and to this has to be added the differences or variations, for any reason, in the parameters for the circuit's components. The justification for such trouble is that in a real chaotic system there is a complexity even worse that in the electronic model. Therefore, the circuit provides a very valuable experience that afterwards can benefit the understanding of the real chaotic attractor.

Recent experimental evidence points to biological neurons and biological neural networks as very likely sources of chaotic attractors. As in the case of chaotic chemical reactions the biological significance given to such behavior is speculative [9] [14]. Nevertheless, the application of the techniques described in this paper might be very useful for interpreting data from neurophysiological experiments where the electrical activity from many neurons is recorded simultaneously.

Similar to Chua's circuit situation, due to the usual difficult conditions of neurophysiological experimentation, in a biological neural network we can only obtain a short duration record of the compound time series (train of action potentials). The individual time series have to be separated and that procedure produces and additional source of uncertainty and every tool available for interpreting the results is welcomed [15].

As a very simple example (for details see companion paper [1]) let us consider the rhythmic firing single neuron *(1)* shown in the upper plot in FIG. 3. This time series was obtained from the simulation of a mathematical model for a single neuron and two of its phase plane portraits for two different physiological parameters ($V - f$, $V - h$) chosen as state variables show very clearly the possibility of chaotic behavior. In the lower plot in FIG. 3 we show another simulation of the rhythmic firing of a neuron *(2)* that belongs to a recurrent-ring network composed of single neurons like the ones given in *(1)*. From the two phase plane portraits we cannot conclude that the neuron *(2)* in the network is chaotic and the conclusion about neuron *(1)* required of an expert. This ambiguity can be surmounted if the Lyapunov exponents are computed for these time series and from there the importance of making such calculations and having experimental testing circuits for improving the confidence in the results.

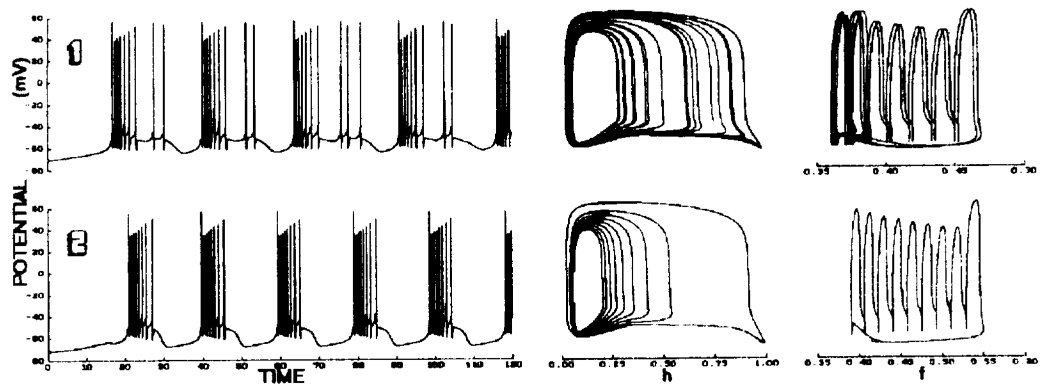

FIG. 3. *(1)* Rhytmically firing single neuron. *(2)* Rhytmically firing neuron belonging to a network formed with type *(1)* neurons.

On the other hand, in biological experiments there are problems similar to the ones present when making measurements in the Chua's circuit. When doing an extracellular recording, the duration of it is limited to a few minutes and afterwards a considerable amount of preprocessing is required to get to the individual contribution of each neuron recorded [16] [17]. The calculation of the Lyapunov exponents for these individual contributions could be added to help understanding the functional role of oscillatory neurons and oscillatory networks. That is the work that we are about to pursue.

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