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NASA Technical Memorandum 106897

P. 14

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Prepared for the
Ninth International Conference on Numerical Methods
in Laminar and Turbulent Flow
sponsored by the Georgia Institute of Technology
Atlanta, Georgia, July 10–14, 1995

National Aeronautics and
Space Administration

THE METHOD OF SPACE-TIME CONSERVATION ELEMENT AND SOLUTION ELEMENT –DEVELOPMENT OF A NEW IMPLICIT SOLVER

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Abstract

The method of space-time conservation element and solution element is a nontraditional numerical method designed from a physicist's perspective, i.e., its development is based more on physics than numerics. It uses only the simplest approximation techniques and yet is capable of generating nearly perfect solutions for a 2-D shock reflection problem used by Helen Yee and others. In addition to providing an overall view of the new method, in this paper we shall introduce a new concept in the design of implicit schemes, and use it to construct a highly accurate solver for a convection-diffusion equation. *It will be shown that, in the inviscid case, this new scheme becomes explicit and its amplification factors are identical to those of the Leapfrog scheme. On the other hand, in the pure diffusion case, its principal amplification factor becomes the amplification factor of the Crank-Nicolson scheme .*

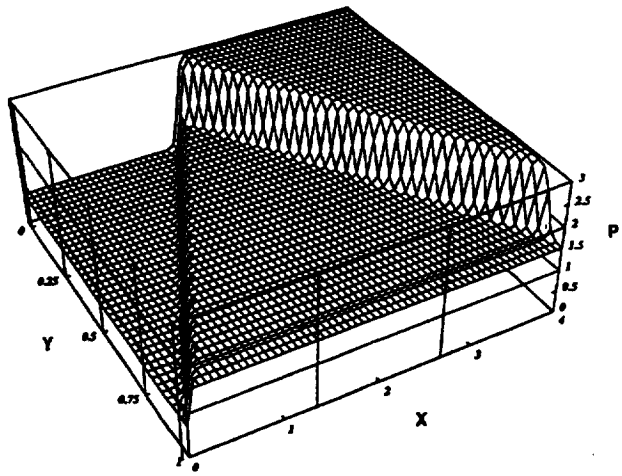

Figure 1.—Pressure distribution for a shock reflection problem (a flow of Mach number 2.9 enters from the left; the shock is reflected by a wall on the back).

1. Introduction

The method of space-time conservation element and solution element is a new numerical discretization method for solving conservation laws [1-11]. It is designed to

overcome several key limitations of the well-established methods–i.e., finite difference, finite volume, finite element, and spectral methods. At the root of its development is a constant drive toward simplicity, generality, and accuracy. To appreciate the significance of this effort, consider a 2-D time-marching Euler solver developed using the new method [2,10]. Even though it does not use (i) any approximation techniques that are more complicated than Taylor's expansion, (ii) any mesh-refinement techniques, (iii) any monotonicity constraints, (iv) any characteristics-based techniques, or (v) any ad hoc techniques that are used only in the neighborhood of a discontinuity, this solver is capable of generating highly accurate solutions for a 2-D shock reflection problem used by Helen Yee and others [12]. As shown in Fig. 1, *both the incident and the reflected shocks can be resolved by a single data point without the presence of numerical oscillations near the discontinuity*.

The Introduction section of [1] begins with a lengthy discussion that focuses more on physics than on numerics. From this discussion, readers can understand (i) the considerations that motivate the new method, and (ii) the key differences that separate it from the traditional methods mentioned above.

By using a set of design principles that are extracted from the above discussion, several two-level explicit schemes were constructed in [1,8] to solve (i) the pure convection equation

$$\partial u/\partial t + a\partial u/\partial x = 0 \tag{1.1}$$

and (ii) the convection-diffusion equation

$$\partial u/\partial t + a\partial u/\partial x - \mu\partial^2 u/\partial x^2 = 0 \tag{1.2}$$

where the convection velocity a, and the viscosity coefficient μ (> 0) are constants. These schemes were then extended to solve the 1-D time-dependent Euler and Navier-Stokes equations of a perfect gas [1,8]. Moreover, except for the Navier-Stokes solver, the above 1-D schemes have been generalized to their 2-D counterparts [2,10]. Because of the inherent simplicity and generality of the current method, the above multidimensional generalization is a straightforward matter. Also, as a result of the similarity in their designs, each of the above 2-D schemes shares with its 1-D version virtually the same fundamental characteristics.

In addition to providing an overall view of the new method, in this paper we shall describe a simple and innovative approach by which accurate implicit time-marching solvers can be constructed using the new method. A striking feature of this new treatment is that the modeling of the diffusion-related terms involves interpolation between neighboring mesh points while that of the convection-related term does not. As a preliminary, first we shall discuss the pros and cons of explicit and implicit schemes.

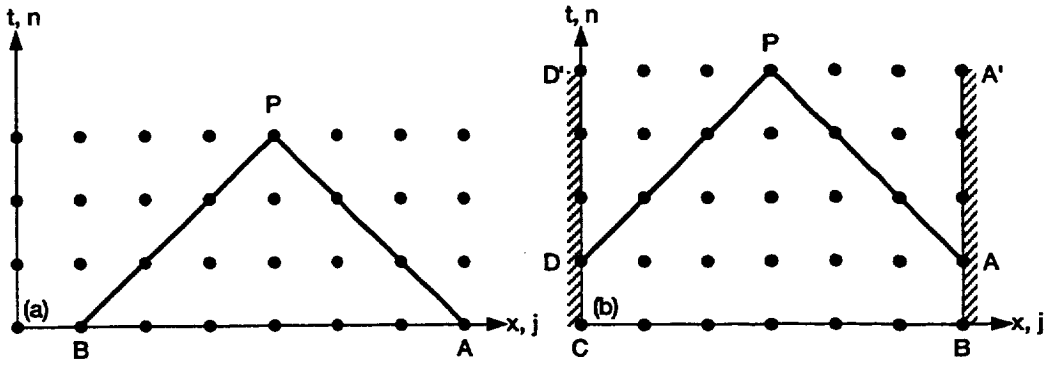

Figure 2.—(a) An initial-value problem. (b) An initial-value/boundary-value problem.

For a two-level explicit scheme, the value of a solution at any mesh point has a

finite domain of dependence at the previous time level. As an example, consider a finite-difference solver for Eq. (1.1). Let u_j^n, the mesh value of u at any mesh point (j, n) (point P in Fig. 2(a)) be determined by u_{j-1}^{n-1}, u_j^{n-1}, and u_{j+1}^{n-1}. Then the domain of dependence of u_j^n at the $(n-1)$th time level contains three mesh points. Also one can see that u_j^n is dependent only on the initial data given on the line segment $\bar{AB}$.

For an initial-value problem, such as a time-dependent Euler problem or a problem involving Eq. (1.1), the solution at any point in space-time also has a finite domain of dependence on the initial plane. As a result, explicit schemes could be ideal solvers for such a problem if they satisfy the requirement that the physical domain of dependence be a subset of the numerical domain of dependence.

On the other hand, the solution of an initial-value/boundary-value problem at any point in space-time is dependent on the initial data and the boundary data up to the time of the point under consideration. As an example, consider a problem involving Eq. (1.2). As shown in Fig. 2(b), the solution at point P is dependent on the initial/boundary data given on $A'B$, BC, and CD' where A', P, and D' are at the same time level. Let this problem be solved using the explicit scheme that was explained using Fig. 2(a). Let P also be a mesh point (j, n). Then u_j^n is dependent only on the initial/boundary data given on AB, BC and CD. *It is completely independent of those data given on AA' and DD'.* Contrarily, if the same problem is solved using an implicit scheme, then u_j^n is dependent on the initial/boundary data given on $A'B$, BC, and CD'. In other words, the numerical domain of dependence of the implicit scheme is consistent with the physical domain of dependence of the problem under consideration.

Two observations can be made as a result of the above discussions.

(a) Generally an explicit scheme is not an ideal solver for an initial-value/boundary-value problem. Because a time-dependent Navier-Stokes problem is such a problem, the above argument implies that an explicit scheme cannot be used to solve a time-dependent Navier-Stokes problem except for the special circumstance in which errors caused by neglecting certain initial/boundary data (such as those given on AA' and DD' in Fig. 2(a)) are relatively small. The factors that help achieve the above special circumstance include: (i) a small time-step size to spatial-mesh interval ratio, (ii) a small time rate of change of boundary data, and (iii) a small contribution of the viscous terms in the Navier-Stokes equations relative to that of the inertial terms. Note that condition (iii) may be met by a high-Reynolds-number flow.

(b) Generally an implicit scheme is not an ideal solver for an initial-value problem. This is because the domain of dependence of the former is far greater than the latter and, as a result, an implicit solution tends to be contaminated by extraneous information.

This completes the discussion of explicit and implicit schemes. A review of an explicit scheme [1,8] will precede the construction of an implicit solver.

2. Review of an Explicit Scheme

Let Eq. (1.1) be in a dimensionless form. Let $x_1 = x$ and $x_2 = t$ be considered as the coordinates of a two-dimensional Euclidean space E_2. By using Gauss' divergence theorem in the space-time E_2, it can be shown that Eq. (1.1) is the differential form of the integral conservation law

$$\oint_{S(V)} \vec{h} \cdot d\vec{s} = 0 \tag{2.1}$$

Here (i) $S(V)$ is the boundary of an arbitrary space-time region V in E_2, (ii) $\vec{h} = (au, u)$ is a current density vector in E_2, and (iii) $d\vec{s} = d\sigma\, \vec{n}$ with $d\sigma$ and $\vec{n}$, respectively, being the area and the outward unit normal of a surface element on $S(V)$. Note that (i) $\vec{h} \cdot d\vec{s}$ is the *space-time* flux of $\vec{h}$ leaving the region V through the surface element $d\vec{s}$,

and (ii) all mathematical operations can be carried out as though E_2 were an ordinary two-dimensional Euclidean space.

At this juncture, note that the conservation law Eq. (2.1) appears in a form in which space and time are unified and treated on the same footing. *This unity of space and time is a key characteristic that distinguishes the current method from most of the traditional methods.*

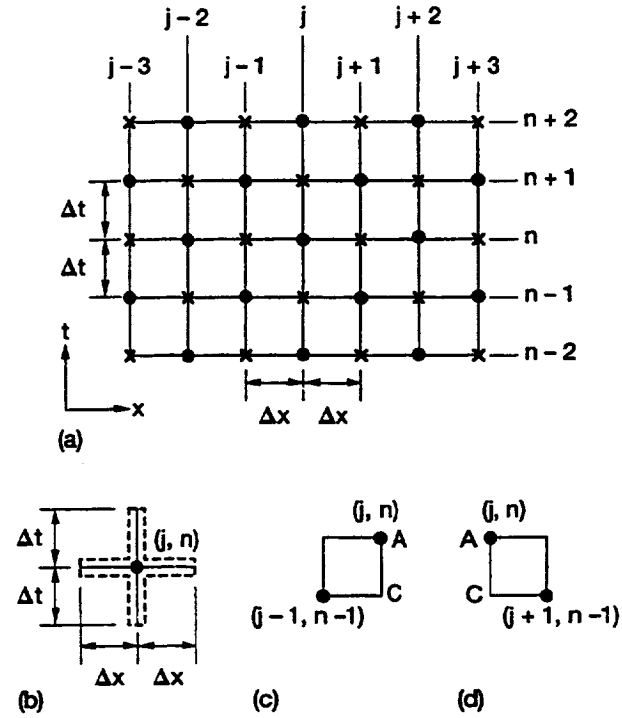

Figure 3.—The SEs and CEs. (a) The relative positions of SEs and CEs. (b) SE(j, n). (c) $CE_-(j, n)$. (d) $CE_+(j, n)$.

In the following, we shall review the inviscid version of the explicit α-μ scheme [1,8]. To achieve consistency of the notations used in this and the next sections, *the notations used here will be slightly different from those used in [1,8].*

Let Ω_1 denote the set of mesh points (j,n) in E_2 (dots in Fig. 3(a)) where $n = 0, \pm1, \pm2, \pm3, \ldots$, and, for each n, $j = n \pm 1, n \pm 3, n \pm 5, \ldots$. There is a solution element (SE) associated with each $(j,n) \in \Omega_1$. Let the solution element SE(j,n) be the *space-time* region bounded by the dashed curve depicted in Fig. 3(b). It includes a horizontal line segment, a vertical line segment, and their immediate neighborhood.

For any $(x,t) \in$ SE(j,n), $u(x,t)$, and $\vec{h}(x,t)$, respectively, are approximated by $u^*(x,t;j,n)$ and $\vec{h}^*(x,t;j,n)$ which we shall define shortly. Let

$$u^*(x,t;j,n) = u_j^n + (u_x)_j^n (x - x_j) + (u_t)_j^n (t - t^n) \tag{2.2}$$

where (i) u_j^n, $(u_x)_j^n$, and $(u_t)_j^n$ are constants in SE(j,n), and (ii) (x_j, t^n) are the coordinates of the mesh point (j,n). Note that

$$u^*(x_j,t^n;j,n) = u_j^n \qquad \frac{\partial u^*(x,t;j,n)}{\partial x} = (u_x)_j^n \qquad \frac{\partial u^*(x,t;j,n)}{\partial t} = (u_t)_j^n \tag{2.3}$$

Moreover, if we identify u_j^n, $(u_x)_j^n$, and $(u_t)_j^n$, respectively, with the values of u, $\partial u/\partial x$, and $\partial u/\partial t$ at (x_j, t^n), the expression on the right side of Eq. (2.2) becomes the first-

order Taylor's expansion of $u(x,t)$ at (x_j,t^n). Thus, u_j^n, $(u_x)_j^n$, and $(u_t)_j^n$ are the numerical analogues of the values of u, $\partial u/\partial x$, and $\partial u/\partial t$ at (x_j,t^n), respectively.

We shall require that $u = u^*(x,t;j,n)$ satisfy Eq. (1.1) within SE(j,n), i.e.,

$$(u_t)_j^n = -a\,(u_x)_j^n \tag{2.4}$$

Combining Eqs. (2.2) and (2.4), one has

$$u^*(x,t;j,n) = u_j^n + (u_x)_j^n\,[(x-x_j) - a\,(t-t^n)] \quad (x,t) \in \mathrm{SE}(j,n) \tag{2.5}$$

Because $\vec{h} = (au,\, u)$, we define

$$\vec{h}^*(x,t;j,n) = \left(au^*(x,t;j,n),\; u^*(x,t;j,n)\right) \tag{2.6}$$

Let E_2 be divided into nonoverlapping rectangular regions (see Fig. 3(a)) referred to as conservation elements (CEs). As depicted in Figs. 3(c) and 3(d), the CE with its top-right (top-left) vertex being the mesh point $(j,n) \in \Omega_1$ is denoted by CE$_-(j,n)$ (CE$_+(j,n)$). Obviously the boundary of CE$_-(j,n)$ (CE$_+(j,n)$) is formed by subsets of SE(j,n) and SE$(j-1,n-1)$ (SE$(j+1,n-1)$). The current approximation of Eq. (2.1) is

$$F_\pm(j,n) \stackrel{\mathrm{def}}{=} \oint_{S(CE_\pm(j,n))} \vec{h}^* \cdot d\vec{s} = 0 \tag{2.7}$$

for all $(j,n) \in \Omega_1$. In other words, the total flux leaving the boundary of any CE is zero. Note that the flux at any interface separating two neighboring CEs is calculated using the information from a single SE. As an example, the interface AC depicted in Figs. 3(c) and 3(d) is a subset of SE(j,n). Thus the flux at this interface is calculated using the information associated with SE(j,n).

Because (i) The CEs associated with Ω_1 can fill any space-time region, and (ii) the surface integration across any interface separating two neighboring CEs is evaluated using the information from a single SE, the local conservation condition Eq. (2.7) leads to a global conservation relation, i.e., *the total flux leaving the boundary of any space-time region that is the union of any combination of CEs will also vanish.*

With the aid of Eqs. (2.5)–(2.7), it can be shown that (see [1,8])

$$F_\pm(j,n)/\Delta x = \pm(1-\nu^2)\left[(u_x^+)_j^n + (u_x^+)_{j\pm1}^{n-1}\right] + (1 \mp \nu)\left(u_j^n - u_{j\pm1}^{n-1}\right) \tag{2.8}$$

where $\nu = a\Delta t/\Delta x$ is the Courant number, and $(u_x^+)_j^n = (\Delta x/2)(u_x)_j^n$. Note that here Δx and Δt, respectively, represent the same mesh interval and time-step size which were denoted by $\Delta x/2$ and $\Delta t/2$ in [1,8]. Using Eqs. (2.7) and (2.8), u_j^n and $(u_x^+)_j^n$, which are considered as *independent* unknowns at the mesh point (j,n), can be solved for in terms of $u_{j\pm1}^{n-1}$ and $(u_x^+)_{j\pm1}^{n-1}$ if $1-\nu^2 \neq 0$. It is shown in [1,8] that, for all $(j,n) \in \Omega_1$,

$$\vec{q}(j,n) = Q_+\,\vec{q}(j-1,\,n-1) + Q_-\,\vec{q}(j+1,\,n-1) \quad (1-\nu^2 \neq 0) \tag{2.9}$$

Here (i) $\vec{q}(j,n)$ is the column matrix formed by u_j^n and $(u_x^+)_j^n$, and (ii)

$$Q_\pm \stackrel{\mathrm{def}}{=} (1/2)\begin{pmatrix} 1 \pm \nu & \pm(1-\nu^2) \\ \mp 1 & -1 \pm \nu \end{pmatrix} \tag{2.10}$$

Eq. (2.9) defines a marching scheme. Because this scheme models Eq. (1.1) which is characterized by the parameter a, hereafter it will be referred to as the a scheme.

The a scheme is the only *two-level explicit* solver of Eq. (1.1) known to the authors to be neutrally stable, i.e., free from numerical dissipation. It also has the simplest stencil, i.e., a triangle with a vertex at the given time level and the other two vertices at the previous time level. Because the flux at an interface separating two neighboring CEs is evaluated using information of a single SE, *no interpolation or extrapolation is required*. Moreover, the a scheme is a two-way marching scheme, i.e., a backward marching scheme in which $\vec{q}(j,n)$ *is determined in terms of* $\vec{q}(j-1,n+1)$ *and* $\vec{q}(j+1,n+1)$ can also be derived from Eq. (2.8), the same relation that gives us the forward marching scheme Eq. (2.9). These and other nontraditional features of the a scheme are discussed in depth in [1,8].

In the above construction of the a scheme, we use the SEs and CEs of the mesh points marked by dots in Fig. 3(a). A similar construction can be performed by using the mesh points marked by crosses in Fig. 3(a). Let Ω_2 denote the set of mesh points (j,n) in E_2 (crosses in Fig. 3(a)) where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, and, for each n, $j = n, n \pm 2, n \pm 4, \ldots$. Let the SEs and CEs of Ω_2 be defined by using Figs 3(b)–(d) with dots replaced by crosses. Obviously (i) the CEs of Ω_2 also fill any space-time region, and (ii) the a scheme can also be constructed using the SEs and CEs of Ω_2. This new scheme is defined by Eq. (2.9) with $(j,n) \in \Omega_2$.

Before we proceed further, some intricate points related to the above constructions will be clarified with the following remarks:

(a) $\mathrm{SE}(j,n)$ may intersect $\mathrm{SE}(j',n')$ if $(j,n) \in \Omega_1$ and $(j',n') \in \Omega_2$.

(b) Let $(j,n) \in \Omega_1$. Then (i) $(j+1,n) \in \Omega_2$, and (ii) $\mathrm{CE}_+(j,n)$ and $\mathrm{CE}_-(j+1,n)$ represent the same rectangle in E_2. However, because the function $\vec{h}^*$ used in the evaluation of $F_+(j,n)$ is tied to a pair of SEs associated with Ω_1, while that used in the evaluation of $F_-(j+1,n)$ is tied to another pair of SEs associated with Ω_2, $F_+(j,n) = 0$ and $F_-(j+1,n) = 0$ represent *two completely independent flux conservation conditions*.

To prepare for the development of the implicit solver to be described in the next section, we shall combine the above two independent schemes into a single scheme. The new scheme, referred to as the $a(2)$ scheme, is defined by Eq. (2.9) with $(j,n) \in \Omega$ where $\Omega \stackrel{\text{def}}{=} \Omega_1 \cup \Omega_2$. Obviously, a solution of the $a(2)$ scheme is formed by two decoupled solutions with each being associated with a mesh that is staggered in time. Several classical schemes also have this property. Among them are the Leapfrog, the DuFort-Frankel, and the Lax schemes [13].

We conclude this section with a review of other 1-D extensions [1,8] of the a scheme:

(a) The a-μ scheme is an explicit solver for Eq. (1.2) ($\mu \geq 0$). It reduces to the a scheme if $\mu = 0$. By using exactly the same procedure by which the $a(2)$ scheme is formed from two independent a schemes, the a-$\mu(2)$ scheme is defined using two independent a-μ schemes.

(b) A scheme that has no numerical dissipation, such as the a scheme, generally can not be extended to solve the Euler equations. Hence, the a scheme is modified to become the a-ϵ scheme. Stability of this scheme is limited by the *CFL* condition and $0 \leq \epsilon \leq 1$ where ϵ is a special parameter that controls numerical dissipation. If $\epsilon = 0$, the a-ϵ scheme reduces to the a scheme. By using exactly the same procedure by which the $a(2)$ scheme is formed from two independent a schemes, the a-$\epsilon(2)$ scheme is defined using two independent a-ϵ schemes.

3. The implicit scheme

An implicit solver for Eq. (1.2), referred to as the a-$\mu(I1)$ scheme, will be discussed in this section. Here "I" stands for "implicit", and "1" is the identification number. This solver is the model for implicit time-dependent Navier-Stokes solvers

under development. It is constructed to meet two requirements given in the following discussion:

(a) With a few exceptions, numerical dissipation generally appears in a numerical solution of a time-marching problem. In other words, the numerical solution dissipates faster than the corresponding physical solution. For a nearly inviscid problem, e.g., flow at a large Reynolds number, this could be a serious difficulty because numerical dissipation may overwhelm physical dissipation and cause a complete distortion of solutions. To avoid such a difficulty, the model solver is required to have the property that *the numerical dissipation shall approach zero as the physical dissipation approaches zero.*

(b) The convection term and the diffusion term in Eq. (1.2) involve the spatial derivatives of first order and second order, respectively. Thus, in a spatial region where a solution is very smooth, the diffusion term is negligible compared with the convection term. As a result, the *effective* physical domain of dependence is more or less dictated by Eq. (1.1). To prevent excessive contamination of the solution by extraneous information, *the implicit solver shall be required to become an explicit solver in the limiting case in which the diffusion term vanishes.*

Because of the requirements set forth in (a) and (b), the implicit solver will be constructed such that it reduces to the $a(2)$ scheme if $\mu = 0$. The former differs from the latter only in the extra modeling involving the diffusion-related terms. Note that the presence of viscosity is felt through (i) the diffusion term $-\mu\partial^2 u/\partial x^2$ in Eq. (1.2), and (ii) the spatial diffusion flux component $-\mu\partial u/\partial x$ in the flux vector

$$\vec{h} = (au - \mu\partial u/\partial x,\ u) \tag{3.1}$$

Note that Eq. (1.2) is the differential form of Eq. (2.1) if $\vec{h}$ is defined according to Eq. (3.1). Also, in this paper, any term in Eq. (1.2) or on the right side of Eq. (3.1) is considered to be convection-related if it is not diffusion-related.

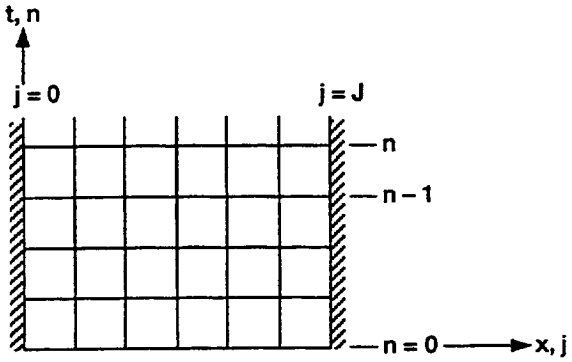

Figure 4.—The space-time mesh for the implict solvers (J = 6).

To construct the a-$\mu(I1)$ scheme, consider the mesh depicted in Fig. 4 ($J \geq 4$). We assume that (i) $u = u_I(x)$ at $t = 0$, (ii) $u = u_L(t)$ at $x = 0$, and (iii) $u = u_R(t)$ at $x = J\Delta x$, where $u_I(x)$, $u_L(t)$, and $u_R(t)$ are given functions. Moreover, for the current case, (i) Ω_1 and Ω_2 are restricted by the conditions $n \geq 0$ and $J \geq j \geq 0$, (ii) $\mathrm{CE}_{\pm}(j,n)$ are not defined if $n = 0$, (iii) $\mathrm{CE}_{-}(j,n)$ is not defined if $j = 0$, and (iv) $\mathrm{CE}_{+}(j,n)$ is not defined if $j = J$. Items (iii) and (iv) imply that *only one conservation condition is associated with a boundary mesh point.* Obviously, the definition of $\mathrm{SE}(j,n)$ also needs to be appropriately modified if $j = 0$, or $j = J$, or $n = 0$.

Eq. (2.2) will still be assumed. We also assume that, for $n = 0, 1, 2, \ldots$,

$$u_0^n = u_L(t^n) \qquad (u_t)_0^n = \dot{u}_L(t^n) \qquad u_J^n = u_R(t^n) \qquad (u_t)_J^n = \dot{u}_R(t^n) \tag{3.2}$$

where $\dot{u}_L(t) \stackrel{\text{def}}{=} du_L(t)/dt$ and $\dot{u}_R(t) \stackrel{\text{def}}{=} du_R(t)/dt$. Thus, only one unknown, i.e., $(u_x)_j^n$, and one conservation condition are associated with a boundary mesh point (j,n).

Furthermore, for an interior mesh point (j,n), we replace Eq. (2.4) with

$$(u_t)_j^n = -a(u_x)_j^n + \frac{\mu}{2\Delta x}\left[(u_x)_{j+1}^n - (u_x)_{j-1}^n\right] \quad J > j > 0 \tag{3.3}$$

Eq. (3.3) is the numerical analogue of the differential condition Eq. (1.2). A comparison between Eqs. (1.2) and (3.3) reveals that *the diffusion term $-\mu\partial^2 u/\partial x^2$ at an interior mesh point $\in \Omega_1$ (Ω_2) is modeled by a central-difference approximation involving the values of u_x at two neighboring mesh points $\in \Omega_2$ (Ω_1).* Eqs. (2.2) and (3.3) imply that, for $J > j > 0$ and $(x,t) \in \mathrm{SE}(j,n)$,

$$u^*(x,t\,;j,n) = u_j^n + (u_x)_j^n(x-x_j) + \left\{\frac{\mu}{2\Delta x}\left[(u_x)_{j+1}^n - (u_x)_{j-1}^n\right] - a(u_x)_j^n\right\}(t-t^n) \tag{3.4}$$

Next, as a result of Eq. (3.1), Eq. (2.6) is replaced by

$$\vec{h}^*(x,t\,;j,n) = \left(au^*(x,t\,;j,n) - \mu u_x^*(x,t\,;j,n),\, u^*(x,t\,;j,n)\right) \tag{3.5}$$

Here, for any $(x,t) \in \mathrm{SE}(j,n)$,

$$u_x^*(x,t\,;j,n) \stackrel{\text{def}}{=} \begin{cases} \left[(u_x)_j^n + (u_x)_j^{n+1}\right]/2, & \text{if } t \geq t^n; \\ \left[(u_x)_j^n + (u_x)_j^{n-1}\right]/2, & \text{if } t < t^n. \end{cases} \tag{3.6}$$

Consider any point (x,t) on the line segment joining the mesh points (j,n) and $(j,n-1)$. Then (x,t) belongs to both $\mathrm{SE}(j,n)$ and $\mathrm{SE}(j,n-1)$. According to Eq. (3.6), for this point (x,t),

$$u_x^*(x,t\,;j,n) = u_x^*(x,t\,;j,n-1) = \left[(u_x)_j^n + (u_x)_j^{n-1}\right]/2 \tag{3.7}$$

Thus, *the same numerical diffusion flux component is assigned to the point (x,t) regardless of whether it is considered as a point in $SE(j,n)$ or a point in $SE(j,n-1)$.*

With the above modifications, the a-$\mu(I1)$ scheme is defined by assuming Eq. (2.7). Note that:

(a) At a mesh point $\in \Omega_1(\Omega_2)$, the diffusion-related terms in Eqs. (1.2) and (3.1) are modeled using *interpolations* that may involve the numerical values of the mesh points $\in \Omega_2(\Omega_1)$. *This contrasts sharply with the modeling of the convection-related terms which uses no interpolation.*

(b) In the $a(2)$ scheme, the two sets of numerical variables associated with Ω_1 and Ω_2 are completely decoupled from each other. Contrarily, *they are "glued" together in the a-$\mu(I1)$ scheme through the interpolations referred to in (a).*

To proceed, let $\alpha = \mu\Delta t/(\Delta x)^2$, and $(u_t^+)_j^n \stackrel{\text{def}}{=} (\Delta t/2)(u_t)_j^n$. Also let

$$\begin{aligned}(S_+)_j^n \stackrel{\text{def}}{=}\ & (1-\nu)u_{j+1}^n + (\frac{\nu}{2} - 1)\alpha(u_x^+)_j^n - (1-\nu^2-\alpha)(u_x^+)_{j+1}^n \\ & - \frac{\nu\alpha}{2}(u_x^+)_{j+2}^n \quad (n = 0,1,2,\ldots;\, j = 0,1,2,\ldots,J-2)\end{aligned} \tag{3.8}$$

$$(S_+)_{j-1}^n \stackrel{\text{def}}{=} (1-\nu)u_j^n - \alpha(u_x^+)_{j-1}^n - (1-\alpha)(u_x^+)_j^n - \nu(u_t^+)_j^n \quad (n = 0,1,2,\ldots) \tag{3.9}$$

$$\begin{aligned}(S_-)_j^n \stackrel{\text{def}}{=}\ & (1+\nu)u_{j-1}^n + (\frac{\nu}{2} + 1)\alpha(u_x^+)_j^n + (1-\nu^2-\alpha)(u_x^+)_{j-1}^n \\ & - \frac{\nu\alpha}{2}(u_x^+)_{j-2}^n \quad (n = 0,1,2,\ldots;\, j = 2,3,4,\ldots,J))\end{aligned} \tag{3.10}$$

$$(S_-)_1^n \stackrel{\text{def}}{=} (1+\nu)u_0^n + \alpha(u_x^+)_1^n + (1-\alpha)(u_x^+)_0^n + \nu(u_t^+)_0^n \quad (n=0,1,2,\ldots) \tag{3.11}$$

By using Eqs. (2.2), (3.2), (3.4)–(3.6), and (3.8)–(3.11), Eq. (2.7) implies that

$$\begin{aligned}&(1\mp\nu)u_j^n \pm (1-\nu^2+\alpha)(u_x^+)_j^n + (\frac{\nu}{2}\mp 1)\alpha(u_x^+)_{j\pm1}^n - \frac{\nu\alpha}{2}(u_x^+)_{j\mp1}^n \\ &= (S_\pm)_j^{n-1} \quad (n=1,2,3,\ldots;j=1,2,\ldots,J-1)\end{aligned} \tag{3.12}$$

$$(1+\alpha)(u_x^+)_0^n - \alpha(u_x^+)_1^n = (S_+)_0^{n-1} - (1-\nu)u_0^n - \nu(u_t^+)_0^n \quad (n=1,2,\ldots) \tag{3.13}$$

$$\alpha(u_x^+)_{J-1}^n - (1+\alpha)(u_x^+)_J^n = (S_-)_J^{n-1} - (1+\nu)u_J^n + \nu(u_t^+)_J^n \quad (n=1,2,\ldots) \tag{3.14}$$

Eqs. (3.12)–(3.14) define the a-$\mu(I1)$ scheme.

Note that Eq. (3.12) represents a pair of equations for each (j,n). Let $1-\nu^2 \neq 0$. Then these equations are equivalent to

$$-\alpha(u_x^+)_{j-1}^n + 2(1-\nu^2+\alpha)(u_x^+)_j^n - \alpha(u_x^+)_{j+1}^n = (1+\nu)(S_+)_j^{n-1} - (1-\nu)(S_-)_j^{n-1} \tag{3.15}$$

and

$$u_j^n = \frac{1}{2}\Big\{(S_+)_j^{n-1} + (S_-)_j^{n-1} + \alpha\left[(u_x^+)_{j+1}^n - (u_x^+)_{j-1}^n\right]\Big\} \tag{3.16}$$

where $j=1,2,3,\ldots,J-1$. In the following discussion, Eqs. (3.12)–(3.14) will be replaced by Eqs. (3.13)–(3.16).

Let the marching variables at the $(n-1)$th time level be given. With the aid of Eq. (3.2), the expressions on the right sides of Eqs. (3.13)–(3.15) can be considered as given source terms. Thus these equations form a tridiagonal system of $J+1$ equations for the $J+1$ unknowns $(u_x^+)_j^n$, $j=0,1,2,\ldots,J$. It can be shown that [3] the coefficient matrix is strictly diagonally dominant in rows and columns, i.e., stability of the Thomas algorithm [13, p.99] is assured, if

$$\nu^2 < 1 \qquad \text{and} \qquad \alpha \geq 0 \tag{3.17}$$

Upon obtaining $(u_x^+)_j^n$, $j=0,1,2,\ldots,J$, the other unknowns u_j^n, $j=1,2,\ldots,J-1$, can be obtained using Eq. (3.16).

Using a procedure described in [1,8], the stability of the a-$\mu(I1)$ scheme was studied using the von Neumann analysis [3]. Because there are two independent marching variables at each (j,n), the a-$\mu(I1)$ scheme has two amplification factors. The principal amplification factor λ_+ and the spurious amplification factor λ_- are given by

$$\lambda_\pm = \frac{-Q \pm \sqrt{Q^2 + (1-\nu^2)^2 - \alpha^2(1-\cos\theta)^2}}{1-\nu^2+\alpha(1-\cos\theta)} \tag{3.18}$$

where θ, $-\pi < \theta \leq \pi$, is the phase angle variation per Δx, and

$$Q \stackrel{\text{def}}{=} \alpha(1-\cos\theta-\nu^2\sin^2\theta) + i\nu(1-\nu^2)\sin\theta \quad (i \stackrel{\text{def}}{=} \sqrt{-1}) \tag{3.19}$$

It is concluded from numerical evaluations of Eq. (3.18) that $|\lambda_\pm| \leq 1$, i.e., the a-$\mu(I1)$ scheme is stable, if Eq. (3.17) is satisfied.

Note that many other implicit solvers are unconditionally stable. However, the price paid for this "desirable" property usually is excessive numerical dissipation. Moreover, the use of a time-step size that is greater than that allowed by Eq. (3.17) generally results in a less accurate time-dependent solution. Thus we do not consider the more restrictive stability condition Eq. (3.17) to be a disadvantage of the a-$\mu(I1)$ scheme.

One can also conclude from Eq. (3.18) that: (i) $\lambda_\pm$ *are reduced to the amplification factors of the Leapfrog scheme if* $\mu = 0$ and (ii) λ_+ *is reduced to the amplification factor of the Crank-Nicolson scheme [13, p.112] and* $\lambda_- = -1$ *if* $a = 0$.

The fact that the amplification factors of the a-$\mu(I1)$ *scheme are related to those of two celebrated classical schemes is only one among a string of similar coincidences.* Other coincidences are summarized in the following remarks [1,3,8]:

(a) Because the a-$\mu(I1)$ scheme reduces to the $a(2)$ scheme if $\mu = 0$, *the amplification factors of the* $a(2)$ *scheme are those of the Leapfrog scheme.*

(b) *The amplification factors of the* a-$\mu(2)$ *scheme reduce to those of the Leapfrog scheme if* $\mu = 0$, *and to those of DuFort-Frankel scheme if* $a = 0$.

(c) *If* $\epsilon = 0$, *the amplification factors of the* a-$\epsilon(2)$ *scheme reduce to those of the Leapfrog scheme. On the other hand, if* $\epsilon = 1$, *these two factors become the same function of the Courant number and the phase angle. It so happens that this function is also the amplification factor of the highly diffusive Lax scheme.*

In summary, *the amplification factors of the Leapfrog, the Crank-Nicolson, the DuFort-Frankel, and the Lax schemes are related to those of the current model schemes.*

In [3], by using the von Neumann analysis, it is shown that the a-$\mu(I1)$ scheme is more accurate if a spurious component is filtered out from its initial conditions. Let $\dot{u}_I(x) = du_I(x)/dx$ where $u_I(x)$ is the given function introduced earlier. Then it is shown in [3] that, after filtering, the initial conditions are (i)

$$u_j^0 = \{u_I(x_{j+1}) + u_I(x_{j-1}) + 2u_I(x_j) + (\Delta x/2)\,[\dot{u}_I(x_{j-1}) - \dot{u}_I(x_{j+1})]\}/4 \qquad (3.20)$$

$$(u_x^+)_j^0 = \{u_I(x_{j+1}) - u_I(x_{j-1}) + (\Delta x/2)\,[2\dot{u}_I(x_j) - \dot{u}_I(x_{j+1}) - \dot{u}_I(x_{j-1})]\}/4 \qquad (3.21)$$

where $j = 1, 2, 3, \ldots, J-1$, (ii) $u_0^0 = u_I(x_0)$ and $u_J^0 = u_I(x_J)$, and (iii)

$$(u_x^+)_0^0 = \{u_I(x_1) - u_I(x_0) + (\Delta x/2)\,[\dot{u}_I(x_0) - \dot{u}_I(x_1)]\}/2 \qquad (3.22)$$

$$(u_x^+)_J^0 = \{u_I(x_J) - u_I(x_{J-1}) + (\Delta x/2)\,[\dot{u}_I(x_J) - \dot{u}_I(x_{J-1})]\}/2 \qquad (3.23)$$

This section is concluded with a brief discussion of another implicit scheme, referred to as the a-$\mu(I2)$ scheme. It differs from the a-$\mu(I1)$ scheme only in one aspect, i.e., Eq. (3.3) is replaced by

$$(u_t)_j^n = -a(u_x)_j^n + \frac{\mu}{(\Delta x)^2}(u_{j+1}^n + u_{j-1}^n - 2u_j^n) \quad J > j > 0 \qquad (3.24)$$

As a result, Eq. (3.4) must be modified accordingly. The details of this scheme will be given in [3]. Note that (i) stability of the a-$\mu(I2)$ scheme is also limited by Eq. (3.17), and (ii) the a-$\mu(I1)$ and a-$\mu(I2)$ schemes are identical if $\mu = 0$ or $a = 0$.

4. Numerical Results

Three test problems will be used to evaluate the accuracy of the a-$\mu(I1)$ scheme. In the first problem, we consider a special case of Eq. (1.2) ($a = 0$ and $\mu = 1$) with $0 \le x \le 1$ and $t \ge 0$. The analytical solution $u = U(x,t)$ with the initial/boundary conditions (i) $u = 0$, $x = 0$ and 1, $t \ge 0$; (ii) $u = 2x$, $0 \le x \le 0.5$, $t = 0$; and (iii) $u = 2(1-x)$, $0.5 \le x \le 1$, $t = 0$, is given on p.15 of [14]. Obviously, $U(0.5 + x', t) = U(0.5 - x', t)$ for $|x'| \le 0.5$.

The numerical solutions given in Table 1 are generated assuming $\Delta x = 0.1$, and $\Delta t = 0.01$. Note that the a-$\mu(I1)$ scheme, like the Crank-Nicolson scheme, is unconditionally stable in this case ($a = 0$). From this table, one concludes that the numerical solutions from these two schemes become closer as time increases. This conclusion is

consistent with the fact that the principal amplification factor of the a-$\mu(I1)$ scheme is identical to the amplification factor of the Crank-Nicolson scheme.

TABLE 1.—COMPARISON OF THE SOLUTIONS OF THE FIRST TEST PROBLEM OBTAINED USING THE ANALYTICAL METHOD, THE a-μ(I1) SCHEME, AND THE CRANK-NICOLSON SCHEME

	x=	0.1	0.2	0.3	0.4	0.5
t = 0.01	analytical	0.1996	0.3966	0.5799	0.7201	0.7743
	a-μ(I1)	0.1992	0.3967	0.5876	0.7286	0.7768
	Crank-Nicolson	0.1989	0.3956	0.5834	0.7381	0.7691
t = 0.1	analytical	0.0934	0.1776	0.2444	0.2873	0.3021
	a-μ(I1)	0.0947	0.1802	0.2481	0.2917	0.3067
	Crank-Nicolson	0.0948	0.1803	0.2482	0.2918	0.3069

In the sceond problem, we consider a special case of Eq. (1.2) ($a = \mu = 1$) with $0 \leq x \leq 1$ and $t \geq 0$. The steady-state solution with the boundary conditions (i) $u = 1$, $x = 0$; and (ii) $u = 0$, $x = 1$, is given on p.155 of [13].

Let (i) $u_I(x) = 1 - x$, (ii) $\Delta x = 0.25$, and (iii) $\nu = 0.5$ (i.e., $\Delta t = 0.125$). After 40 time steps, the numerical values of u at $x = 0.25$, 0.5, and 0.75 are 0.8356, 0.6234, and 0.3504, respectively. The corresponding exact steady-state values are 0.8347, 0.6225, and 0.3499, respectively. The numerical results are accurate to almost *three* significant figures even though only *three* interior mesh points are used.

In the third problem, again we consider a special case of Eq. (1.2) ($a = 1$ and $\mu = 0.1$) with $0 \leq x \leq 1$ and $t \geq 0$. The functions $u_I(x)$, $u_L(t)$ and $u_R(t)$ are defined such that they are consistent with a special solution to Eq. (1.2), i.e.,

$$u = u_e(x,t) \stackrel{\text{def}}{=} \exp(-4\pi^2\mu t)\sin[2\pi(x - at)] \tag{4.1}$$

Let $u_{ex}(x,t) \stackrel{\text{def}}{=} \partial u_e(x,t)/\partial x$. At any time $t = t^n$, let

$$L_1(u) \stackrel{\text{def}}{=} \frac{1}{(J-1)\exp(-4\pi^2\mu t^n)} \sum_{j=1}^{J-1} |u_j^n - u_e(x_j,t^n)| \tag{4.2}$$

$$L_1(u_x) \stackrel{\text{def}}{=} \frac{1}{(J+1)2\pi\exp(-4\pi^2\mu t^n)} \sum_{j=0}^{J} |(u_x)_j^n - u_{ex}(x_j,t^n)| \tag{4.3}$$

$L_1(u)$ and $L_1(u_x)$ are two error norms (per mesh point) which are normalized by the decay factors of $u_e(x,t)$ and $u_{ex}(x,t)$, respectively.

Let $J = 80$ (i.e., $\Delta x = 1/80$) and $\nu = 0.8$ (i.e., $\Delta t = 0.01$). Then $L_1(u) = 0.2312 \times 10^{-2}$ and $L_1(u_x) = 0.1363 \times 10^{-1}$ at $t = 1$ (i.e., $n = 100$). Through numerical experiments, it has been shown that both $L_1(u)$ and $L_1(u_x)$ at a given time t are reduced by a factor of 4 if both Δx and Δt are reduced by half.

5. Conclusions and Discussions

In the explicit a-μ scheme [1,8], the diffusion term in Eq. (1.2) is not modeled, i.e., Eq. (2.4) is assumed. Also the diffusion term in Eq. (3.1) is modeled with no interpolation or extrapolation. Obviously, it can be used only when the diffusion term

is small compared with the convection term. Contrarily, the diffusion terms in both Eqs. (1.2) and (3.1) are modeled in the current implicit solvers.

A more complete discussion of the a-$\mu(I1)$ and a-$\mu(I2)$ schemes will be given in [3]. In this paper, we shall (i) study the stability, dissipation, dispersion, consistency, and truncation error of the above two schemes, and (ii) discuss other explicit solvers for Eq. (1.2) in which the diffusion terms in both Eqs. (1.2) and (3.1) are modeled.

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	April 1995	Technical Memorandum

4. TITLE AND SUBTITLE

The Method of Space-Time Conservation Element and Solution Element–Development of a New Implicit Solver

5. FUNDING NUMBERS

WU–505–62–52

6. AUTHOR(S)

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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135–3191

8. PERFORMING ORGANIZATION REPORT NUMBER

E–9545

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Washington, D.C. 20546–0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

NASA TM–106897

11. SUPPLEMENTARY NOTES

Prepared for the Ninth International Conference on Numerical Methods in Laminar and Turbulent Flow sponsored by the Georgia Institute of Technology, Atlanta, Georgia, July 10–14, 1995. S.C. Chang, NASA Lewis Research Center; X.Y. Wang and C.Y. Chow, University of Colorado, Boulder, Colorado 80309–0429; A. Himansu, Cleveland Telecommunications Corporation, Solon, Ohio 44139. Responsible person, S.C. Chang, organization code 2660, (216) 433–5874.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified - Unlimited
Subject Category 64

This publication is available from the NASA Center for Aerospace Information, (301) 621–0390.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

The method of space-time conservation element and solution element is a nontraditional numerical method designed from a physicist's perspective, i.e., its development is based more on physics than numerics. It uses only the simplest approximation techniques and yet it capable of generating nearly perfect solutions for a 2-D shock reflection problem used by Helen Yee and others. In addition to providing an overall view of the new method, in this paper we shall introduce a new concept in the design of implicit schemes, and use it to construct a highly accurate solver for a convection-diffusion equation. It will be shown that, in the inviscid case, this new scheme becomes explicit and its amplification factors are identical to those of the Leapfrog scheme. On the other hand, in the pure diffusion case, its principal amplification factor becomes the amplification factor of the Crank-Nicolson scheme.

14. SUBJECT TERMS

Space-time; Conservation element; Solution element

15. NUMBER OF PAGES
14

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	