ANALYSIS OF DEVELOPING LAMINAR FLOWS IN CIRCULAR PIPES USING A HIGHER-ORDER FINITE-DIFFERENCE TECHNIQUE

Ching L. Ko and Douglas E. Boddy
Oakland University
Rochester, Michigan

and

Herbert J. Gladden
NASA Lewis Research Center
Cleveland, Ohio

SUMMARY

A higher-order finite-difference technique is developed to calculate the developing-flow field of steady incompressible laminar flows in the entrance regions of circular pipes. Navier-Stokes equations governing the motion of such a flow field are solved by using this new finite-difference scheme. This new technique can increase the accuracy of the finite-difference approximation, while also providing the option of using unevenly spaced clustered nodes for computation such that relatively fine grids can be adopted for regions with large velocity gradients. The velocity profile at the entrance of the pipe is assumed to be uniform for the computation. The velocity distribution and the surface pressure drop of the developing flow then are calculated and compared to existing experimental measurements reported in the literature. Computational results obtained are found to be in good agreement with existing experimental correlations and therefore, the reliability of the new technique has been successfully tested.

INTRODUCTION

Due to the effect of viscous dissipation, velocity and pressure distributions in fluid flows normally vary non-uniformly. The flow velocity typically has a large spatial variation near a wall and a relatively small variation in a region far away from wall surfaces. To calculate flow characteristics, the classical finite-difference method discretizes the mathematical domain into uniform-size meshes. In order to obtain accurate results without resolving to using extremely fine meshes, the physical domain is preferable to be discretized into unenvenly spaced clustered nodes such that fine meshes can be adopted in regions with large velocity gradients and coarse meshes can be used in regions with small velocity gradients. As was discussed by Anderson et al (ref. 1), a physical domain discretized by using unevenly spaced clustered nodes can be transformed into a mathematical domain consisting of evenly spaced nodes by using the method of coordinate transformation such that the classical finite-difference technique can be applied. However, to find a proper mathematical function to transform coordinates of a physical domain discretized by using arbitrarily clustered nodes into new coordinates for a mathematical domain with uniformly spaced meshes is practically infeasible because of the complex nature of this type of transformation. In addition,transformed governing equations for the mathematical domain can become highly transcendental because of the nonlinear behavior of fluid flows. Hence, it is highly desirable to develop a finite-difference technique which can be applied directly to a mathematical domain discretized by using arbitrarily spaced clustered nodes such that the transformation of governing equations can be avoided.

Several existing methods which utilize the classical finite-difference formulation to solve partial differential equations have been major tools in computational fluid mechanics. The conventional explicit method, Crank-Nicholson method, and the Box method of Keller (refs. 2 to 4) are major finite-difference techniques that have been widely used in computational fluid mechanics. However, theses methods are only second-order accurate and are not appropriate to be applied to cases with unevenly spaced clustered nodes without using the coordinate transformation technique. In view of these shortcomings, the objective of this work is to develop a fourth-order explicit finite-difference scheme such that clustered nodes can be directly used in a mathematical domain. In addition to having the capability of allowing untransformed governing equations be applied directly to unevenly spaced clustered nodes in a physical domain, this new technique determines the first four derivatives of dependent variables with respect to any independent variable consistently to the fourth-order accuracy. Therefore, it should be more accurate than the classical second-order finite difference method.

In order to test the reliability of the new explicit finite-difference technique, it is used to solve the flow-development problems of fluid flows in the entrance region of a circular tube as well as in the leading-edge region between parallel plates. The well known solutions of Couette flow and of plane Poiseuille flow are applicable to the fully developed regions of these problems. As were described by Sparrow et al (ref. 5), several analytical techniques such as linearized methods and boundary-layer approaches have also been developed to approximately model flows in the entrance regions of these two problems. Sparrow et al (ref. 5) used a linearized method to solve the developing flow problems for both cases. In addition, Bodoia and Osterle (ref. 6) also utilized the classical finite-difference method to solve these flow-development problems. However, they applied Prandtl's momentum equation for the boundary layer instead of Navier-Stokes equations to these problems. In order to properly model the actual developing process, the present analysis applies Navier-Stokes equations to the entire domain and utilizes an iterative sweeping technique to calculate nonlinear terms. Therfore, the present mathematical approach is different from any existing analyses.

HIGHER-ORDER FINITE-DIFFERENCE FORMULATION

Five different types of higher-order finite-difference formulations which allow the usage of clustered nodes can be developed by using Taylor's series expansion of functions up to the fourth-order accuracy. Nodal intervals for these five types of formulation, namely, the central difference, the partially forward difference, the fully forward difference, the partially backward difference, and the fully backward difference, are shown in Figure 1. All five types can be used for different nodes in the same domain and their selection for each node depends upon the distribution of unknown dependent variables surrounding the node under consideration. The first, the second, the third, and the fourth derivatives of a dependent variable with respect to an independent variable evaluated at this particular node (node i) can be expressed algebraically in terms of the nodal values of the same dependent variable associated with the five neighboring clustered nodes. The coefficients of these linear algebraic relationships can be calculated for any values of nodal intervals by solving four simultaneous linear algebraic equations relating to Taylor's series expansion. As an example, central-difference relationships for the case with uniform intervals ($h_1 = h_2 = h_3 = h_4 = h$) can be expressed as

$$\left[\frac{du}{dx}\right]_{x=x_i} = \frac{1}{12h}(u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}) \tag{1}$$

$$\left[\frac{d^2u}{dx^2}\right]_{x=x_i} = \frac{1}{12h^2}(-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}) \tag{2}$$

$$\left[\frac{d^3u}{dx^3}\right]_{x=x_i} = \frac{1}{2h^3}(-u_{i-2} + 2u_{i-1} - 2u_{i+1} + u_{i+2}) \tag{3}$$

$$\left[\frac{d^4u}{dx^4}\right]_{x=x_i} = \frac{1}{h^4}(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}) \tag{4}$$

where u_{i-2}, u_{i-1}, u_i, u_{i+1} and u_{i+2} are values of u evaluated at x_{i-2}, x_{i-1}, x_i, x_{i+1} and x_{i+2}, respectively.

FORMULATION OF THE PROBLEM

Two physical problems are considered to test the higher-order finite-difference technique. one is the axisymmetric developing incompressible laminar flow in a circular pipe and the other is the two-dimensional developing incompressible laminar flow between parallel plates. Flow velocity and the pressure at the inlet region are assumed to be uniform and the no-slip boundary conditions are imposed on all wall surfaces. By utilizing a switching constant m for both the two-dimensional problem (m = 0) and the axisymmetric problem (m = 1), governing equations for both problems in different regions can be expressed as follows:

(A) General Region of Fluid Flow ($0 < y^* < 1$)

(1) Continuity Equation

$$b\frac{\partial u^*}{\partial x^*} + \frac{mv^*}{y^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5}$$

(2) Momentum Equations

$$bu^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -ab\frac{\partial p^*}{\partial x^*} + \frac{b^2}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{m}{y^*Re}\frac{\partial u^*}{\partial y^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{6}$$

$$bu^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -a\frac{\partial p^*}{\partial y^*} + \frac{b^2}{Re}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{m}{y^*Re}\frac{\partial v^*}{\partial y^*} - \frac{mv^*}{y^*Re} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}} \tag{7}$$

(B) Centerline Region ($y^* = 0$)

(1) Continuity Equation

$$b\frac{\partial u^*}{\partial x^*} + (m+1)\frac{\partial v^*}{\partial y^*} = 0 \tag{8}$$

(2) Momentum Equations

$$bu^* \frac{\partial u^*}{\partial x^*} = -ab \frac{\partial p^*}{\partial x^*} + \frac{b^2}{Re} \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{(1+m)}{Re} \frac{\partial^2 u^*}{\partial y^{*2}} \tag{9}$$

$$v^* = 0 \tag{10}$$

(C) Wall-Surface Region ($y^* = 1$)

(1) No-Slip Conditions

$$u^* = v^* = 0 \tag{11}$$

(2) Momentum Equation

$$ab \frac{\partial p^*}{\partial x^*} + a \frac{\partial p^*}{\partial y^*} = \frac{b^2}{Re} \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{m}{y^* Re} \frac{\partial u^*}{\partial y^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}} + \frac{b^2}{Re} \frac{\partial^2 v^*}{\partial x^{*2}} + \frac{m}{y^* Re} \frac{\partial v^*}{\partial y^*} - \frac{m v^*}{y^* Re} + \frac{1}{Re} \frac{\partial^2 v^*}{\partial y^{*2}} \tag{12}$$

where

$$b \equiv \frac{H}{L}, \quad x^* \equiv \frac{x}{L}, \quad y^* \equiv \frac{y}{H}, \quad u^* \equiv \frac{u}{U_\infty}, \quad v^* \equiv \frac{v}{U_\infty}, \quad p^* \equiv \frac{p}{p_\infty},$$

$$a \equiv \frac{p_\infty}{\rho U_\infty^2}, \quad Re \equiv \frac{\rho H U_\infty}{\mu}.$$

Here, x and y are either the Cartesian coordinates in the longitudinal and the transverse directions for the parallel-plate problem or the cylindrical coordinates in the axial and the radial directions for the circular-pipe problem. The origins of these coordinate systems are located at the flow inlets. The x axis is located either at the centerline of the parallel planes for the two-dimensional problem or at the center of the pipe for the axisymmetric problem. The flow domain in the x and y directions are denoted as L and H, respectively. Hence, H represents either the half distance between two parallel planes or the radius of the circular pipe. The axial velocity and the pressure at the inlet are specified as U_∞ and p_∞ respectively. Velocity components in the x and the y directions as well as the pressure, the density, and the viscosity of the flow are denoted as u, v, p, ρ, and μ, respectively. The x-momentum equation applied at x = 0 (centerline region) is derived by applying L'Hospital rule to the Navier-Stokes equation. In addition, equation (12) is obtained by combining both the x and the y components of the momentum equations such that the number of unknowns can be identical to the number of equations at each surface node.

Governing equations for the entire domain, equations (5-12), are discretized in the y^* direction by using the present higher-order finite-difference technique to adapt to unevenly spaced clustered nodes. However, they are discretized in the x^* direction by applying the classical finite-difference formulation to evenly spaced nodes such that an iterative sweeping technique can be used to calculate nonlinear terms. The initial value of a nonlinear term associated with a given node can be estimted by using values of terms evaluated at upstream nodes at the same y^* location such that governing equations can be solved line by line from the upstream to the downstream by using the classical backward difference formulation in the x^* direction. To reduce the inaccuracy due to this approximation, results calculated for each line are iterated to a convergent value before the next marching sequence of the sweeping process is undertaken. To start the process, the first two lines have to be solved simultaneously by using the central difference for the first line and the backward difference for the second line in the x^* direction. For all subsequent sweeping calculations, only one line of nodes are involved in the process.

NUMERICAL RESULTS

The normalized domains for both the parallel-plate and the circular pipe problems are discretized by dividing the domain in the x^* direction into 30 evenly spaced intervals and the domain in the y^* direction into 20 uneven clustered intervals. These unevenly speced intervals in the y^* direction vary from 0.0025 at the wall to 0.1 at the centerline. Table 1 compares non-dimensional axial velocities of the circular pipe between those obtained by the present analysis and those measured by Nikuradse (ref. 7) as well as those calculated by Sparrow et al (ref. 5). Table 2 shows the comparison of non-dimensional longitudinal velocities for the parallel-plate problem between those calculated by using the present technique and those determined by Sparrow et al as well as by Bodia and Osterle (refs. 5 and 6). Table 3 summarizes the comparison of the non-dimensional centerline pressure drop, $[2(p_\infty - p)]/(\rho U_\infty^2)$, of the pipe flow between those calculated by using the present analysis and the Schiller's experimental correlation reported by Prandtl and Tietjens (ref. 7). Results obtained by using the present technique are shown to be in good agreement with those determined by existing techniques. The non-dimensional entrance length of the circular pipe estimated by locating the cross section with its centerline velocity being 99% of the fully developed centerline velocity is found to be approximately equal to $x/(DRe_D) = 0.0593$ or $x/(HRe) = 0.237$ which is roughly consistent with the value of $x/(DRe_D) = 0.05$ suggested by Kays (ref. 8) or the value of $x/(HRe) = 0.26$ reported by Prandtl and Tietjens (ref. 7) for pipe flows.

CONCLUSIONS

A higher-order finite-difference technique which allows the usage of clustered nodes has been successfully developed. Numerical results obtained in the present analysis have also verified the reliability of this technique. Therefore, it can be a very useful tool in computational fluid mechanics because of its accuracy and the need to use unevenly spaced clustered nodes for modeling fluid flows.

REFERENCES

1. Anderson, D.A.; Tannehill, J.C.; and Pletcher, R.H.: Computational Fluid Mechanics and Heat Tansfer. Hemisphere, 1984.
2. Bradshaw, P.; Cebecci, T.; and Whitelaw, J.H.: Engineering Calculation Methods for Turbulent Flow, Academic Press, 1984.
3. Crank, J.; and Nicholson, P.: A Practical Method for Numerical Evaluation of Solutions of Partial-Differential Equations of the Heat-Conduction Type. Proc. Cambridge Phil. Soc., vol. 43, 1947, pp. 50-67.
4. Keller, H.B.: A New Difference Scheme for Parabolic Problems. Numerical Solutions of Partial Differential Equations. (edited by Bramble, J.) Academic Press, 1970.
5. Sparrow, E.M.; Lin, S.H.; and Laudgren, T.S.: Flow Development in the Hydrodynamic Entrance Region of Tubes and Ducts. Phys. Fluids, vol. 7, no. 3, Mar. 1964, pp. 338-347.
6. Bodoia, J.R.; and Osterle, J.F.: Finite Difference Analysis of Plane Poiseuille and Couette Flow Developments. Appl. Sci. Res. sec. A, vol. 10, 1961, pp. 265-276.
7. Prandtl, L; and Tietjens, O.G.: Applied Hydro- and Aeromechanics. Dover, 1957, p. 25.
8. Kays, W.M.: Convetive Heat and Mass Transfer. McGraw-Hills, 1966, p. 63.

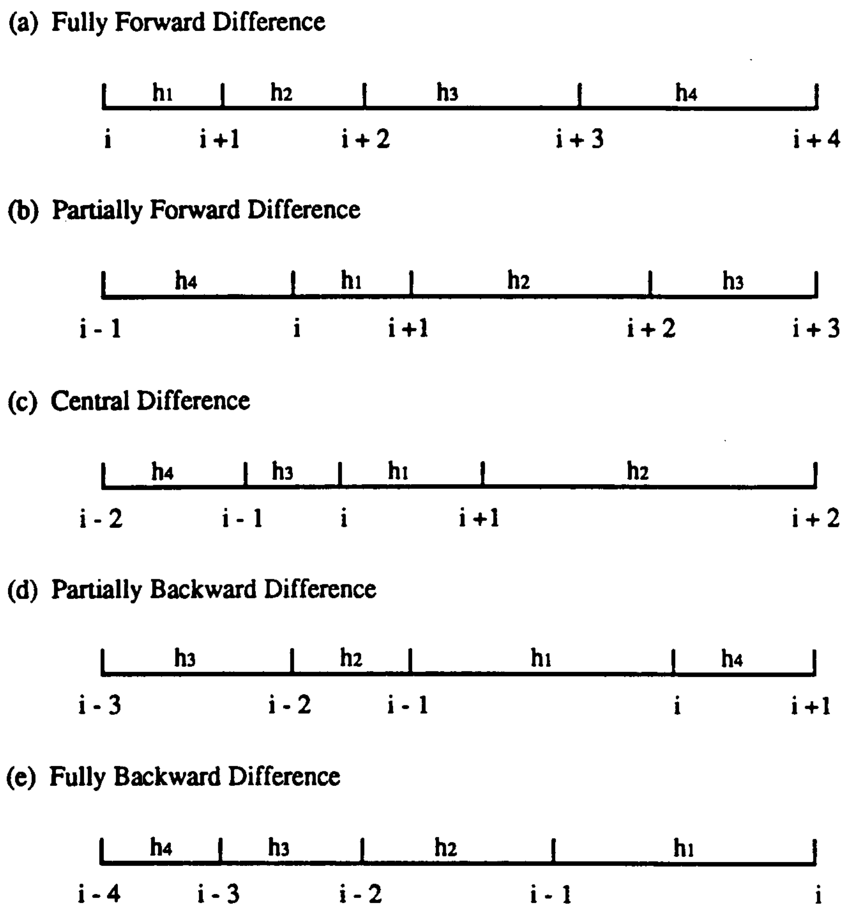

Figure 1. Higher-order finite-difference nodes

TABLE 1 - Comparison of the non-dimensional axial velocity (u/U_∞) of pipe flows

$\frac{x}{HRe}$	Present Analysis				Nikuradse's Expermental Measurement (ref.7)				Analysis by Sparrow et al (ref. 5)			
	y/H				y/H				y/H			
	0.0	0.4	0.8	0.9	0.0	0.4	0.8	0.9	0.0	0.4	0.8	0.9
0.01	1.34	1.32	0.99	0.63	1.30	1.32	1.12	0.71	1.32	1.31	1.00	0.59
0.02	1.44	1.41	0.92	0.53	1.43	1.44	0.95	0.58	1.44	1.42	0.92	0.51
0.03	1.53	1.48	0.87	0.48	1.53	1.53	0.89	0.52	1.53	1.49	0.87	0.47
0.04	1.60	1.52	0.84	0.46	1.60	1.58	0.86	0.50	1.60	1.53	0.84	0.45
0.05	1.66	1.56	0.81	0.44	1.66	1.60	0.83	0.48	1.66	1.57	0.81	0.44
0.10	1.84	1.63	0.76	0.40	1.83	1.65	0.77	0.41	1.85	1.63	0.76	0.40
0.20	1.97	1.67	0.73	0.38	1.97	1.67	0.72	0.38	1.97	1.67	0.72	0.38

TABLE 2 - Comparison of the non-dimensional axial velocity (u/U_∞) of flows between parallel plates

$\frac{x}{HRe}$	Present Analysis				Finite-Difference Calculation by Bodia and Osterle (ref. 6)				Calculation by Sparrow et al (ref. 5)			
	y/H				y/H				y/H			
	0.0	0.5	0.7	0.9	0.0	0.5	0.7	0.9	0.0	0.5	0.7	0.9
0.01	1.20	1.15	1.00	0.50	1.16	1.16	1.05	0.50	1.18	1.16	1.03	0.48
0.02	1.25	1.18	0.96	0.42	1.22	1.18	0.99	0.41	1.24	1.18	0.96	0.40
0.03	1.30	1.18	0.91	0.37	1.27	1.18	0.93	0.37	1.29	1.18	0.91	0.37
0.04	1.33	1.18	0.88	0.35	1.31	1.18	0.89	0.35	1.33	1.17	0.88	0.35
0.05	1.36	1.17	0.86	0.34	1.34	1.17	0.86	0.33	1.36	1.16	0.86	0.33
0.10	1.44	1.15	0.80	0.30	1.44	1.14	0.80	0.30	1.44	1.14	0.80	0.30
0.20	1.49	1.13	0.77	0.29	1.49	1.13	0.77	0.29	1.49	1.13	0.77	0.29

TABLE 3 - Comparison of the non-dimensional centerline pressure drop, $2(p_\infty - p)/\rho U_\infty^2$, of pipe flows.

$\frac{x}{HRe}$	Present Analysis	Schiller's Correlation (ref. 7)
0.01	0.89	0.66
0.02	1.21	1.03
0.03	1.49	1.27
0.04	1.74	1.48
0.05	1.96	1.80
0.10	2.94	2.68
0.12	3.30	3.18
0.16	3.98	3.70