A PROBABILSTIC MODEL OF A POROUS HEAT EXCHANGER

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ABSTRACT

This paper presents a probabilistic one-dimensional finite element model for heat transfer processes in porous heat exchangers. The Galerkin approach is used to develop the finite element matrices. Some of the submatrices are asymmetric due to the presence of the flow term. The Neumann expansion is used to write the temperature distribution as a series of random variables, and the expectation operator is applied to obtain the mean and deviation statistics. To demonstrate the feasibility of the formulation, a one-dimensional model of heat transfer phenomenon in superfluid flow through a porous media is considered. Results of this formulation agree well with the Monte-Carlo simulations and the analytical solutions. Although the numerical experiments are confined to parametric random variables, a formulation is presented to account for the random spatial variations.

INTRODUCTION

Porous heat exchangers are key components in many engineering systems such as high performance regenerative heat exchangers, thermal energy storage systems, cryocoolers, and packed beds. Several techniques have been developed to analyze these systems. These techniques include, among others, analytical techniques, such as separation of variables [1,2], Riemann method [3] and similarity transformation [4]; semi-analytical techniques such as orthogonal collocation [5] and collocation-perturbation [6,7]; and numerical methods such as numerical integration [8], shooting and Runge-Kutta integration [9], and finite element methods [10]. In these techniques the above systems are considered as deterministic. That is, the problems are formulated in terms of mean-values of the properties neglecting variations in the mean values. Experimental measurements, however, show that the properties of the systems may vary significantly in a random fashion, especially near a low temperature. Given the stringent demand on the design of modern heat exchangers, these deterministic models may not be adequate.

Random properties can be incorporated in the above techniques using the probability theories and the theories of differential equations [11-14]. In many applications, it is difficult to solve the resulting differential equations in closed form even

when the randomness is not considered [15]. In the last 30 years, Finite Element Methods (FEMs) have successfully been applied to solve a large class of deterministic systems, and more recently, probabilistic systems. Current trends for analyzing random systems in engineering are given by Iyengar and Dash [16], Ibrahim [17], Shinozuka [18], and Benaroya and Rehak [19]. Chamis and coworkers [20, 21] have developed a general purpose finite element computer code called PICAN for probabilistic analysis of composite materials.

The numerical techniques available to solve problems consisting of random variables and functions may be broadly classified into two categories; statistical and nonstatistical. Most statistical techniques rely on numerical simulations among which Monte Carlo simulation has been widely used [22]. The nonstatistical techniques include perturbation methods [23-26], spectral decomposition methods [15, 27], and basis random variable methods [28, 29].

In addition to the FEMs, several investigators have used the Boundary Element Methods (BEMs). The random operator problems were analyzed by Ettouney et al. [30-32], and more recently by Manolis and Shaw [33]. Burczyński [34] employed the direct boundary element method to develop two distinct procedures for the treatment of random potential problems. Cheng and Lafe [35] employed the indirect boundary element method to obtain stochastic integral equations for boundary potentials and fluxes in terms of fictitious boundary sources. Other applications of the boundary element method include the first order perturbation method developed by Drewniak [36] for the analysis of heat conduction problems with random heat transfer and random heat conduction coefficient, respectively, and a procedure for the analysis of time dependent problems in the frequency domain developed by Burczyński and John [37].

The probabilistic methods discussed here, however, are largely confined to structural systems, and very little effort has been made to develop methods for porous heat exchangers. In this paper, a probabilistic one-dimensional finite element model for heat transfer process in porous heat exchangers is presented. The formulation is based on the Galerkin method, the spectral decomposition of random processes, and the Neumann expansion.

MATHEMATICAL FORMULATION

This section is divided into three parts: deterministic finite element model, probabilistic finite element models for parametric randomness and stochastic processes, and Monte-Carlo models. These models are considered next.

Deterministic Finite Element Model.- In order to develop a deterministic analytical model for the heat transfer process in fluid flow through porous media, consider the schematic of a one dimensional heat exchanger as shown in Figure 1. It is assumed that the system is stationary; that is, the system parameters are not

changing with time. Using energy balance, one can derive the following differential equations

$$k_f \frac{d^2T}{dx^2} + hs(t - T) = c\frac{dT}{dx} \tag{1}$$

$$k_s \frac{d^2t}{dx^2} + hs(T - t) = 0 \tag{2}$$

where k_f and k_s are the thermal conductivity coefficients of the fluid and the solid, h is the convective heat transfer coefficient, s is the porosity coefficient, T and t are the temperature distributions in the fluid and the solid, and the coefficient c accounts for the energy transfer due to fluid flow. In addition to these equations, the boundary conditions are also required. The formulation presented here is applicable to various boundary conditions. However, for simplicity, it is assumed that the terminal temperatures of the fluid and the solid at the two ends are prescribed. These conditions are written as

$$T(x = 0) = T_0, \quad t(x = 0) = t_0 \quad T(x = L) = T_L, \quad \text{and} \quad t(x = L) = t_L \tag{3}$$

where L is the length of the heat exchanger which is discretized into several finite elements. If N is the vector of the shape functions defined over the elements, then the temperatures T and t may be approximated as

$$T = N^T T_n \qquad \text{and} \qquad t = N^T t_n \tag{4}$$

where T_n and t_n represent the vector of the nodal temperatures of the fluid and the solid, and the superscript T represents the transpose. Observe that both solid and fluid regions have been discretized into an equal number of elements, and the same shape functions have been used for both temperature distributions. This is not necessary and a formulation that considers different numbern of elements and different shape functions is possible. Using the Galerkin approach, equations (1) and (2) may be written as

$$\int_0^L N \left[k_f \frac{d^2T}{dx^2} + hs(t - T) - c\frac{dT}{dx} \right] dx = 0 \tag{5}$$

$$\int_0^L N \left[k_s \frac{d^2t}{dx^2} + hs(T - t) \right] dx = 0 \tag{6}$$

Performing integration by parts on some terms of equations (5) and (6), and rearranging the terms, one obtains

$$E \begin{bmatrix} T_n \\ t_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \tag{7}$$

where E is the global coefficient matrix defined as

$$E = \begin{bmatrix} -k_f A - hsB - cC & hsB \\ hsB & -k_s A - hsB \end{bmatrix} \tag{8}$$

and vector b appears due to partial integration of some terms in Eq. (3) and the boundary conditions. Matrices A, B, and C in equation (8) are defined as

$$A = \int_0^L (\frac{dN}{dx})(\frac{dN}{dx})^T dx \tag{9}$$

$$B = \int_0^L NN^T dx \tag{10}$$

$$C = \int_0^L N(\frac{dN}{dx})^T dx \tag{11}$$

Equation (7) provides the desired deterministic finite element model. Observe that matrices A and B are symmetric positive definite finite element matrices, whereas matrix C is an asymmetric matrix. This makes matrix E asymmetric. Therefore, one should not use a symmetric simultaneous equation solver to solve equation (7).

Probabilistic Finite Element Model.- As stated earlier, two types of random behavior may appear in the system; parametric and spatial. These two random processes will be considered separately.

Parametric Randomness.- For simplicity, only k_s is considered as a random parameter. If other parameters also vary randomly, then the formulation can be extended appropriately. The random parameter k_s may be written as $k_s = k_{s0} + \epsilon$, where k_{s0} is the mean value of k_s, and ϵ represents the random variations with mean zero and standard deviation σ (i.e. $< \epsilon >= 0$, and $< \epsilon^2 >= \sigma^2$). Substituting the expression for k_s into Eq. (7), one obtains

$$[E_1 + \epsilon E_2] \begin{bmatrix} T_n \\ t_n \end{bmatrix} = b \tag{12}$$

where matrix E_1 is the same as matrix E in equation (8) except that k_s is replaced by k_{s0} and matrix E_2 is given as

$$E_2 = \begin{bmatrix} 0 & 0 \\ 0 & -A \end{bmatrix} \tag{13}$$

Observe that T_n and t_n are now vectors of random variables. Using the Neumann expansion, the temperature vector can be written as

$$\begin{bmatrix} T_n \\ t_n \end{bmatrix} = \left[I - \epsilon(E_1^{-1}E_2) + \epsilon^2(E_1^{-1}E_2)^2 - \epsilon^3(E_1^{-1}E_2)^3 + \cdots\right] E_1^{-1}b \tag{14}$$

provided that $||\epsilon(E_1^{-1}E_2)|| < 1$ is satisfied, which is reasonable for most practical systems. From equation (14), the expected value of the temperature vector is

$$< \begin{bmatrix} T_n \\ t_n \end{bmatrix} >= \left[I - < \epsilon > (E_1^{-1}E_2) + < \epsilon^2 > (E_1^{-1}E_2)^2 - < \epsilon^3 > (E_1^{-1}E_2)^3 + \cdots\right] E_1^{-1}b \tag{15}$$

where $<>$ is the expectation operator. Observe that $E_1^{-1}E_2$ is constant. Furthermore, given the probability distribution, $< \epsilon^i >$ $(i = 1, 2, \cdots,)$ can be computed numerically and in some cases analytically. Substituting these values in equation (15), the expected values of the temperatures can be obtained. Similarly, the second order characteristics of T_n and t_n can be obtained as follows:

$$< \begin{bmatrix} T_n^T & t_n^T \end{bmatrix} \begin{bmatrix} T_n \\ t_n \end{bmatrix} >=< b^T E_1^{-T} \left[I + \epsilon E_1^{-1}E_2\right]^{-T} \left[I + \epsilon E_1^{-1}E_2\right]^{-1} E_1^{-1}b > \tag{16}$$

Once again, the matrices containing ϵ in equation (16) can be expanded in Neumann series to obtain the covariance matrix for temperature distribution.

Equations (15) and (16) provide a probabilistic model for parametric randomness in k_s. A similar approach can be used for other random parameters.

Spatial Randomness.- Consider that k_s varies randomly from point to point along the length of the heat exchanger and that other properties are constant. k_s can be written as $k_s = k_{sm} + k_{sr}$, where k_{sm} represents the mean function and k_{sr} represents the stationary Gaussian process with zero mean functions and specified correlation function $R(x, u)$, which is symmetric and positive definite. Using the Karhunen-Loeve (KL) expansion, k_{sr} can be represented as [38]

$$k_{sr} = \sum_i \epsilon_i \phi_i(x) \tag{17}$$

where $\{\phi_i | i = 1, 2, \ldots\}$ is a set of orthogonal eigenfunctions of certain differential equations and ϵ_i $(i = 1, 2, \ldots)$ are uncorrelated random variables. These eigenfunctions satisfy the following integral equation

$$\lambda_i \phi_i(x) = \int_0^L R(x, u)\phi_i(u)du, \tag{18}$$

where λ_i is an eigenvalue associated with $\phi_i(x)$. Furthermore, the coefficients ϵ_i $(i = 1, \cdots, \infty)$ satisfy the following identity

$$\lambda_i \delta_{ij} =< \epsilon_i \epsilon_j >, \tag{19}$$

where $< \epsilon_i^2 >= \lambda_i$ gives measure of randomness along the $\phi_i(x)$ coordinate. One of the advantages of the series expansion is that it provides a second moment characterization of k_{sr} in terms of uncorrelated random variables.

Using Eqs. (5), (6), and (17), and following the approach presented above, one obtains

$$\begin{bmatrix} -k_f A - hsB - cC & hsB \\ hsB & -\sum \epsilon_i D_i - hsB \end{bmatrix} \begin{bmatrix} T_n \\ t_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \tag{20}$$

where matrix D_i is defined as

$$D_i = \int_0^L \phi_i(x)(\frac{dN}{dx})^T(\frac{dN}{dx})dx \tag{21}$$

Equation (20) is very similar to Eq. (12). Therefore, the statistical characteristics of the temperature vector $\begin{bmatrix} T_n^T & t_n^T \end{bmatrix}^T$ can be obtained using the Neumann expansion and the procedure discussed for parametric randomness.

The above discussion provides a probabilistic model when k_s represents a stochastic process. A similar approach can be used for other stochastic processes.

Monte-Carlo Method.- Monte-Carlo simulations rely on equations (12) and (20). In this technique, a random number generator is used to obtain a large set of random numbers that represent the desired probability distribution curve of the random variables. This process is repeated for each random variable. Depending on parametric or spatial randomness, equation (12) or (20) is used to obtain an equal number of sets of nodal temperatures, which are then used to obtain the statistics for the nodal temperatures. For an accurate answer, this scheme requires a large number of numerical tests. This number can be reduced using the following approach: (1) grouping the random data, (2) performing only one test for each group, and (3) using the probability information to account for other data in the group. This approach can significantly reduce the number of numerical runs for accurate results.

NUMERICAL RESULTS AND DISCUSSIONS

To validate the formulation developed here, a dilution refrigerator heat exchanger consisting of superfluid Helium II as fluid and the sintered copper as the solid was considered. The system response was obtained using this scheme and an analytical scheme. For numerical simulations, the following parameters were considered: $K_f = 7X10^4 W/(m.K)$, $K_s = 500W/(m.K)$, $h = 1200W/(m^2K)$, $c = 11.75W/(m.K)$, and $s = 0.2792m$ which are typical of this system. The value of L was taken as 10 m. For convenience, the non-dimensional temperatures of the fluid T_f and the solid t_s were defined as follows: $T_f = (T - T_0)/(T_L - T_0)$ and $t_s = (t - T_0)/(T_L - T_0)$. The following boundary conditions were taken: $T_f(0) = 0.0$, $t_s(0) = 0.8$, and $T_f(L) = t_s(L) = 1.0$. The values of k_s and h were varied to study the effects of these parameters on the temperature distribution. To study the probabilistic effects, the system statistical response was obtained using Monte-Carlo simulations, the analytical schemes, and the proposed scheme. Results of this study

are shown below.

Figure 2 compares the temperature distributions obtained using the analytical and the proposed schemes. The two results agree very well. The fluid temperature changes almost linearly due to the large conductivity of superfluid HeII. The solid media temperature decreases faster near the inlet point than inside the exchanger because of the strong convection near the inlet. Since the outlet point temperature is higher than the inlet temperature, the temperature of both fluid and solid increases after some distance. Once the temperature of the fluid and the solid become equal, convection stops and the temperature of both media increases at the same rate.

Figure 3 shows the temperature profiles of the solid and the fluid for k_s = 250, 500, 1000, 1500, and 2000 W/(m.K). It is clear that an increase in solid thermal conductivity causes the temperature of the solid to increase. This is because convection becomes less dominant at higher values of k_s. Due to large fluid thermal conductivity, the fluid temperature profile remains unchanged.

The temperature profiles for h = 600, 1200, 2400, and 12,000 W(m^2.K) are shown in figure 4. As expected, increase in convective heat transfer coefficient causes the solid temperature to decrease rapidly and merge with the fluid temperature sooner.

In this investigation, parameter h was considered as a random variable, and all other parameters were kept the same. Using a random number generator, 20,000 random sample points having 10 % variations of mean convective heat transfer coefficient with 90 % confidence were generated. These data were used in the Monte-Carlo method, the exact solution, and this scheme to predict the mean response of the temperature profile. All three schemes gave the same results. To compare the relative accuracy of the current scheme with the Monte-Carlo scheme, the percentage errors for the two schemes were computed. Results are shown in Figure 5. It can be observed that both schemes overpredict fluid temperature while they underpredict solid temperature, and the difference between the two schemes is small. However, from the formulation, it is clear that this scheme requires fewer number of computations than the Monte-Carlo scheme.

CONCLUSION

A deterministic and a probabilistic one-dimensional finite element model for heat transfer processes in porous heat exchangers has been presented. A set of numerical experiments have been performed to validate the model. This formulation leads to an asymmetric global coefficient matrix. Numerical experiments show that this scheme agrees well with the analytical and the Monte-Carlo methods. However, for mean and standard deviations, this scheme requires fewer number of computations in comparison to the Monte-Carlo scheme.

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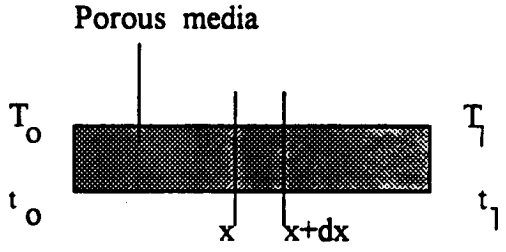

Figure 1. One dimensional porous heat exchanger

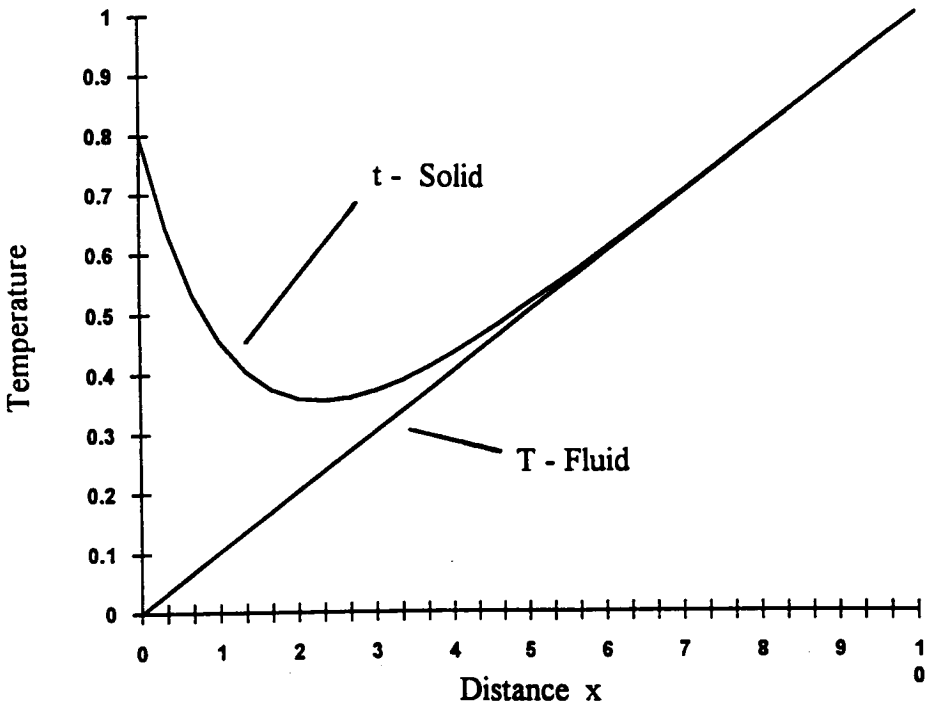

Figure 2. Temperature distribution

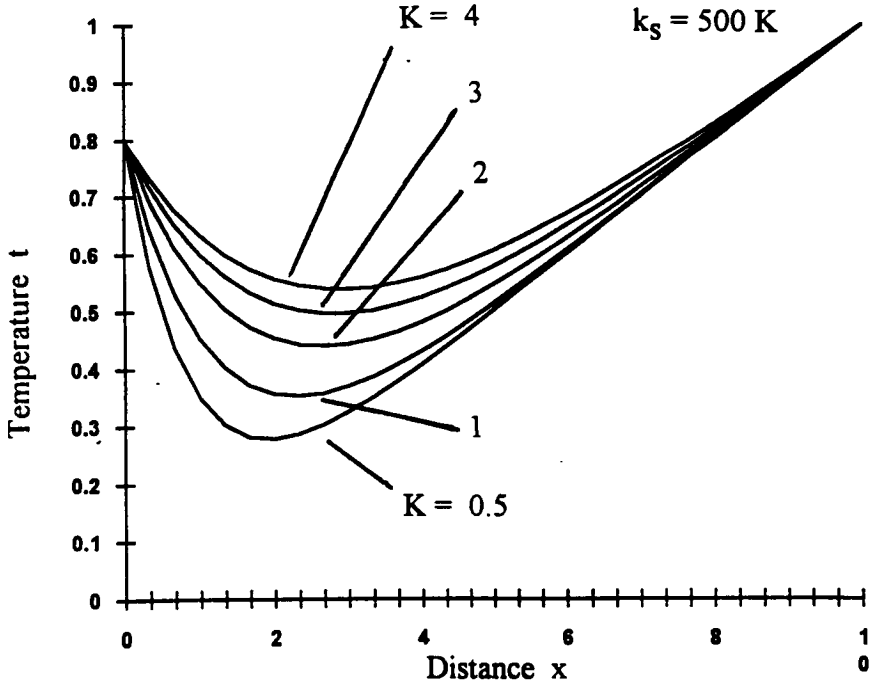

Figure 3. Effect of k_S on temperature distribution t

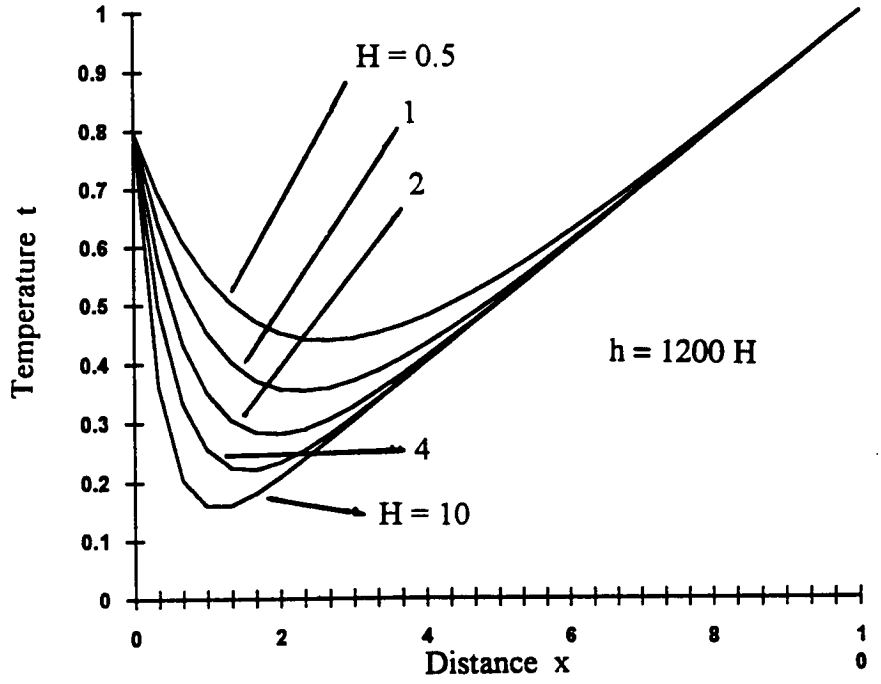

Figure 4. Effect of h on temperature distribution t

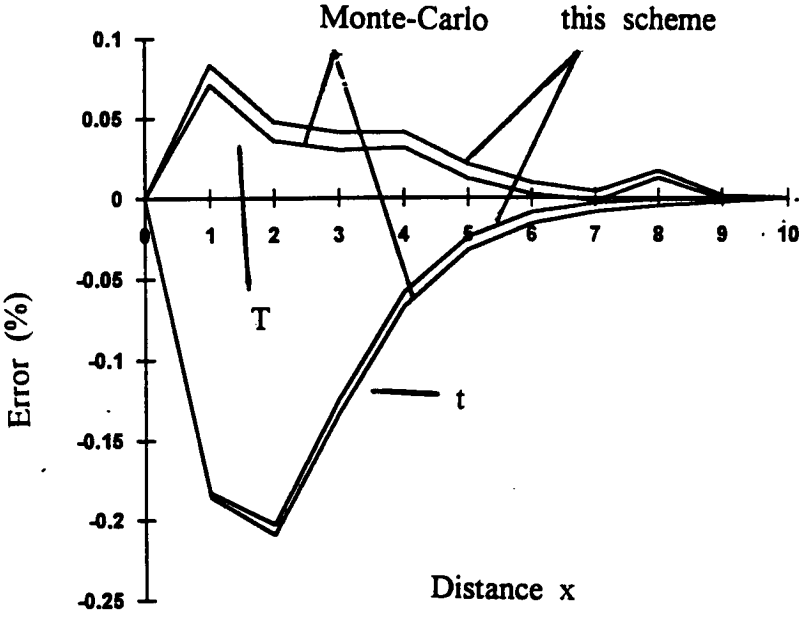

Figure 5. Error in mean values for temperatures of Monte-Carlo and this scheme