AN IMPROVED PLATE THEORY OF ORDER {1,2}

FOR THICK COMPOSITE LAMINATES

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SUMMARY

A new {1, 2}-order theory is proposed for the linear elasto-static analysis of laminated composite plates. The basic assumptions are those concerning the distribution through the laminate thickness of the displacements, transverse shear strains and the transverse normal stress, with these quantities regarded as some weighted averages of their exact elasticity theory representations. The displacement expansions are linear for the inplane components and quadratic for the transverse component, whereas the transverse shear strains and transverse normal stress are respectively quadratic and cubic through the thickness. The main distinguishing feature of the theory is that all strain and stress components are expressed in terms of the assumed displacements prior to the application of a variational principle. This is accomplished by an a priori least-square compatibility requirement for the transverse strains and by requiring exact stress boundary conditions at the top and bottom plate surfaces. Equations of equilibrium and associated Poisson boundary conditions are derived from the virtual work principle. It is shown that the theory is particularly suited for finite element discretization as it requires simple C^0- and C^{-1}-continuous displacement interpolation fields. Analytic solutions for the problem of cylindrical bending are derived and compared with the exact elasticity solutions and those of our earlier {1, 2}-order theory based on the assumed displacements and transverse strains.

INTRODUCTION

Designing of aerospace and ground-vehicle structures with thick-section organic-matrix composites is necessarily associated with the analytical modeling of the structural response and the prediction of failure under service loads. For such applications, viable analytical models are those that can properly account for transverse shear and transverse normal deformations. These effects, which are negligibly small in thin laminates, can be significant when the laminate thickness and the wavelength of loading are the same order of magnitude. Moreover, the reduced stiffness and strength in the transverse shear and transverse normal material directions can contribute to significant deformations and matrix dominated failure modes such as delamination and transverse cracking.

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The majority of finite element methods for composite laminates are based upon the {1,0}-order theories (commonly referred to as 1st-order theories). These account in some average sense for transverse shear but ignore the effects due to transverse normal deformations. The use of three-dimensional elements, including all three-dimensional effects, are generally computationally prohibitive even on a supercomputer, unless employed in some global-local fashion in a relatively small domain of interest. Although post-processing methods have been devised leading in some cases to adequate three-dimensional stress and strain recovery from the {1,0}-order theory computations, the static and dynamic response predictions obtained with this order of approximation are often inadequate for thick laminates (e.g., refs. (1-7) and references thereof).

Numerous efforts to generate laminate theories of higher-order that also include transverse normal deformations failed to produce sufficiently accurate and computationally suitable formulations to model thick-section composites. Recently, Tessler (ref. 1) proposed a {1,2}-order theory for the elasto-static analysis of homogeneous orthotropic plates which was demonstrated to be ideally suited for finite element approximations. Subsequent developments included the analytic and finite element analyses of elastic beams (refs. 2-4), an extension to the general {1,2}-order orthotropic shell theory (ref. 5), and an extension to laminated composite plates for elasto-statics (ref. 6) and elasto-dynamics (ref 7). These theories assume linear inplane displacements and a quadratic transverse displacement across the thickness. The key aspect in the approximation is the independent expansions for transverse strains which allow exact stress boundary conditions at the top and bottom plate faces to be satisfied. These strains are also least-square compatible with those derived directly from the displacement assumptions.

In this paper we propose a new {1,2}-order theory for the linear elasto-static analysis of laminated composite plates following the methodology established in (ref. 6). The present formulation departs from (ref. 6) in that the transverse normal stress is independently expanded across the laminate thickness instead of the transverse normal strain. The transverse shear strains are taken to be parabolic satisfying traction-free boundary conditions. The transverse normal stress is assumed to be a cubic function through the thickness, also satisfying the consequence of zero shear tractions at the top and bottom laminate surfaces; specifically, the vanishing on those surfaces of the transverse normal stress gradient taken with respect to the thickness coordinate. These latter conditions are exact according to three-dimensional elasticity theory when the body and inertial forces are neglected. The resulting equilibrium equations are associated with exclusively Poisson boundary conditions. As its predecessor theory, the present theory requires only C^0- and C^1-continuous displacement interpolation fields and thus lends itself well to the development of simple, robust, and computationally efficient plate formulations similar to Mindlin-type elements (refs. (9-11)).

In assessing the accuracy of the proposed theory, we resort to the problem of cylindrical bending of an infinite laminated plate for which an exact elasticity solution is available (ref. 8). We compare the results of the present theory for several lamination patterns and span-to-thickness ratios with those of our previous {1,2}-order laminate plate theory (ref. 6) and the exact solutions.

NOMENCLATURE

A_{ij}	membrane plate rigidities
$C_{ij}^{(k)}$	elastic stiffness coefficients for the kth ply
B_{ij}	membrane-bending coupling plate rigidities
D_{ij}	bending plate rigidities
G_{ij}	transverse shear plate rigidities
C^0	continuous functions with discontinuous derivatives at element interfaces
C^{-1}	continuous functions that are discontinuous at element interfaces
$2h$	plate thickness
N_{ij}, Q_i	inplane and transverse shear force resultants
M_{ij}	bending moment resultants
q^+	applied transverse load at z=h
u, v	midplane displacements along x and y axes
u_x, u_y, u_z	Cartesian displacement components
w, w_i	components of the transverse displacement
x,y and z	inplane and transverse coordinates
δ	variational operator
$\varepsilon_{ij}, \kappa_{ij}$	strain and curvature components
θ_i (i=x,y)	bending rotations about x and y axes
ξ	dimensionless thickness coordinate
σ_{ij}, τ_{ij}	stress components

{1, 2}-KINEMATIC PLATE ASSUMPTIONS

In a laminated composite plate constructed of perfectly bonded orthotropic plies whose transverse constitutive properties do not differ appreciably, the displacement vector $u = (u_x, u_y, u_z)$ can be approximated with functions that vary continuously across the total thickness. The lowest-order displacement expansions accounting for transverse shear and transverse normal deformations involve linear thickness variations for the inplane components u_x and u_y and a quadratic one for the transverse displacement u_z (ref. 6) (i.e, {1,2}-order theory):

$$u_x(x,y,z) = u(x,y) + h\xi\,\theta_y(x,y), \quad u_y(x,y,z) = v(x,y) + h\xi\,\theta_x(x,y)$$

$$u_z(x,y,z) = w(x,y) + \xi\,w_1(x,y) + (\xi^2 - 1/5)\,w_2(x,y)$$

(1)

where $\xi = z/h \in [-1, 1]$ is the dimensionless thickness coordinate and $\xi=0$ identifies the reference midplane position; $u(x,y)$ and $v(x,y)$ are the midplane displacements along the x and y axes, and $\theta_x(x,y)$ and $\theta_y(x,y)$ are the rotations of the normal about the x and y axes (see fig. 1). The w_1 and w_2 variables can be interpreted as the normalized strain and curvature in the thickness direction:

$$w_1/h = u_{z,z}\big|_{z=0}, \quad w_2/h^2 = \tfrac{1}{2}u_{z,zz}$$

(1.1)

The 3-D Hooke's law for any kth orthotropic ply is expressed in the *mixed* form[2]

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \varepsilon_{zz} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)}
=
\begin{bmatrix}
\hat{C}_{11} & \hat{C}_{12} & \hat{C}_{16} & R_{13} & 0 & 0 \\
\hat{C}_{12} & \hat{C}_{22} & \hat{C}_{26} & R_{23} & 0 & 0 \\
\hat{C}_{16} & \hat{C}_{26} & \hat{C}_{66} & R_{63} & 0 & 0 \\
-R_{13} & -R_{23} & -R_{63} & S_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & C_{45} \\
0 & 0 & 0 & 0 & C_{45} & C_{55}
\end{bmatrix}^{(k)}
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \sigma_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}
$$

(2)

with

$$\hat{C}_{ij}^{(k)} = C_{ij}^{(k)} - C_{i3}^{(k)}C_{j3}^{(k)}/C_{33}^{(k)}$$

$$R_{i3}^{(k)} = C_{i3}^{(k)}/C_{33}^{(k)}, \quad S_{33}^{(k)} = 1/C_{33}^{(k)} \quad (i = 1,2,6)$$

(2.1)

where $C_{ij}^{(k)}$ are the elastic stiffness coefficients corresponding to the x-y coordinates (ref. 12).

The components of strain and stress can be expressed in terms of the plate strain and curvature variables that are independent of the thickness coordinate:

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[2] The variables superscribed with the k index are ply-dependent, whereas those without the k index are some average representations across the laminate thickness and thus are independent of the individual ply properties.

$$\varepsilon_o^T = [\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{zo}, \gamma_{xyo}] = [u_{,x}, v_{,y}, w_1/h, u_{,y} + v_{,x}]$$

$$\kappa_o^T = [\kappa_{xo}, \kappa_{yo}, \kappa_{zo}, \kappa_{xyo}] = [\theta_{y,x}, \theta_{x,y}, w_2/h^2, \theta_{x,x} + \theta_{y,y}] \tag{3}$$

$$\gamma_o^T = [\gamma_{xzo}, \gamma_{yzo}] = [w_{,x} + \theta_y, w_{,y} + \theta_x]$$

The inplane strains, obtained from eq. (1) in accordance with elasticity theory, are

$$\varepsilon_{xx} = \varepsilon_{xo} + z\,\kappa_{xo}, \quad \varepsilon_{yy} = \varepsilon_{yo} + z\,\kappa_{yo}, \quad \gamma_{xy} = \gamma_{xyo} + z\,\kappa_{xyo} \tag{4}$$

The present theory departs from that in ref. 6 only in the manner in which the transverse normal strain and stress are developed. Here, we independently expand the transverse shear strains and the transverse normal stress. The motivation for the latter assumption stems from a careful examination of a series of 3-D exact elasticity stress solutions (ref. 8) for a cylindrical bending problem (see fig. 2). Figure 3 shows 3-D solutions for the transverse shear and transverse normal stresses corresponding to the four distinct laminations: [0], [30/-30]s, [0/90]s and [0/90]. It is quite revealing that the form of the transverse normal stress remains relatively consistent from lamination to lamination whereas the transverse shear stress changes dramatically. These elasticity results clearly suggest that an independent, continuously varying expansion for the transverse normal stress should be an improvement over an analogous expansion for the transverse normal strain as used in (ref. 6). On the other hand, it is unlikely that one would gain any advantage by expanding the transverse shear stresses in a continuous manner across the laminate over that concerning the transverse shear strain expansions (ref. 6) since the solutions for these stresses (fig. 3) cannot be reproduced with a relatively low order approximation such as a parabola.

The transverse normal stress and transverse shear strains are expanded independently across the total laminate thickness as[3]

$$\sigma_{zz} = \sum_{n=0}^{3} \sigma_{zn}\xi^n, \quad \gamma_{iz} = \sum_{n=0}^{2} \gamma_{in}\xi^n \quad (i = x, y) \tag{5}$$

The expansion coefficients $\sigma_{zn} = \sigma_{zn}(x,y)$ and $\gamma_{in} = \gamma_{in}(x,y)$ are determined by requiring the stress field to satisfy exact traction conditions at the top and bottom plate surfaces

$$\tau_{iz}^{(k)}(x, y, \pm h) = \sigma_{zz,z}(x, y, \pm h) = 0 \quad (i = x, y) \tag{6}$$

along with the transverse strains to be least-square compatible with those obtained directly from elasticity theory using our approximations for the displacements (1):

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[3] The present approach and that in ref. 6 are equivalent for homogeneous plates (ref. 1).

$$\text{minimize} \quad \int_{-h}^{h} [\varepsilon_{zz}^{(k)} - u_{z,z}]^2 dz \tag{7}$$

$$\text{minimize} \quad \int_{-h}^{h} [\gamma_{iz} - (u_{z,i} + u_{i,z})]^2 dz \quad (i = x, y)$$

where the minimizations are performed with respect to the σ_{zn} and γ_{in} variables. The resulting transverse normal stress and transverse shear strains can be expressed as

$$\sigma_{zz} = \sigma_{zo}(\varepsilon_o, \kappa_o) + \sigma_{zl}(\varepsilon_o, \kappa_o)(\xi - \xi^3/3), \quad \gamma_{iz} = \tfrac{5}{4}(1 - \xi^2)\gamma_{io} \quad (i = x, y) \tag{8}$$

Application of the virtual work principle results in the 10th-order plate equations of equilibrium and associated Poisson boundary conditions. The principle also serves as a variational framework for finite element approximations. The resulting two-dimensional variational statement has the form

$$\int_{S_m} [N^T \delta\varepsilon_o + M^T \delta\kappa_o + Q^T \delta\gamma_o] dxdy - \delta W_e(u, v, w, \theta_x, \theta_y, w_1, w_2) = 0 \tag{9}$$

where δW_e denotes the virtual work of external forces; $N = \{N_{ij}\}$, $M = \{M_{ij}\}$ and $Q = \{Q_i\}$ are vectors of the plate stress resultants which are related to the plate strains (eqs. (3)) via the constitutive relations

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_o \\ \kappa_o \\ \gamma_o \end{Bmatrix} \tag{10}$$

where $A = [A_{ij}]$, $B = [B_{ij}]$, $D = [D_{ij}]$ and $G = [G_{ij}]$ are the plate constitutive matrices.

REMARKS ON FEM APPROXIMATIONS

The variational principle (9) provides a convenient framework for developing efficient plate bending elements for the analysis of thin and thick composite laminates. In eq. (9), the kinematic variables u, v, w, θ_x and θ_y possesses spatial derivatives that do not exceed order one, thus requiring only C^0-continuous finite element trial functions; w_1 and w_2 do not have spatial gradients thus needing only C^{-1} approximations.

The simplest laminate plate element, which is also the most desirable from the standpoint of adaptive mesh refinement, is a three-node anisoparametric triangle (ref. 6). The element employs linear (3-node) parametric functions for u, v, θ_x and θ_y, an anisoparametric quadratic field for w and uniform fields for w_1 and w_2. The latter variables only contribute two degrees-of-freedom per element; in static problems, they can be conveniently condensed out at the element level using static condensation.

RESULTS AND DISCUSSION

The present theory will be qualitatively assessed with respect to its application to the problem of cylindrical bending of an infinite carbon/epoxy laminate subjected to a sinusoidal transverse pressure $q^+ = q_o \sin(\pi x/L)$. The exact elasticity solution for this problem was first determined by Pagano (ref. 8). We shall consider an orthotropic [0] laminate, two symmetric laminates — a cross-ply $[0/90]_s$ laminate and an angle-ply $[30/-30]_s$ laminate — and an antisymmetric cross-ply laminate, [0/90]. The ply material properties are taken as

$$E_L = 25\times10^6 psi, \quad E_T = 10^6 psi, \quad G_{LT} = 0.5\times10^6 psi$$

$$G_{TT} = 0.2\times10^6 psi, \quad v_{LT} = v_{TT} = 0.25$$

(11)

where L and T denote the longitudinal and transverse ply material directions, respectively.

As noted previously, this theory differs from ref. 6 only in the manner in which the transverse normal stress and strain are approximated. Thus, it is only natural to expect that the differences in the predictions with the two theories will mostly affect these transverse normal variables. Indeed, the displacement predictions by both theories are virtually identical, and so are the inplane and transverse shear stresses and strains.

Figure 4 depicts the percent error in the maximum deflection computed at the midplane of the laminate versus the length-to-thickness ratio L/2h. The results, which pertain to the present and ref. 6 theories, clearly show that as far as the deflection predictions are concerned, the engineering accuracy (i.e., error $\leq 5\%$) is attained for laminates with the ratio L/2h $\geq$ 4. These results demonstrate the adequacy of the overall plate stiffness representation.

Figures 5 through 8 compare the stress and strain thickness distributions for thin (L/2h=40) and thick (L/2h=4) laminates obtained with the present theory (designated as HOT-S), our previous {1,2}-theory (HOT-E, ref. 6), and the exact solutions obtained in this effort using the 3-D elasticity approach (ref. 8). Figure 5 shows the transverse normal stress distributions across the laminate thickness. Note that while both HOT-E and HOT-S are nearly equally accurate in the $[30/-30]_s$ lamination predictions — with HOT-E exhibiting a slight discontinuity at the interface between 30 deg. and -30 deg. plies — the HOT-S prediction for the cross-ply [0/90] laminate is much superior. The transverse normal strain distributions are depicted in fig. 6, where the present theory exhibits proper discontinuous character at the ply interfaces. Except on the bottom surface, both HOT-E and HOT-S produce appreciable errors in the [0/90] thick laminate.

The transverse shear stress (τ_{xz}) distributions are compared in fig. 7. The HOT-S and HOT-E stresses are computed by integrating appropriate inplane stress gradients in the 3-D equations of equilibrium (ref. 6). Throughout, these predictions compare well with the exact elasticity solutions. Figure 8 depicts thickness distributions of the normal stress (σ_{xx}). These show that for the [30/-30]s thick laminate the stresses in the outer plies are underestimated.

This is naturally the result of linear assumptions for the inplane displacements (1). Interestingly, the σ_{xx} results for the [0/90] thick laminate are excellent.

CONCLUDING SUMMARY

In this paper we have discussed a new {1, 2}-order theory for the elasto-static analysis of laminated composite plates in which the assumed variables are the components of the displacement vector, the transverse shear strains and the transverse normal stress. The transverse stresses satisfy exact stress boundary conditions at the top and bottom plate surfaces. The virtual work principle produces a set of equilibrium equations and associated Poisson boundary conditions. As was the previous theory, this theory is particularly suited for finite element approximation with simple C^0- and C^{-1}-continuous displacement interpolation fields. The analytic solutions - obtained for a wide range of laminations and thicknesses - showed that the theory is applicable to thin and thick laminated composites and has some advantages over our previous {1, 2}-order theory.

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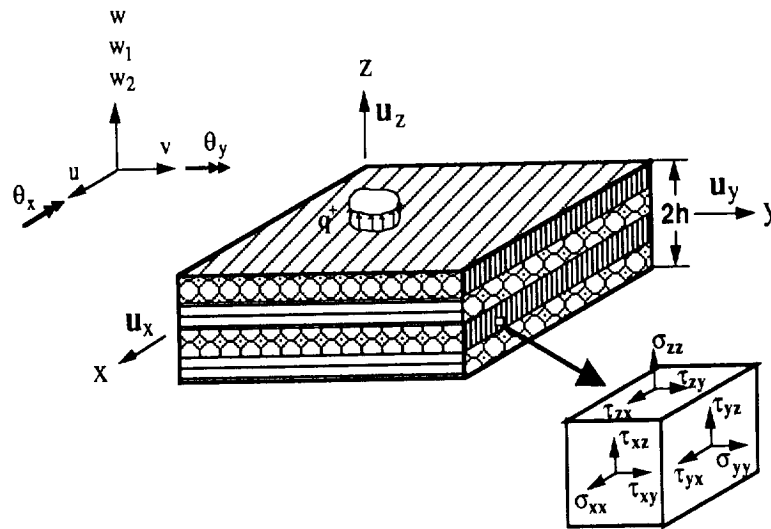
Figure 1. Notation for {1,2}-order plate theory.
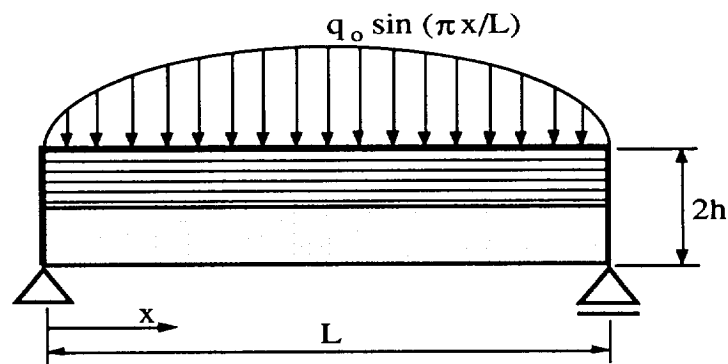
Figure 2. Cylindrical bending of infinite laminated plate.
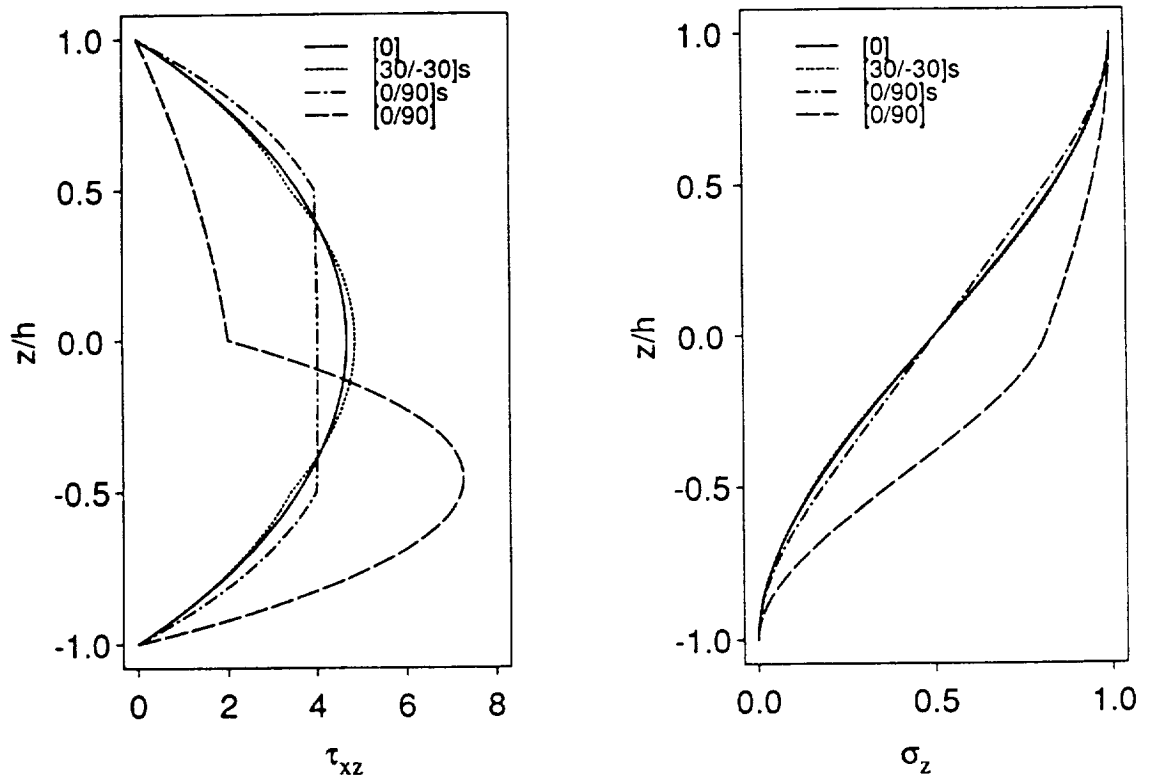
Figure 3. Exact distributions of $\tau_{xz}(0, z)$ and $\sigma_z(L/2, z)$ across thickness in Gr/Ep laminates, $L/2h=10$.
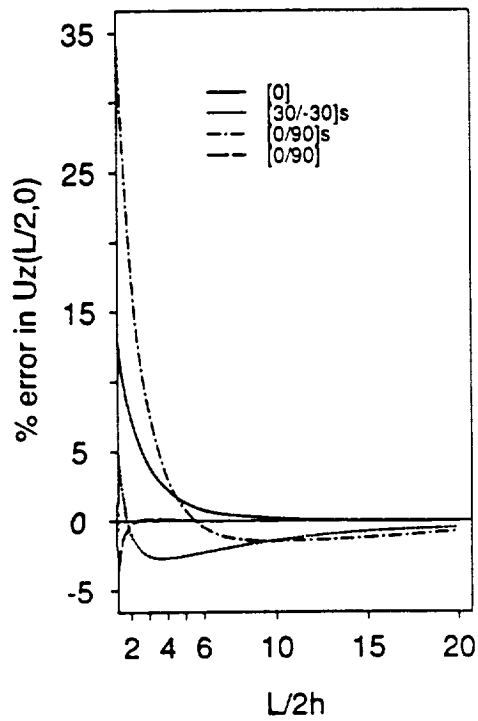

Figure 4. Percent error in maximum midplane deflection vs. $L/2h$ ratio for various Gr/Ep laminates.
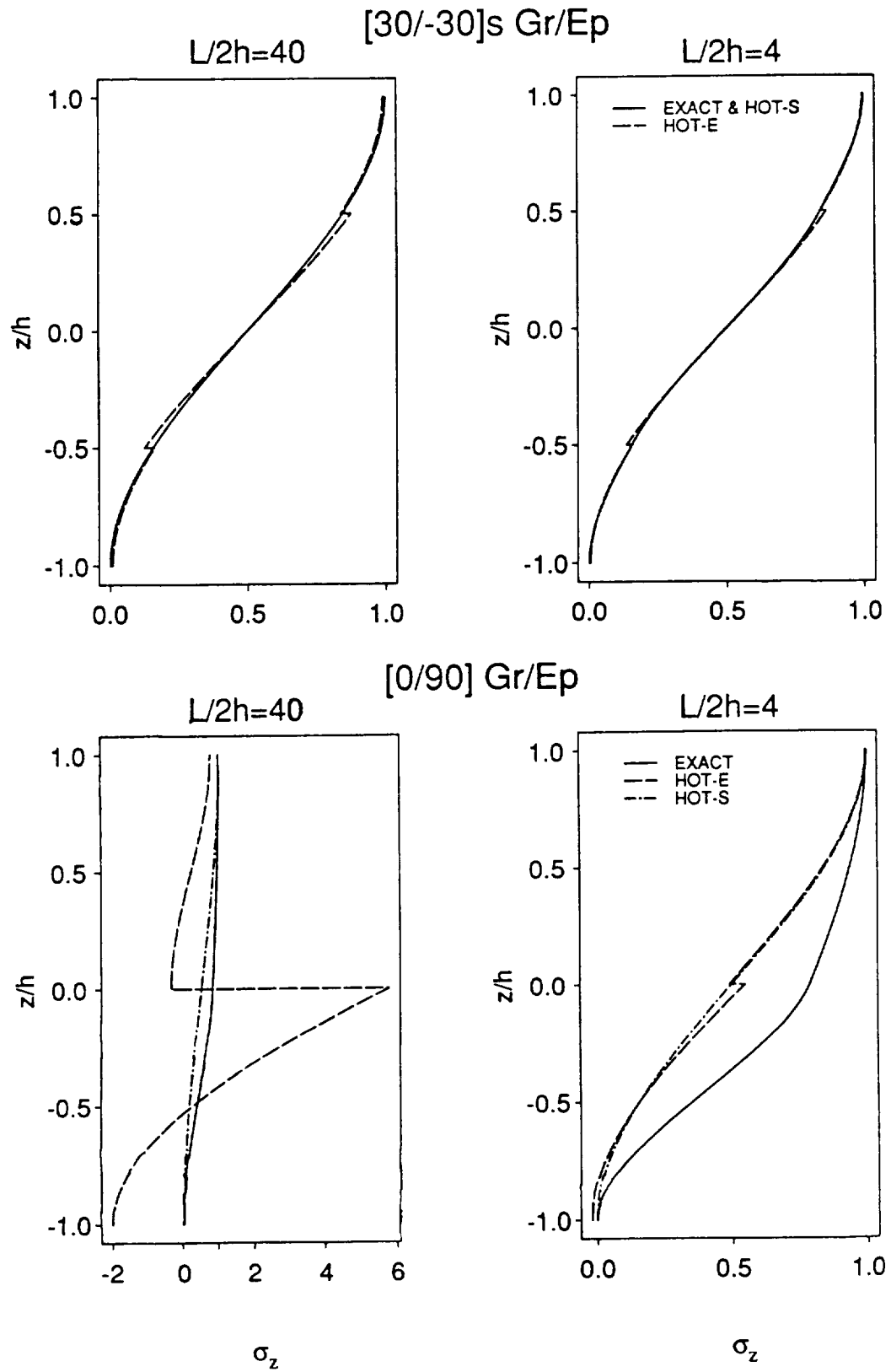
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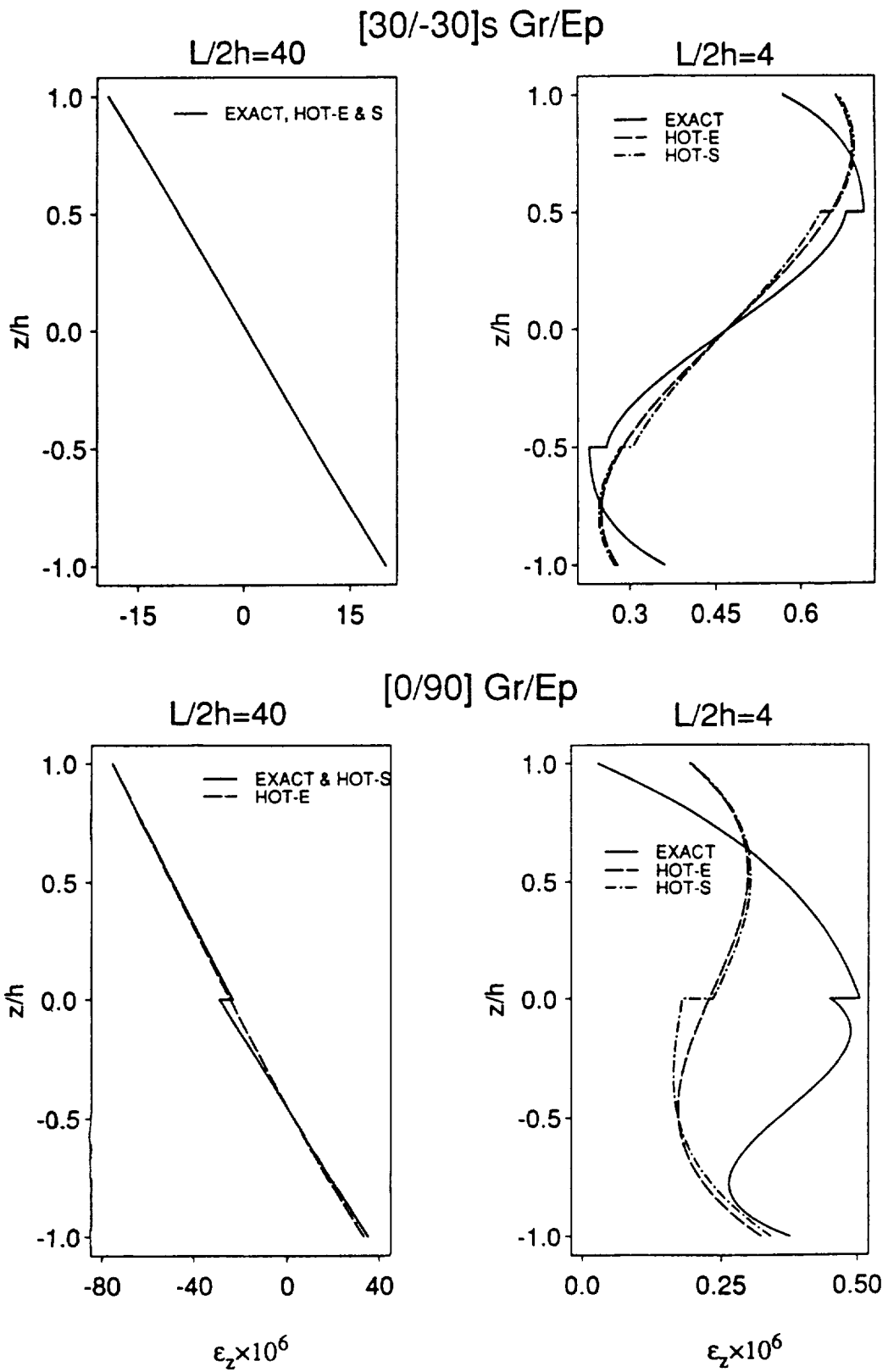Figure 5. Distributions of $\sigma_z(L/2, z)$ across thickness in Gr/Ep laminates; $L/2h = 40, 4$.

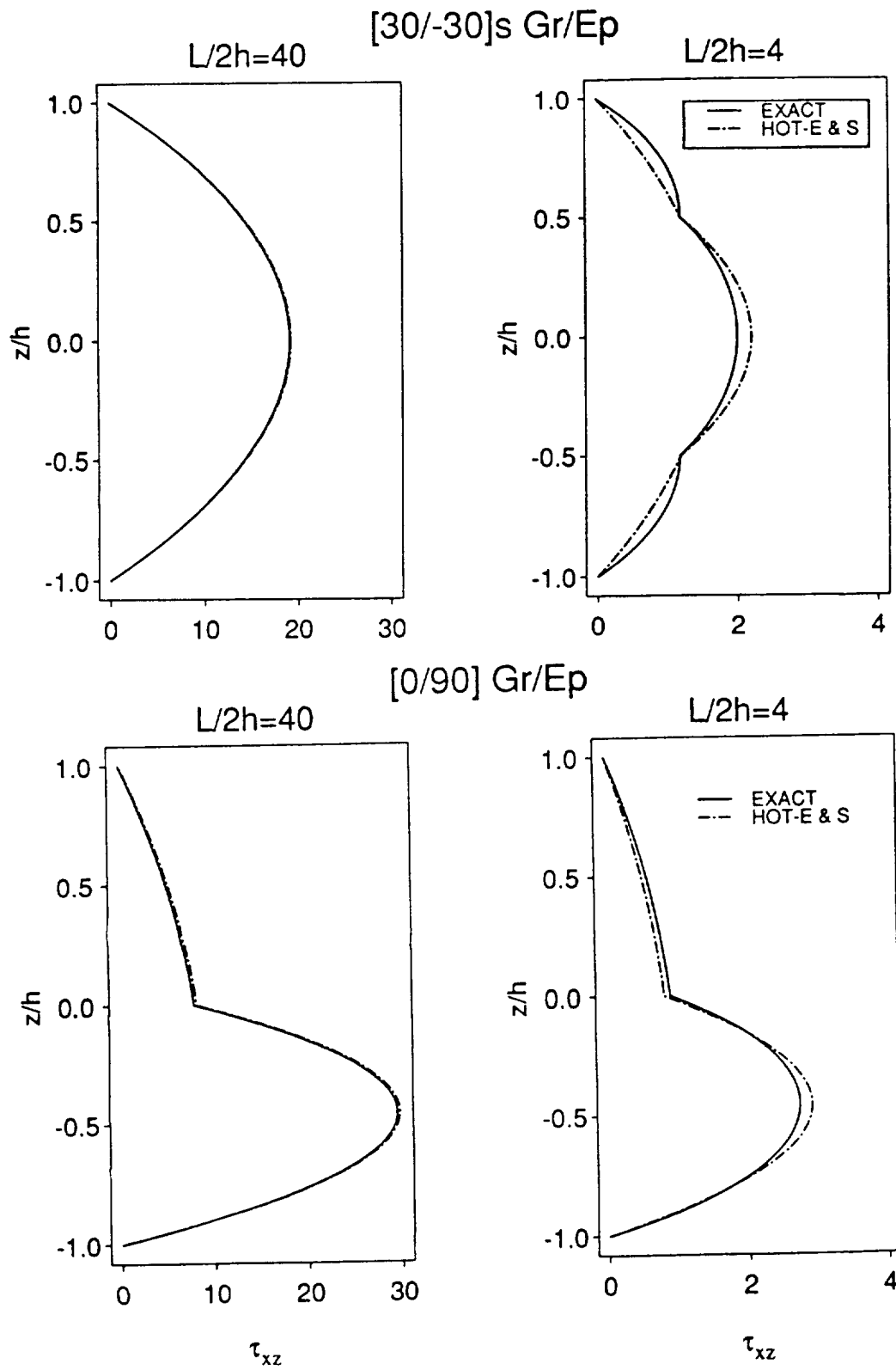Figure 6. Distributions of $\varepsilon_z(L/2, z)$ across thickness in Gr/Ep laminates; $L/2h = 40$, 4.

Figure 7. Distributions of $\tau_{xz}(0, z)$ across thickness in Gr/Ep laminates; L/2h = 40, 4.
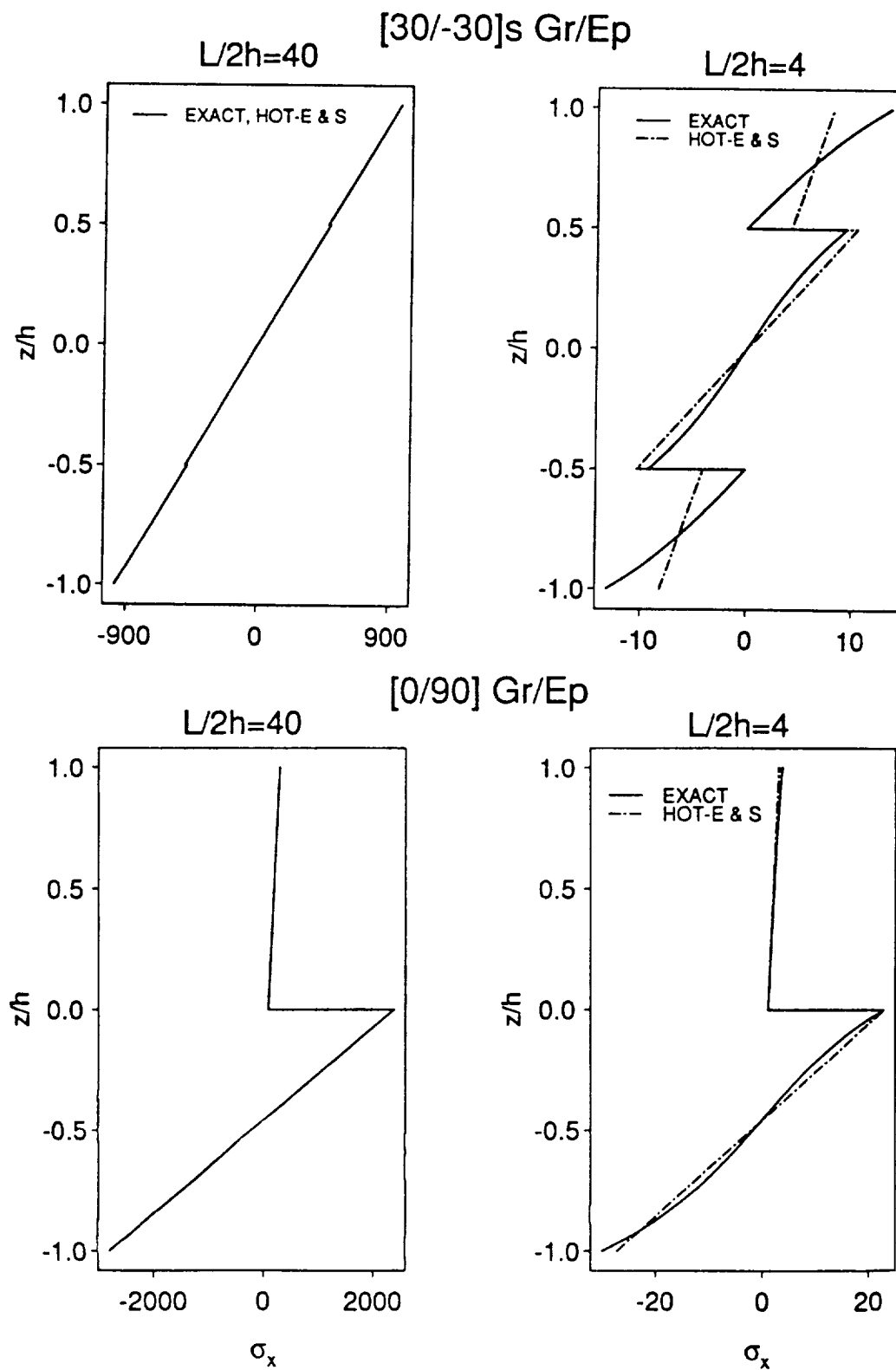
Figure 8. Distributions of $\sigma_x(L/2, z)$ across thickness in Gr/Ep laminates; $L/2h = 40, 4$.