NASA Technical Memorandum 110163

An Extension of the Lighthill Theory of Jet Noise to Encompass Refraction and Shielding

H. S. Ribner
Langley Research Center, Hampton, VA

May 1995

National Aeronautics and
Space Administration
Langley Research Center
Hampton, Virginia 23681-0001

AN EXTENSION OF THE LIGHTHILL THEORY OF JET NOISE TO ENCOMPASS REFRACTION AND SHIELDING

by

Herbert S. Ribner

University of Toronto Institute for Aerospace Studies and NASA Langley Research Center

ABSTRACT

A formalism for jet noise prediction is derived that includes the refractive 'cone of silence' and other effects; *outside the cone it approximates the simple Lighthill format.* A key step is deferral of the simplifying assumption of uniform density in the dominant 'source' term. The result is conversion to a *convected wave* equation *retaining the basic Lighthill source term.* The main effect is to amend the Lighthill solution to allow for refraction by mean flow gradients, achieved via a frequency-dependent directional factor. A general formula for power spectral density emitted from unit volume is developed as the **Lighthill-based** value **multiplied** by a squared 'normalized' **Green's function** (the directional factor), referred to a *stationary* point source. The convective motion of the sources, with its powerful amplifying effect, also directional, is already accounted for in the Lighthill format: *wave convection and source convection are decoupled*. The normalized **Green's function** *appears to be near unity outside the refraction dominated 'cone of silence'*, this validates our long term practice of using Lighthill-based approaches outside the cone, with extension inside via the Green's function. The function is obtained either experimentally (injected 'point' source) or numerically (computational aeroacoustics). Approximation by unity seems adequate except near the cone and except when there are shrouding jets: in that case the difference from unity quantifies the *shielding effect.* Further extension yields dipole and monopole source terms (cf. Morfey, Mani, and others)) when the mean flow possesses density gradients (e.g., hot jets).

INTRODUCTION

Lighthill, in his seminal papers[1,2] posed the problem of flow noise in terms of a wave equation for a 'virtual medium at rest': the actual flow was effectively incorporated into right hand side terms; these were interpreted as sources of sound. Although the equation is exact, *approximations* to these source terms have the effect of suppressing sound convection (hence refraction and shielding) by the mean jet flow[3,4]. (Some effects of refraction were pointed out by Powell[5] even before this connection was made.) Equivalent equations for a moving medium have been put forward[6-10]; they allow for the sound convection. Of these, Lilley's equation[10] has received much attention: it has been developed by Mani[11-13], Balsa[14,15], and others[16-22] into a quantitative predictive formalism for properties of jet noise. It entails a formidable derivation or calculation of a Green's function for highly idealized models of a jet flow. By contrast, the Lighthill procedure, as developed by Ribner[23], Pao and Lowson[24], and others based on Csanady[7,25-27], is relatively simple. These various formalisms appear to yield comparable predictive accuracy outside the 'cone of silence' opening downstream of the jet[28,29]. Within this cone the Lighthill-based theory fails completely--it predicts no attenuation--whereas the Lilley-based theory exhibits good to poor accuracy, depending on frequency.

It is now well known[3,4,7]that expansion of the basic Lighthill source term leads to extra terms that may be shifted to yield a convected wave equation: it was the implicit discarding of these extra terms that was cited above. In 1977 the expansion was exploited[28,29] to demonstrate a considerable equivalence between the Lighthill- and Lilley-based approaches outside the 'cone of silence'. In the course of the present study it was realized that *the dominant residual source terms,* as finally modelled, *were fully equivalent to the Lighthill term.* Taken together, these findings led to the notion that *the Lighthill solution could be amended to allow for convected wave effects: extended in simple fashion into the 'cone of silence'.* The basis of the extension lies in replacing the ordinary oscillatory Green's function, $e^{ikr}/4\pi r$, by that for the convected wave equation. These are both for a *stationary* point source, in contradistinction to the Lilley procedures; this decouples convection of the sources and the sound waves, permitting the cited simplifications in the theory. The following analysis develops the mathematical implementation. Since the Green's function is frequency-dependent, the formalism is directed toward the power spectral density, in particular, from unit volume. Then quadratures can provide the full jet spectrum and the broadband noise intensity. All of these are direction-dependent.

The survey paper, reference 29, from which the present study evolved, illuminates certain facets only briefly touched on herein; moreover, it displays graphically several comparisons of theory and experiment merely cited here. For a fuller understanding and perspective as to how the present notions relate to other theories of jet noise, that paper should be consulted as well.

PARTIAL OVERVIEW

Lighthill manipulated the conservation equations of fluid dynamics into the form of a wave equation forced by nonlinear terms in the unsteady velocity v_i on the right hand side; these were interpreted as sources of sound. For practical prediction, only the dominant source term was retained, to yield

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j} \quad (1a)$$

As a further approximation, the fluid density was approximated as uniform ($\rho = \rho_o$) in the source term, resulting in

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_o \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \quad (1)$$

The early assumption of uniform density is premature, however: it completely suppresses refraction of the sound by the flow[3,4]. Herein, gradients in the density figure in the expansion of the original source term of (1a), *after* which the density is replaced by its mean. This restores the refractive capability in terms of a convective wave equation: in the case of a transversely sheared mean flow it takes the simple form

$$\frac{1}{c_o^2}\frac{\bar{D}^2 p}{Dt^2} - 2\rho_o \frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \rho_o \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \quad (2a)$$

(where u_{2ac} is defined by

$$\rho_o \frac{\bar{D} u_{2ac}}{Dt} = - \frac{\partial p}{\partial x_2} \quad (2b))$$

as a refinement of the unconvected wave equation (1): the degree of approximation in the step $\rho=\rho_o$ here is less. This bears a close relationship to Lilley's well-known equation[10,12,13]. It is, however, vastly simpler, with the special feature of being forced by *the same source term as in Lighthill's uniform density format,* if, compatibly with $\rho = \rho_o$, $\partial v_i/\partial x_i$ is taken to be zero. The convection implicit in $\bar{D}/Dt$ is by the jet velocity, approximated locally as $U(x_2)$; this is the time average of the instantaneous velocity v_i, the fluctuations being designated u_i.

The solution of Eq.(2) for a single frequency component of the pressure p is obtained via a Green's function, the right hand side being replaced by a *stationary* oscillatory point 'source'. Guided by an elegant relation due to Balsa[15] (but like Lighthill, he used a *moving* point source), we develop a general expression for the power spectral density of the sound pressure emitted from unit volume of the flow as

$$\frac{d\Phi(x|\omega)}{d\,\mathrm{Vol}} \propto |G|^2 * \text{ 4-dimensional Fourier transform of space-time correlation of } (\rho_0 \partial^2 v_i v_j / \partial x_i \partial x_j) \quad (3a)$$

where $|G|$ is the absolute value of the oscillatory Green's function, and it applies to a large distance x (far field). (An earlier attempt by Schubert[8,9], in retrospect oversimplified, may be mentioned.) Eq.(3a) can be reduced to the much neater form

$$\frac{d\Phi(x|\omega)}{d\,\mathrm{Vol}} \propto \kappa^4 * |G|^2 * \text{ 4-dimensional Fourier transform of space-time correlation of } (\rho_0 v_\kappa^2) \quad (3b)$$

where κ is a wave vector (with absolute value κ) governing the phase of G, and v_κ is the instantaneous velocity component in the direction of κ. This reduction parallels a similar one in a space-time domain formalism (for $\overline{p^2}$) due to Proudman[45].

The four dimensional transform may be reduced to a one dimensional transform (time ⇒ frequency); this involves the widely used approximation of the turbulent 'eddies' as 'acoustically compact'. *This is a great simplification, but at a significant price in accuracy:* the amplifying effect[1,2] of eddy convection is progressively overestimated with increasing jet velocity[3,4,30,38].

An alternative version of Eq.(3) is obtained by transformation of the correlation to a convected frame of reference, following Chu[30,31]. This resembles equations of Lilley[10] and Balsa[15] based on a *moving* point Green's function in connection with a moving frame. (Cf. also references 24, 38, 46-48 for developments in terms of a four dimensional Fourier transform.) In either case, use of the moving frame simplifies extraction of the amplifying effect of convection of the source pattern.

For the simple wave equation (1), the magnitude of the Green's function at a large distance x is simply $(1/4\pi x)$. For the convected wave equation (2), on the other hand, $|G|$ will have a form approximating $(1/4\pi x)$ times a directional factor[8,9]. In terms of these results, *a solution of Eq.(1) corresponding to (3b)* is obtained by replacing $\kappa^4|G|^2$ in Eq.(3b) by $k^4(4\pi x)^{-2}$, where $k = \omega/c$. Then, on taking the ratio of the two equations,

$$\underset{\text{convected wave eq.}}{d\Phi(x|\omega)/d\mathrm{Vol}} = [(\kappa^2/k^2)\,(4\pi x)|G|]^2 \underset{\text{Lighthill wave eq}}{d\Phi(x|\omega)/d\mathrm{Vol}} \quad (4)$$

where the factor in brackets is defined as $|G_N|$ and for the purposes herein is designated the **normalized Green's function.**

Eq.(4) generalizes the power spectral density from unit volume based on the simple Lighthill wave equation to account, via a more realistic wave equation, for the influence of flow. It exhibits the result as the **Lighthill-based** value **multiplied** by a squared 'normalized' **Green's function.** *The Green's function multiplier incorporates the flow-acoustic interaction: it yields a frequency-dependent alteration of the directional pattern.* The interaction has been described as a refractive effect[32,33] due to mean flow or sound speed gradients and also as a shielding effect[11,12,15] due to the mean flow itself. The term **'Lighthill-based** value' in the above relation is not restricted to a formulation in terms of 4-dimensional transforms. **All valid estimation schemes used in the past are encompassed;** these are usually based on an approach in the space-time domain, rather than the wave number-frequency domain.

Equations (2), (3), and especially their corollary Eq.(4) are the key results of this paper. They provide the basis for reinterpretation of early procedures of the author's group -- experimental[32,33] and numerical[8,9,34] -- that effectively evaluated an approximation to the **Green's function**. The normalized magnitude, $|G_N|$, appears near unity outside the refraction dominated 'cone of silence': *this validates our practice of using Lighthill-based approaches outside the cone, with extension inside via the Green's function.*

A further extension in an Appendix yields dipole and monopole source terms (cf. Morfey[35], Mani[13], Michalke and Michel[49, 50]) when the mean flow possesses density gradients (e.g., hot jets).

MODIFIED LIGHTHILL EQUATION

Unconvected Wave Equation: Virtual Medium at Rest

Lighthill's wave equation[1,2] is equivalent to

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j} + \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 \rho}{\partial t^2} \qquad (5)$$

wherein the pressure has replaced the density that he used as the dependent variable; additionally, the fluid has been approximated as inviscid and non-heatconducting. The right hand side is interpreted as a spatial distribution of sources of sound. In the usual approximation, the last two terms are taken to cancel, and the fluid density is taken to be a constant ($\rho = \rho_o$) in the first. This leads to

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho_o \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \tag{6}$$

A rich literature[4,29] has dealt with applications of this equation for the prediction of properties of jet noise.

The early replacement of ρ by ρ_o to yield (6) is premature, however; it has the effect of suppressing wave convection (and refraction) by the flow[3,4]. This is shown most simply by expansion of the *original* first source term under the specification of a *unidirectional, transversely sheared, mean flow $U(x_2)$*. (This is known as the 'locally parallel' approximation when applied to a real spreading jet. That more complicated case is dealt with in Appendix A, along with the derivation of additional source terms[13,35] that arise in, e.g., heated jets, from density gradients.) The instantaneous local velocity is written as the mean plus a perturbation u_i,

$$v_i = U_i + u_i\,; \quad U_i = (U(x_2),0,0), \tag{7}$$

and the expansion changes (5) to

$$\frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U}{\partial x_2}\frac{\partial \rho u_2}{\partial x_1} + \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{\bar{D}^2 \rho}{Dt^2} \tag{8}$$

using the definition $\quad \bar{D}/Dt = \partial/\partial t + U\partial/\partial x_1 \quad$ (9)

as a convective derivative following the mean flow. Both (5) and (8) are exact.

Convected Wave Equation: Actual Medium with Flow

At this point we approximate $\bar{D}^2\rho/Dt^2$ as $\bar{c}^{-2}\bar{D}^2 p/Dt^2$, where $\bar{c}$ is a local time-average sound speed (Appendix B). On shifting the term to the left hand side, equation (8) goes over to

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - \nabla^2 p = \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U}{\partial x_2}\frac{\partial \rho u_2}{\partial x_1} \tag{10}$$

Thus *in place of the unconvected wave equation (6), we now have the more accurate convected wave equation (10).* The difference is traceable to the deferral of the incompressible flow assumption in the first source term of Eq.(5): this had suppressed the expansion of $\partial^2\rho/\partial t^2$ into the convective form $\bar{D}^2\rho/Dt^2$.

This is not yet in final form. The second source term, involving mean flow shear $\partial U/\partial x_2$, is linear in u_2. For this reason, it has been argued[8-10,36] that the term participates in wave propagation and so should be on the left hand side. (See Goldstein[37], pp. 389-391, for a further discussion.) *This applies, however, only to a small acoustic (or compressible) component associated with wave propagation.* Within a subsonic flow the overwhelming part of u_i is induced by the turbulence vorticity; being small compared with the soundspeed, it may be approximated as incompressible. Thus we split off the acoustic component and place it on the left hand side. (The acoustic component of the first source term, on the other hand, is of higher order and may be neglected). *Thus it is at this point that we may justifiably apply Lighthill's approximation $\rho = \rho_o$ to the remaining right hand side source terms.* Consistently, the u_i are taken to have zero divergence. Equation (10) then goes over to

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - 2\rho_o\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \rho_o\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\rho_o\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1} \qquad (11)$$

where u_{2ac} is defined via the momentum equation

$$\rho_o\frac{\bar{D}u_{2ac}}{Dt} = -\frac{\partial p}{\partial x_2} \qquad (11a)$$

The corresponding equation for the Green's function, $G(\mathbf{x},\mathbf{y}|\omega)$, is

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - 2\rho_o\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \delta(\mathbf{x}-\mathbf{y})\, e^{-i\omega t} \qquad (12)$$

and the solution of the pair of equations (11a) and (12) for p is $Ge^{-i\omega t}$.

Recall now that Eqs. (8) to (12) relate to turbulence u_i superposed on a transversely sheared mean flow $U_i = (U(x_2), 0, 0)$, Eq. (7). For this scenario we examine the Lighthill source term $\rho_o\partial v_i v_j/\partial x_i x_j$ of Eq.(6), where $v_i = U_i + u_i$. On carrying out the differentiation, the term expands *exactly* into the two terms on the right hand side of Eq.(11). Thus that equation is exactly equivalent to

$$\frac{1}{\bar{c}^2}\frac{\bar{D}^2 p}{Dt^2} - 2\rho_o\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \rho_o\frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \qquad (13)$$

In the expansion we have used the *incompressibility* relation $\partial v_i/\partial x_i = 0$ implied by taking $\rho = \rho_o$; as noted, the approximation of incompressibility is made only in the source terms.

This is a major new result, approximated as Eq (2) in the OVERVIEW. It is slightly generalized here, with $\bar{c}^2(\mathbf{x})$ replacing c_o^2. (A further generalization in Appendix A, (i) replaces the restrictive Eq. (7) by a more realistic mean flow $\mathbf{U}(\mathbf{x})$ on the left hand side, and (ii) allows for mean flow density gradients; these give rise to important additional source terms in the cases of hot jets and jets of nonambient gas[13,35,49,50].)

Now according to Lighthill's arguments[1,2], $\rho_o \partial v_i v_j/\partial x_i x_j$ is a valid source term for flow noise. From the above it is clear that *both* the right hand source terms of Eq.(11), *being equivalent* for the sheared mean flow, are likewise valid source terms for that scenario: the 'shear noise' term $2\rho_o(\partial U/\partial x_2)(\partial u_{2ac}/\partial x_1)$ as well as the 'self noise' term $\rho_o \partial u_i u_j/\partial x_i x_j$. Furthermore, the velocities u_i therein are incompressible to the same approximation as in Lighthill's term. All of this supports the earlier argument concerning the split of the shear term in Eq.(11): a compressible acoustic component has been excised (before the step $\rho \Rightarrow \rho_o$) and placed on the left hand side, leaving an incompressible component on the right hand side (presumably much larger). The shear term with u_{2ac} on the left hand side serves in a propagation role, whereas the shear term with u_2 on the right hand side serves in a generation role.

Equation (11) with deletion of the last term, together with Eq. (11a), are roughly equivalent to Lilley's equation[10,12,13] (he combined them into a single third order wave equation[37]). However, Lilley, in effect, moved the *entire* shear term to the left hand side wave operator. *The 'shear noise' source term so lost from the right hand side is of major importance.* This was shown indirectly by Ribner[4,23], and later more directly by Pao and Lowson[24] in terms of Lighthill's equation. Aside from the arguments of the last paragraph, credibility is afforded by comparisons with experiment: the 'self noise' spectrum, downshifted in frequency, shows a predicted match to the 'shear noise' spectrum (**Fig.1**). We will return to this point later.

The successive modifications of the wave equation for flow noise, proceeding from Eq.(5) (virtual medium at rest) to Eq.(13) (actual medium with flow) may be summarized at this point. Deferring the approximation $\rho = \rho_o$, as in the derivation of Eq.(13), leads to propagative terms that may be moved to the left hand side. This is a conversion into a convective wave operator, making explicit the role of the mean flow in convecting the sound waves. On the other hand, the convective role of the mean flow is only implicit in Eq.(5): it resides in the gradients of ρ in the source term. These gradients are suppressed in the approximation $\rho=\rho_0$ prematurely applied in Eq. (6). At very high frequencies wave convection should reduce[7,11,54,55] the well known amplification predicted[1,2], in the absence of flow, for moving sources; the reduction is referred to as 'fluid shielding'[11, 14]. But at jet noise frequencies the most dramatic effect has to do with the flow gradients: the radial decrease in mean-flow velocity in the jet; this turns or refracts the sound rays away from the axis to yield the well known 'cone of [relative] silence'.

The allowance for refraction in the generalized Lighthill equation (13) comes at a price: the point source solution (Green's function, utilizing eq. (11) as well) is enormously more complex than that for the simple wave operator of Lighthill's equation (1). It has been found, however[8,9], that there is near agreement at the larger angles from the jet axis. This implies that *the simple Lighthill solution for rms sound pressure will approximate that predicted via eq. (13) outside the 'cone of silence'.* The basis for this is elaborated below.

FORMULAS FOR POWER SPECTRAL DENSITY

General Relations

The formulas that follow are based on the Green's function for a stationary, oscillatory point source. The approach parallels that of Balsa[15] based on a moving, oscillatory point source, with missing steps being inferred. We seek the power spectral density $\Phi(\mathbf{x}|\omega)$ of the radiated sound pressure dictated by equations (1) or (13); they may be written symbolically as

$$L[\partial/\partial t, \partial/\partial \mathbf{x}; \mathbf{a}(\mathbf{x})]\, p(\mathbf{x}, t) = Q(\mathbf{x}, t) \quad (14)$$

where L may be either the unconvected wave operator of Eq (1) or the convected wave operator of Eq (13). The $\mathbf{a}(\mathbf{x})$ are the coefficients; for the convected wave operator they allow for the local mean flow, taken as $U(x_2)$, and a space-variable sound speed. (In a generalized version of Eq. (13) in Appendix A, $U(x_2)$ goes over to $\mathbf{U}(\mathbf{x})$,and ρ_0 to $\bar{\rho}(\mathbf{x})$, where $\bar{\rho}$ is a local time average. Equation (14) applies to this version as well.) Correspondingly,

$$L[-i\omega, \partial/\partial \mathbf{x}; \mathbf{a}]\, p(\mathbf{x}|\omega) = Q(\mathbf{x}|\omega) \quad (15)$$

where p and Q are defined in Fourier transform pairs:

$$p(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(\mathbf{x},t)\, e^{i\omega t}\, dt; \qquad p(\mathbf{x},t) = \int_{-\infty}^{\infty} p(\mathbf{x}|\omega)\, e^{-i\omega t}\, d\omega \tag{16}$$

$$Q(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\mathbf{x},t)\, e^{i\omega t}\, dt\,; \quad Q(\mathbf{x},t) = \int_{-\infty}^{\infty} Q(\mathbf{x}|\omega)\, e^{-i\omega t}\, dt \tag{17}$$

Equation (15) has a solution

$$p(\mathbf{x}|\omega) = \int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y}|\omega)\, Q(\mathbf{y}|\omega)\, d^3\mathbf{y} \tag{18}$$

in terms of a Green's function $G(\mathbf{x},\mathbf{y}|\omega)$ that is the solution of

$$L[-i\omega, \partial/\partial\mathbf{x}; a]\, G(\mathbf{x},\mathbf{y}|\omega) = \delta(\mathbf{x} - \mathbf{y}) \tag{19}$$

The (two-sided) power spectral density of the sound pressure is evaluated as

$$\Phi(\mathbf{x}|\omega) = \langle p(\mathbf{x}|\omega)\, p^*(\mathbf{x}|\omega)\rangle \tag{20}$$

where $\langle\ \rangle$ signifies an ensemble average. Inserting (18), with $\mathbf{y}$ replaced by $\mathbf{y}'$ and $\mathbf{y}''$,

$$\Phi(\mathbf{x}|\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y}'|\omega) G^*(\mathbf{x},\mathbf{y}''|\omega)\langle Q(\mathbf{y}'|\omega)\, Q^*(\mathbf{y}''|\omega)\rangle\, d^3\mathbf{y}'\, d^3\mathbf{y}'' \tag{21}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{y}+\xi/2|\omega) G^*(\mathbf{x},\mathbf{y}-\xi/2|\omega)\langle Q(\mathbf{y}+\xi/2|\omega) Q^*(\mathbf{y}-\xi/2|\omega)\rangle d^3\xi\, d^3\underline{\mathbf{y}} \tag{22}$$

where $\xi = \mathbf{y}' - \mathbf{y}''$ and $\underline{\mathbf{y}} = (\mathbf{y}' + \mathbf{y}'')/2$.

Reduction for Far Field

The Green's function of Eq. (22) may be written in the form

$$G(\mathbf{x},\mathbf{y}\,|\omega) = |G(\mathbf{x},\mathbf{y}\,|\omega)|\, e^{i\psi(\mathbf{x},\mathbf{y}\,|\omega)} \tag{23}$$

We now restrict $\mathbf{x}$ to the far field defined by $|\mathbf{x}|$ being very much greater than both $|\mathbf{y}|$ *and the largest wavelengths of concern;* $\mathbf{y}$ is limited to the region of nonzero source strength Q. A sufficient approximation for the phase, which seems to be implied in Balsa's Appendix[15], is then

$$\psi(\mathbf{x},\mathbf{y}'|\omega) = \psi(\mathbf{x},\mathbf{y}''|\omega) - \kappa\bullet(\mathbf{y}'-\mathbf{y}'') \qquad (24)$$

where the wave vector

$$\kappa = -(\nabla_y \psi)_{\text{far field}} \qquad (25)$$

is proportional to ω. There seems to be the further reasonable assumption, which we make also, that the variation in amplitude of G is negligible compared with that of the phase as ξ of eq. (22) ranges within the source region Q. Then ξ may be dropped in comparison with $\mathbf{y}$ in the amplitude so that

$$G(\mathbf{x},\mathbf{y}+\xi/2|\omega)\, G^*(\mathbf{x},\mathbf{y}-\xi/2|\omega) \approx |G(\mathbf{x},\mathbf{y}|\omega)|^2 e^{-i\kappa\bullet\xi} \qquad (26)$$

The other factor in (22) is the frequency-domain correlation

$$\boldsymbol{R}(\mathbf{y}, \xi|\omega) \equiv \langle \boldsymbol{Q}(\mathbf{y}+\xi/2|\omega)\, \boldsymbol{Q}^*(\mathbf{y}-\xi/2|\omega)\rangle \qquad (27)$$

This is the Fourier transform of the time-domain correlation

$$\mathrm{R}(\mathbf{y}, \xi,\tau) = \langle \mathrm{Q}(\mathbf{y}+\xi/2, t+\tau)\, \mathrm{Q}(\mathbf{y}-\xi/2, t)\rangle \qquad (28)$$

(which is independent of t); specifically,

$$\boldsymbol{R}(\mathbf{y}, \xi|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\mathrm{R}(\mathbf{y}, \xi,\tau)d\tau \qquad (29)$$

Insertion of equations (26) and (29) into (22) yields

$$\Phi(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\mathbf{x},\mathbf{y}|\omega)|^2\, d^3y \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{R}(\mathbf{y},\xi,\tau)\, e^{-i\kappa\bullet\xi+i\omega\tau}\, d^3\xi d\tau \qquad (30)$$

Equation (30) is the desired general result for the power spectral density $\Phi(\mathbf{x}|\omega)$ (cf. Eq.(3) of the Overview). The inner integral can be recognized as a four-dimensional Fourier transform of the two-point space-time correlation $\mathrm{R}(\mathbf{y}, \xi,\tau)$. Alternatively, it is a three-dimensional transform of the cross-spectral density $\boldsymbol{R}(\mathbf{y}, \xi|\omega)$. This transform multiplied by the square of the amplitude of the Green's function (frequency domain) has a simple interpretation: it is the contribution of unit volume of the sources Q to the power spectral density of the sound pressure radiated to the field point $\mathbf{x}$.

Moving Reference Frame

Experimentally the correlation $R(\mathbf{y},\xi,\tau)$ in a jet flow has a form describing a moving, fluctuating pattern. This is dealt with most neatly by transforming to a reference frame moving with the pattern convection velocity, taken as U_c. (But the Green's function, unlike that in Balsa's relation[15], still refers to a point source at rest.) Following Chu[31], we take

$$\xi_m = \xi - \mathbf{U}_c\tau ; \qquad \mathbf{U}_c = (U_c,0,0) \tag{31}$$

so that

$$\kappa \bullet \xi_m = \kappa \bullet \xi - \kappa \bullet \mathbf{U}_c \tau \tag{32}$$

and reexpress R in terms of ξ_m, using (31), as

$$R_m(\mathbf{y},\xi_m,\tau) = R(\mathbf{y},\xi,\tau) \tag{33}$$

Then in Eq. (30)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mathbf{y},\xi,\tau)\, e^{-i\kappa\bullet\xi+i\omega\tau} d^3\xi d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_m(\mathbf{y},\xi_m,\tau)\, e^{-i\kappa\bullet\xi_m+i(\omega-\kappa\bullet U_c)\tau} d^3\xi_m d\tau \tag{34}$$

since the Jacobian of the transformation $\xi \rightarrow \xi_m$ is unity. We may further define

$$\varpi = \omega - \kappa \bullet \mathbf{U}_c \tag{35}$$

as the effective source frequency in the moving frame to yield an observer frequency ω at $\mathbf{x}$ in the stationary frame (far field). Inserting these last two equations converts Eq. (30) into

$$\Phi(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\mathbf{x},\mathbf{y}|\omega)|^2\, d^3y \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_m(\mathbf{y},\xi_m,\tau)\, e^{-i\kappa\bullet\xi_m+i\varpi\tau}\, d^3\xi_m d\tau \tag{36}$$

This is an alternative form for the power spectral density; it is more useful in that the effects of source convection are more easily brought out. In Eq. (30) the space-time source field correlation is referred to a stationary coordinate frame and is designated R. In Eq. (36) this same correlation is referred by transformation to a coordinate frame moving with velocity U_c, and is designated R_m. In both cases *the two points being correlated are stationary.*

Source Term of Form $a(y)\partial^2 b_{ij}/\partial y_i \partial y_j$

Suppose the (monopole) source strength distribution has the form of the right hand side of Eq. (13), slightly generalized:

$$\mathbf{Q}(\mathbf{y}, t) = a(y)\partial^2 b_{ij}(y,t)/\partial y_i \partial y_j \qquad \text{(summed over i, j = 1 to 3)} \qquad (37)$$

It can be shown that the Fourier transform of this is

$$\mathcal{Q}(y|\omega) = a(y)\partial^2 b_{ij}(y|\omega)/\partial y_i \partial y_j$$

The

The cross spectral density of the source field between points **y'** and **y"** is then

$$\langle \mathcal{Q}(y'|\omega)\mathcal{Q}^*(y''|\omega)\rangle = \langle a(y')a(y'')\frac{\partial^2 b_{ij}(y'|\omega)}{\partial y'_i \partial y'_j}\frac{\partial^2 b^*_{kl}(y''|\omega)}{\partial y''_k \partial y''_l}\rangle$$

which is equivalent to

$$\langle \mathcal{Q}(y'|\omega)\mathcal{Q}^*(y''|\omega)\rangle = a^2(y)\,\frac{\partial^4\langle b_{ij}(y'|\omega)b^*_{kl}(y''|\omega)\rangle}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l} \qquad (38)$$

where $\xi \equiv \xi_1, \xi_2, \xi_3 = \mathbf{y}' - \mathbf{y}''$, if $a(y')a(y'')$ is approximated $a^2((y'+y'')/2) = a^2(y)$.

Insertion of equations (26) and (38) into (22) yields the power spectral density of the radiated sound pressure as

$$\Phi(x|\omega) = \int_{-\infty}^{\infty} |G(x,y|\omega)|^2\, a^2(y)\,\frac{\partial^4\langle b_{ij}(y'|\omega)b^*_{kl}(y''|\omega)\rangle}{\partial\xi_i\partial\xi_j\partial\xi_k\partial\xi_l}\, e^{-i\kappa\bullet\xi}\, d^3\xi d^3y \qquad (39)$$

which is replaceable by

$$\Phi(x|\omega) = \int_{-\infty}^{\infty} |G(x,y|\omega)|^2\, a^2(y)\ \kappa_i\kappa_j\kappa_k\kappa_l\langle b_{ij}(y'|\omega)b^*_{kl}(y''|\omega)\rangle\, e^{-i\kappa\bullet\xi}\, d^3\xi d^3y \qquad (40)$$

But the source cross-spectral density designated by $\langle\ \rangle$ is the Fourier transform of a two-point correlation:

$$\langle b_{ij}(y'|\omega)b^*_{kl}(y''|\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle b_{ij}(y', t+\tau)b_{kl}(y'', t)\rangle\, e^{i\omega\tau}\, d\tau$$

so that the power spectral density of the sound generated by sources of type (37) may be written

$$\Phi(x|\omega) = \int_{-\infty}^{\infty} |G(x,y|\omega)|^2\, a^2(y)\ \kappa_i\kappa_j\kappa_k\kappa_l\langle b_{ij}(y', t+\tau)b_{kl}(y'', t)\rangle\, e^{-i\kappa\bullet\xi + i\omega\tau} d\tau d^3\xi d^3y \qquad (41)$$

Suppose we identify $a(y)b_{ij}(y,t)$ with the Lighthill source term $\bar{\rho}(y)v_iv_j$ (wherein the uniform ρ_0 is generalized to $\bar{\rho}(y)$ to allow for a space-variable time average density: cf. Appendix A). Then, on taking $\kappa_i\kappa_j\kappa_k\kappa_l$ inside the $\langle\ \rangle$ of Eq.(41), we have terms like $\kappa_i\kappa_j b_{ij} = \kappa_i\kappa_j v_i v_j$. But the summation $\kappa_i v_i$ is κ times the component of $\mathbf{v}$ along κ, which we designate v_κ. Writing κ as the magnitude of κ, this is summarized as

$$\kappa_i\kappa_j b_{ij} = \kappa_i\kappa_j v_i v_j = \kappa^2 v_\kappa^2 \tag{42}$$

It follows that, in Eq.(41) the summation

$$\kappa_i\kappa_j\kappa_k\kappa_l\langle b_{ij}(y', t+\tau)b_{kl}(y'', t)\rangle = \kappa^4 v_\kappa^2\langle(y', t+\tau)v_\kappa^2(y'', t)\rangle \equiv \kappa^4 R_\kappa(y,\xi,\tau) \tag{43}$$

where R_κ is a two-point correlation of v_κ^2. With this replacement Eq.(41) simplifies to

$$\Phi(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\kappa^4\bar{\rho}^2(y)|G(\mathbf{x},\mathbf{y}|\omega)|^2\, d^3\mathbf{y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_\kappa(y,\xi,\tau)\, e^{-i\kappa\bullet\xi + i\omega\tau}\, d^3\xi d\tau \tag{44}$$

where, it is noted, the two points being correlated in R_κ are referred to a *stationary* reference frame. Correspondingly, Eq. (36) becomes

$$\Phi(\mathbf{x}|\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\kappa^4\bar{\rho}^2(y)|G(\mathbf{x},\mathbf{y}|\omega)|^2\, d^3\underline{y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{\kappa m}(y,\xi_m,\tau)e^{-i\kappa\bullet\xi_m + i\varpi\tau} d^3\xi_m d\tau \tag{45}$$

where the correlation $R_{\kappa m}$ designates the *same* correlation with respect to the moving frame (velocity $\mathbf{U}_c$); it is obtained from R_κ via the transformation (31). *Eq.(45) is a key result stated in words as Eq.(3b) of the Overview (but generalized with* $\bar{\rho}(\mathbf{y})$ *replacing* ρ_o*).*

These latest results for the power spectral density $\Phi(x|\omega)$ may be put in perspective: they are all expressed in terms of the Green's function for a *stationary*, oscillatory point source in an arbitrary flow $\mathbf{U}_c(\mathbf{x})$. Equations (30) and (36) refer to a general source strength function **Q**; Eqs.(43) and (45), on the other hand, refer to a source strength of the generalized Lighthill form $\mathbf{Q} = \bar{\rho}(y)\partial^2 v_i v_j/\partial y_i \partial y_j$ (summed), to which $\bar{\rho}(y)\kappa^2 v_\kappa^2$ is equivalent.

It is noteworthy, as he pointed out, that this double divergence form of Q implies that the sources are of *quadrupole* nature. In the time domain this was associated with an operator $\partial^2/\partial t^2$ in the far field format; in the present wave-number domain the factor κ^2 plays an equivalent role.

Virtual Medium at Rest

This is Lighthill's scenario[1,2]: the fluid flow is incorporated into the source term. The original region of flow is now treated as a 'virtual medium at rest'. For this case the oscillatory Green's function is simply

$$G(\mathbf{x},\mathbf{y}|\omega) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} e^{ik|\mathbf{x}-\mathbf{y}|}; \qquad k = \omega/c_o \tag{46}$$

When the observer point **x** is in the far field (cf. after Eq. (23)), a sufficient approximation is, with $x = |\mathbf{x}|$,

$$G(\mathbf{x},\mathbf{y}|\omega) = \frac{1}{4\pi x} e^{ikx-ik\mathbf{x}\bullet\mathbf{y}/x}; \qquad |G(\mathbf{x},\mathbf{y}|\omega)| = \frac{1}{4\pi x} \tag{47}$$

Thus, in this case, the vector $\kappa = -(\nabla_y \psi)_{\text{far field}}$ of Eq. (25) may be identified with the wave vector **k** given by

$$\kappa = \mathbf{k} = k\mathbf{x}/x = \omega\mathbf{x}/c_o x; \qquad k = |\mathbf{k}| \tag{48}$$

Also, with the sources convected parallel to the x_1-axis, $\mathbf{U}_c = (U_c,0,0)$, Eq. (35) yields

$$\varpi = \omega[1-(U_c/c_o)(x/x)] = \omega(1-M_c\cos\theta) \equiv \omega\Theta \tag{49}$$

This is just the Doppler shifted source frequency that will yield an observed frequency ω at **x**.

Reduction to Chu's Equation

Equation (45)--correlation referred to moving reference frame, Q of form $\rho_o\partial^2 v_i v_j/\partial x_i \partial x_j$--may be applied to Chu's scenario[30,31] by *specializing to a fluid at rest.* This implies invoking Eqs. (47) to (49). His power spectral density $\Phi_1(\mathbf{x}|\omega)$ is one-sided (limited to positive values of ω), so that it is twice our $\Phi(\mathbf{x}|\omega)$. He notes further that, since $R_{km}(\mathbf{y},\xi_m,\tau)$ is even in ξ and τ, the Fourier transforms may be replaced by cosine transforms. Equation (45) then takes the form, after doubling,

$$\Phi_1(\mathbf{x}|\omega) = \frac{\rho_o{}^2 k^4}{\pi} \int_{-\infty}^{\infty} |G(\mathbf{x},\mathbf{y}|\omega)|^2 d^3y \int_{-\infty}^{\infty} \cos \mathbf{k}\bullet\xi_m d^3\xi_m \int_{-\infty}^{\infty} R_{km}(\mathbf{y},\xi_m,\tau)\cos\varpi\tau \, d\tau \tag{50}$$

But we note that, from the equation for the time delay $\tau_m{}^*$ above Eq. (8) of ref.31, we have the equivalence

$$\omega\Theta\tau_m{}^* = \mathbf{k}\bullet\xi_m \tag{51}$$

employing our Eq. (49). Inserting this and the appropriate values of $|G|$, and ϖ from Eqs. (47) and (49)

$$\Phi_1(x|\omega) = \frac{\rho_o{}^2\omega^4}{16\pi^3 c_o{}^4 x^2}\int_{-\infty}^{\infty} d^3y \int_{-\infty}^{\infty} \cos\omega\Theta\tau_m{}^* d^3\xi_m \int_{-\infty}^{\infty} R_{km}(y,\xi_m,\tau)\cos\omega\Theta\tau d\tau \tag{52}$$

This recovers Chu's result for the one-sided power spectral density (Eq.(2.2.7) in ref.(30) and Eq.(8) in ref. 31).

The *simplifying assumption that the turbulent 'eddies' are 'acoustically compact' permits a marked simplification*: cos $\omega\Theta\tau_m{}^*$ in Eq. (52) (and similarly $e^{-i\kappa\bullet\xi_m}$ in Eq. (45)) can be replaced by unity. *The effect is to reduce four dimensional Fourier transforms to one dimensional transforms*: the space-wavenumber transform becomes merely a volume integral. The replacement is explained along with other observations concerning Eq. (52) in the following quotation from Chu[30] (annotations shown in brackets; our notation and reference numbering):

"Firstly, we can identify ϖ as the frequency in the turbulence; the corresponding radiated frequency ω is then the Doppler-shifted frequency (i.e., $\omega = \varpi/\Theta$). This is a logical result which one could have obtained on physical grounds.. Secondly, Lighthill's criterion for neglecting retarded time shows up automatically in the cos $\omega\Theta\tau_m{}^*$ term. According to Lighthill, retarded time can be neglected if $\omega L/c_0$ is small so that the eddy size L is small compared with the wave-length of the sound that it generates *['acoustically compact']*. If this condition is met...then for $\xi_m \leq L$ the term cos $\omega\Theta\tau_m{}^*$ can be approximated as unity. Thirdly, if retarded time [difference across an 'eddy'] is neglected, Lighthill's convection factor $(1-M_c\cos\theta)^{-5}$, which accounts for the main effect of convection at limited speeds is exhibited as a vertical shift $(1-M_c\cos\theta)^{-4}$ plus a Doppler shift $(1-M_c\cos\theta)^{-1}$ in the power spectrum.....This concept is a low-speed version of a similar idea presented in ref. 4. Fourthly, although the Lighthill's convection factor is not applicable for high-speed convection because of its singularity where $1-M_c\cos\theta = 0$, [an example given by Chu, discussed above Eq. (60)] will show that Eq. (52) is still valid for high-speed convection if retarded time [difference] is not neglected. In fact, this moving frame integral with proper account of retarded time posses a zeroo that exactly cancels the $(1-M_c\cos\theta)^{-5}$ singularity and replaces it by a nonsingular convection factor (cf. also refs. 3 and 38)."

Actual Medium vs Virtual Medium

Here we compare the noise emission from the actual medium, allowing for the effect of the fluid flow on propagation, to that predicted for the virtual medium at rest. We will show how the former differs from the latter (the Lighthill scenario) in being an extension to allow for flow-acoustic interaction effects: e.g., refraction that bends sound rays away from the jet axis to create a 'cone of silence' opening downstream. And we will show a close to asymptotic approach to the Lighthill case outside the cone of silence.

It will be convenient to restrict attention to the power spectral density emitted from unit volume at **y**. For the scenario of Eq. (45) ('actual medium'), with the appropriate Green's function and anticipating that κ may be approximated as **k** at jet noise frequencies (Appendix C),

$$\frac{\partial^3\Phi(x|\omega)}{\partial y^3} = \rho_o{}^2k^4\,|G(\mathbf{x},\mathbf{y}|\omega)|]^2\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{km}(y,\xi_m,\tau)e^{-i\mathbf{k}\bullet\xi_m+i\varpi\tau}d^3\xi_m d\tau\} \qquad (53)$$

The corresponding result, using the respective Green's function and wave vector $\kappa = \mathbf{k}$ for a virtual medium at rest (Lighthill format), is

$$\left[\frac{\partial^3\Phi(x|\omega)}{\partial y^3}\right]_{VM} = \rho_o{}^2k^4\,(1/4\pi x)^2\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{km}(y,\xi_m,\tau)e^{-i\mathbf{k}\bullet\xi_m+i\varpi\tau}d^3\xi_m d\tau\} \qquad (54)$$

The ratio of these two can be put in the form

$$\frac{\partial^3\Phi(x|\omega)}{\partial y^3} = [(4\pi x)\,|G(\mathbf{x},\mathbf{y}|\omega)|]^2\left[\frac{\partial^3\Phi(x|\omega)}{\partial y^3}\right]_{VM} \qquad (55)$$

It will simplify discussion if we refer to the factor in brackets as a 'normalized Green's function'

$$|G_N(x/x,y|\omega)| = (4\pi x)|G(\mathbf{x},\mathbf{y}|\omega)| \qquad (56)$$

so that (54) may be written

$$\frac{\partial^3\Phi(x|\omega)}{\partial y^3} = |G_N(x/x,y|\omega)|^2\left[\frac{\partial^3\Phi(x|\omega)}{\partial y^3}\right]_{VM} \qquad (57)$$

with $|G_N|$ dependent solely on the *direction* of **x** by virtue of the $4\pi x$ normalization and the $1/x$ decay of $|G|$ in the far field.

Normalized Green's Function, $|G_N|$: Single Choice for Entire Jet

The 'normalized Green's function' designated in Eq. (56) is nominally a function of source location. In this section we develop evidence that the value for a single choice of source position **y** on the axis, designated y_0, may serve as an effective average.

This was concluded in the context of a series of experiments[32,33] on the far-field directivity pattern of a 'point source' of sound immersed in a subsonic jet. Except for some uncertainty as to the accuracy of simulation of a point source, *the measurements effectively yielded values of $|G|^2$* normalized by the value at 90°. This result was deemed equivalent to $|G_N|^2$, on the ground that the effect of wave convection was expected to be nil at 90°; that is, $|G|$ should reduce to $(1/4\pi x)$ there. It was found that the geometric average of the ϑ–dependence for symmetric off-axis positions $+\vartheta$ or $-\vartheta$ differed little from that of the on-axis source position for a given y_1. Further, the variation with y_1 was small. This justifies referring $|G_N|^2$ to a single location, which will greatly simplify both utility and interpretation.

Noting that the direction **x**/x may be designated ϑ,φ in polar coordinates for a general, non-round, jet, the effective average of the squared 'normalized Green's function' may be defined as $|G_N|^2$; thus

$$|G_N(\vartheta,\varphi,y_{ref}|\omega)|^2 = \langle(4\pi x)^2|G(\mathbf{x},\mathbf{y}|\omega)|^2\rangle_{ave} \qquad (58)$$

where y_{ref} is a representative value of y_1 along the jet axis. (The dependence on φ, of course, disappears for a round jet.) According to the arguments above, this refers to an average over **y**. In practice, however, it would be used as a surrogate for a weighted average; that is, equivalent to a single average value of $|G(\mathbf{x},\mathbf{y}|\omega)|^2$ taken outside the integral of Eq. (45) to replace the value inside that varies with **y**. The use of a surrogate single Green's function has also been the practice in solutions of the Lilley equation[12-15]; Mani[13] referred to the same experiments cited above as justification. Replacement of the **y**-dependent Green's function of Eq. (57) by the single **y**-independent Green's function of Eq. (58) allows immediate integration of (56) into

$$\Phi(\mathbf{x}|\omega) = |G_N(\vartheta,\varphi,y_0|\omega)|^2\,[\Phi(\mathbf{x}|\omega)]_{VM} \qquad (59)$$

This states that *the same frequency-dependent $|G_N|^2$ that applies to unit volume is, to a sufficient approximation, applicable to the jet as a whole.*

DIRECTIVITY OF JET NOISE

$|G_N|^2$ Defines Flow-Acoustic Interaction.

The form of Eq. (59) tells us that the flow-acoustic interaction effects, refraction and shielding, are embodied *in the normalized Green's function $|G_N|^2$.* This factor modifies the smooth directional pattern of intensity otherwise predicted ($|G_N|^2$ taken as unity). Both experimental (injected 'point' source)[32,33] and numerical (computational aeroacoustics)[8,9,34] attempts to evaluate $|G_N|^2$ display this. The most striking effect is a progressive reduction of intensity within a 'cone of silence' or 'refraction valley' opening downstream along the jet axis: this is owing to the sound having been refracted outward by the jet velocity gradients. *For filtered jet noise there is a similar 'cone of silence' that is matched by the experimental $|G_N|^2$ impressively well*[28,29]*, at each frequency* **(Fig. 2).**

For the computed $|G_N|^2$ there is a qualitative match[8,9], with fairly good agreement at low frequencies and Mach numbers. However, the computed depth in decibels of the 'refraction valley' is much exaggerated at high frequencies and high subsonic Mach numbers. We would attribute this overprediction to the use of the mean flow only in the convective terms of the wave operator (the turbulence being averaged out): the neglected large scale, low frequency, velocity field distortions would spoil the assumed axisymmetry during the transit time of a wave. It is intuitively evident that perfect axisymmetry is required to yield a very high refractive attenuation along the axis (e.g., 54 decibels in Schubert's[8,9] most extreme case.)

The frequencies for which *geometric acoustics* is applicable are many-fold higher[8,9] than those of jet noise. It was found[8,34] that *at these high frequencies the computed valley depth is grossly exaggerated* (e.g., 90 dB prediction at M=0.3). Despite this, many studies (e.g., references[7, 17]) attempt to quantify the 'cone of silence' via geometric acoustics.

For heated or cooled jets, or jets of foreign gases, sound-speed gradients come into play[8,9,32-34]. Heating enhances the outward refraction, hence increases the depth of the refraction valley. Cooling has an opposite effect. If cold enough, the temperature gradients could dictate inward refraction strong enough to overpower the outward refraction imposed by the velocity gradients. This would give rise to some 'focusing' enhancement of noise intensity along the jet axis: *$|G_N|^2$ should exhibit an axial lobe in place of a 'cone of silence'.* This expectation was dramatically confirmed in the experiments of Grande[29,33]: the enhancement lobe was 9 dB at 3000 Hz for an M=.112 jet of nitrogen at -180 C. *An almost identical lobe was found in his measurements of the* *jet noise* in a narrow filter band at the same frequency **(Fig.2)**. Schubert's approximate numerical calculations of $|G_N|^2$ showed a similar, albeit exaggerated, lobe[8,9].

This similarity of behavior between jet noise and experimental and computational approximations to $|G_N|^2$, for both ambient and cold jets, shows that $|G_N|^2$ dominates the intensity pattern in a conical downstream zone about the jet axis; elsewhere it is near unity. These facts, together with Eqs. (57) and (59), permit the following interpretation: $|G_N|$, the normalized Green's function, **serves to extend the Lighthill-based calculations into the refractive zone near the jet axis:** the 'cone of silence', or focused lobe, as the case may be. Approximation of $|G_N|^2$ by unity outside this zone yields the Lighthill-based pattern: this seems an adequate approximation except when there are shrouding jets.

The calculations **need not be formulated in terms of the four-dimensional Fourier transform** of Eq. (54), despite its figuring in the derivation: other formulations deemed to be equivalent may be used. In practice, relatively crude approximations--implicitly for the correlation R_{km}--have been used. Some approximate developments of this kind, normally resulting from an analysis in the space-time domain, are discussed in a later section and in Appendix D.

Convective Amplification and Basic Directivity

Lighthill[1,2] showed that convection of the sources (but not the sound waves) gives rise to amplification of the noise in the downstream direction. To quantify this, Chu[30] modeled R_{km} as a Gaussian function in the convected frame and carried out the four dimensional Fourier transform to evaluate Eq. (52) or the equivalent Eq. (54). The resulting power spectral density per unit volume was then integrated over frequency to yield the broad band radiated intensity per unit volume. This displayed the $M_c \rightarrow 0$ intensity as multiplied by a *convective amplification* factor, C^{-5}, given by

$$C^{-5} \equiv [(1 - M_c \cos\vartheta)^2 + \alpha^2 M_c^2]^{-5/2} \qquad (60)$$

in terms of the source pattern convection Mach number M_c. This factor was originally obtained by Ffowcs Williams[38] and by Ribner[3] via different formalisms; it is a refinement of the Lighthill-based relation[39] $\mathbf{(1 - M_c \cos\vartheta)^{-5}}$ that is nonsingular when $M_c \cos\vartheta = 1$. (The basis for this was elaborated under Eq. (52).) The directional pattern of intensity radiated by the sources (the **Basic Directivity**), before multiplication by this factor, was tacitly taken to be spherical.

Ribner[23] made allowance for the **two** source terms of Eq.(11), leading to a *nonspherical* basic directional pattern; he chose a more defensible (but still somewhat unrealistic) form for R_{km}: it was modelled as compatible with isotropic turbulence superposed on a transversely sheared mean flow. This could be expressed as a 'narrow band' directional spectrum

$$C^{-5} * \{ C[a(C\omega) + b(C\omega)(\cos^4\vartheta + \cos^2\vartheta)/2] \}$$

$$= \mathbf{C^{-5} * Basic\ Directivity}\ \text{(quasi-elliptic)} \qquad (61)$$

emitted from unit volume; the two terms $a(C\omega)$ and $b(C\omega)$ correspond respectively to the two source terms (*'self noise'* and *'shear noise'*) on the right hand side of Eq.(11). $C\omega$ is an 'effective' Doppler shifted source frequency[3,4], and the outer factor C disappears in the integral over ω (cf. also remarks under Eq. (52)).

The format of Eq.(61) was effectively confirmed in the work of Pao and Lowson[24]. Their approach was, in fact, superior to Ribner's in that it yielded the convective amplification factor C^{-5} in an internally consistent fashion. Ribner, on the other hand, used the 'compact eddy' assumption in a compatible $M \to 0$ approximation to derive the Basic Directivity. Then he generalized this to finite M by assuming that the broadband factor C^{-5} applied as a multiplier, to yield Eq.(61).

The shear noise, given here by the $b(C\omega)$ term, has a dipole-like directivity. Thus it has been argued that the convective amplifier should be the C^{-3} of a dipole, rather than the C^{-5} of a quadrupole. Pao and Lowson[24] obtain a C^{-5} multiplier, but they attribute the change from -3 to -5 to the choice of turbulence model. "The spectrum in the low wave number region follows a k^2 law, which artificially raises the power dependence from [-3 to -5]. In actual measurements of jet turbulence, the spectrum in this region is usually flat." We observe, however, that the k^2 law refers to a 3D spectrum, whereas the hot wire measures only a 1D spectrum, and the flat 1D spectrum at small k is fully compatible with a k^2 3D spectrum. Thus there is no artificiality here, and the convective amplifier C^{-5} would appear to be applicable to the shear noise as well as to the self noise. Comparisons with experiment on this basis seem to support that conclusion.

To our approximation, the magnitudes of C^{-5} and $|G_N|^2$ apply equally to unit volume and to the jet as a whole. This is not true for the $a(C\omega)$ and $b(C\omega)$ of Eq. (61): the magnitudes differ for the two applications, but the ratio b/a is unaltered. An important prediction of the theory[4,23] that connects these two spectra is

$$b(C\omega) = \beta\, a(2C\omega) \qquad (62)$$

where β is a constant of order 2. The implications of this are elaborated below.

The 'basic directivity' of Eq.(61) is compounded of $b(C\omega)$ and $a(C\omega)$ in different proportions, depending on direction; credibility is afforded by comparisons with experiment. First, the predicted overall directional pattern over a range M=0.37 to M=0.9 (outside the 'cone of silence') is about right[28,29]: this is a minor point, since the contribution of the basic directivity is weak compared with that of the convective amplification. Second, and much more significant, *the prediction in Eq.(62) of separate spectra of "shear noise' $b(C\omega)$ and 'self noise' $a(2C\omega)$ of almost identical shape, but with one octave relative shift, was strikingly confirmed*[29,40]**(Fig. 1).** These are extracted from measurements at angles of 45 and 90 degrees from the jet axis, without adjustable constants. When normalized to unity peak, curves of shear noise $b(C\omega)$ and self noise $a(2C\omega)$ virtually collapse on one another.

The third point has to do with correlations of *two* microphones located on a large sphere centered on the jet nozzle: with one microphone fixed, the other was displaced either along a meridian or a circle of latitude. It was with a series of such measurements carried out by Maestrello[41] that the theory was compared. In particular, correlations of two microphones are sensitive to details of the source *instantaneous* directivity, whereas the single microphone mean square response is not. The Lighthill-Ribner theory[23] leading to Eq.(61) was extended to deal with this case[42,43]. It was found that prediction of two-microphone cross-correlations along circles of latitude showed good qualitative agreement over a range of angular separations, and for different latitudes. This was true both in broadband[42] and the more demanding narrow band[43]; see also references 51 and 52. (Microphones located *along a meridian* exhibit a cusp-like correlation in broadband, decaying sharply with separation[41]. *The agreement of the theory here was particularly striking*[42]; although only weakly relevant to Eq. (61), it makes a strong case for the credibility of key aspects of the basic theory[23].)

Basic Directivity x Convective Amplification x $|G_N|^2$

The product of the weak *basic directivity* and the strong *convective amplification,* C^{-5}, Eq.(61), yields the directivity of the jet noise of frequency ω on the Lighthill-Ribner model. Multiplication by the normalized Green's function, $|G_N|^2$, modifies this to allow for the flow-acoustic interaction (refraction[3-5] and shielding[7,11,12,44]). $|G_N|^2$ dominates in a cusp-like fashion near the axis to produce the refractive downstream 'cone of silence'. The broad fan-shaped amplification, C^{-5}, with its maximum downstream as well, amplifies not far from uniformly across the 'valley', scarcely modifying its shape. This accounts for the relatively close match between the measured $|G_N|^2$ and the jet noise pattern near the axis **(Fig.1)**. *The resultant of the three factors is the well known heart-shaped directional pattern of jet noise.* The opposing effects of C^{-5} and $|G_N|^2$, in their relative strengths, determine the location and magnitude of the maxima. The progression of patterns from **Basic Directivity** to **Basic x C^{-5}** to **Basic x C^{-5} x $|G_N|^2$** is schematically shown in **Figure 3**; (the factors are additive on the decibel scale of the figure).

Directivity vs Spectrum

We have been discussing the directivity of jet noise in frequency bands; that is, the variation of $\Phi(\mathbf{x}|\omega)$ with ϑ, the angle of the observer vector **x** with the jet axis. The directivity of the mean square sound pressure $\overline{p^2}(\mathbf{x})$, being an integral over ω, is a weighted average of these. The spectra at fixed angles ϑ are, of course, cross-plots.

The *directivity* of the emission from a typical unit volume ($\partial^3\Phi(\mathbf{x}|\omega)/\partial y^3$ and its integral over ω, $\partial^3\overline{p^2}/\partial y^3$) and the directivity of the entire jet, discussed above, are much the same (normalized to values at $\vartheta = 90$ degrees). But the *spectral shape* emitted from unit volume differs greatly from that of the entire jet. The former is estimated as less than 2 octaves wide, the latter is measured[54] as some 5 octaves, taken between the 6 dB down points. Thus calculations for a single unit volume may serve for the directivity of the entire jet, but an integration over the jet is required for estimating the spectrum. The directivity is only slightly sensitive to the asumed turbulence model (it affects the basic directivity); the spectrum requires a detailed estimate of turbulence properties throughout the jet. These refer to *shapes* : prediction of absolute levels requires a higher level of accuracy in the turbulence data and its modelling, and in any simplifying assumptions.

APPROXIMATE QUANTITATIVE PREDICTION

We turn now to the approximate quantitative prediction of jet noise properties. The central element is the four dimensional Fourier transform (4DFT) of the source term correlation function R_{km} (e.g., Eqs. (3) and (53)). For illustration we limit attention to the self noise term, the shear term involving rather more complication. Neither theory nor measurements (so far) adequately describe the correlation function over space and time. If it is dealt with at all (in place of scaling arguments), a simplistic form (e.g., a Gaussian) is often assumed to ease the mathematics. It follows that the space transform (3DFT: $\xi \rightarrow \mathbf{k}$) is of dubious accuracy, and the mathematical difficulty it entails seems thereby unwarranted. Thus the 3DFT is normally bypassed (an exception is ref. 24) by means of the 'compact eddy' assumption: this reduces the transform to just a volume integral (over ξ), as noted earlier.

The 4DFT is then a volume integral times a 1DFT ($\tau \rightarrow \omega$). Moreover, the 1DFT defines a spectrum between one and two octaves wide. For simplicity this is commonly approximated as a single line: a δ-function of local frequency ω_{se} and amplitude A. Combining these yields the *rather drastic approximation*

$$\text{4DFT (correlation function)} \approx A((\overline{u_k^2})^2 L_1 L_2 L_3 \delta(\omega-\omega_{se})) \qquad (62)$$

where the volume integral (of the normalized correlation function) is by definition the product $L_1L_2L_3$ of the three orthogonal scales of turbulence; normalization is by the peak correlation $(\overline{u_k^2})^2$), and u_k refers to the component of turbulence in the **k**-direction (aligned with the **x**-direction). All these parameters are functions of **y**. Insertion in Eq.(54) yields the power spectral density emitted from unit volume at **y** as

$$\left[\frac{\partial^3\Phi(\mathbf{x}|\omega)}{\partial \mathbf{y}^3}\right]_{VM,\,SE} \approx A\frac{\rho_o^2\omega^4(\overline{u_k^2})^2}{C^5 c_o^4(4\pi x)^2} L_1L_2L_3\delta(\omega-\omega_{se}) \qquad (63)$$

as applied to the *self noise*. This refers to a virtual medium at rest. The format of Eq.(53) generalizes this to a jet flow, encompassing refractive and shielding effects, via the factor $|G_N|^2$ (using Eq.(56)):

$$\left[\frac{\partial^3\Phi(\mathbf{x}|\omega)}{\partial \mathbf{y}^3}\right]_{SE} \approx A|G_N|^2\frac{\rho_o^2\omega^4(\overline{u_k^2})^2}{C^5 c_o^4(4\pi x)^2} L_1L_2L_3\delta(\omega-\omega_{se}) \qquad (64)$$

Mathematically, the appropriate choices for the amplitude A and local frequency ω_{se} are these: (i) A should be the integral of the 1DFT spectrum (cf. above Eq.(62)) over ω, and (ii) ω_{se} should be the centroid or first moment of that spectrum. Then the δ-function approximation will yield minimum error in computing the overall power spectral density $\Phi(\mathbf{x}|\omega)$ by integrating Eqs. (63) or (64) over **y**. In practice, heuristic scaling laws are often used for evaluating both A and ω_{se}. These and the other parameters must be estimated as **y** ranges throughout the jet. A representative example[27] of such an estimation, leading to rather good spectral prediction, is given in Appendix D.

CONCLUDING REMARKS

Central Result

The central result of the paper may be restated in simplified terms. Lighthill posed his aerodynamic sound sources as radiating into a 'virtual medium at rest.' Refraction of sound (creating the axial 'cone of silence') was suppressed by approximating the density in the dominant source term as constant ($\rho = \rho_o$). But by deferring the step $\rho = \rho_o$, we can pose the radiation as being emitted *into the actual jet flow*. This brings the refractive effect of the flow into play. Moreover, the *residual sound source term is the same.*

Mathematically, the only change is replacement of the solution for a pure tone point source in a medium at rest by the solution for the source in a jet flow. The former can be written down by inspection as $e^{ikr}/4\pi r$; the latter is a complicated solution, $G(\mathbf{x},\mathbf{y}|\omega)$, of a convected wave equation. But, at frequencies characteristic of jet noise, we find from both experiment[32,33] and calculation[8,9] that G reduces in the far field to $e^{ikr}/4\pi r$ (with a phase shift) times a directional factor. That directional factor, for a single round jet, is near unity for angle ϑ greater than some value ϑ_M. For smaller angles it decreases sharply to a minum on the jet axis, $\vartheta = 0$. This describes the 'cone of silence' **(Figs. 2,3).** In summary, in the far field the new G differs in amplitude from the Lighthill $e^{ikr}/4\pi r$ significantly only within the 'cone of silence'. *Use of G thus serves to extend the Lighthill-based solutions into this refractive zone. But outside it may be dispensed with, with little error.*

Relationship to Other Approaches

A variety of Lighthill-based solutions — formalisms for jet noise prediction — have been used, e.g., Refs. 23-27. They were all approximations. As discussed, they usually involved simplistic replacements for the four dimensional Fourier transform formalism. The turbulence correlation function, if it was modeled at all (rather than bypassed by heuristic assumptions), was normally taken as separable in space and time (implicit in Eqs.(62)-(64)). Chu[30] improved the model, removing the spurious separability: he used data from his own comprehensive program of very careful measurements by hot wire. He did, however, avoid the 4DFT by invoking the 'small eddy' assumption. Neverthless, despite these deemed improvements, his predictive accuracy fell far short of the best of refs. 23-27. Hindsight suggests the capabilities of his data may not have been optimally exploited. A revisit in the light of the present formalism could be profitable.

The predictive problem is compounded by the difficulty of a four-dimensional Fourier transform. As noted, the 'compact eddy' approximation reduces this to a simple volume integral. But that step compromises the accuracy of the prediction of convective amplification. The approach has, indeed, led to fairly accurate predictions, both for round and more complex jet configuration. However, even with this approximation, they could be improved as indicated under Eq.(64). The further inclusion of the factor $|G_N|^2$ will extend the solution into the 'cone of silence' (small θ region), and even improve the accuracy outside this region (by the amount $|G_N|^2$ differs from unity).

Lilley's wave equation, in the hands of Balsa,[15] leads to a similar formula $|G_B|^2$ x four-dimensional Fourier transform of source correlation function. Here again, in practice the Fourier transform is bypassed by an approximation. There are other important differences . A major component, the shear term included in Eq.(11), is missing from the source term: the consequences are discussed in the text. And the squared normalized Green's function, $|G_B|^2$, refers to a *moving* point source. The effects of source convection and wave convection — respectively governing amplification and refraction (via velocity gradients) — are thereby combined. In contradistinction, these effects are decoupled herein by the use of the value of $|G_N|^2$ for a *stationary* source That is, $|G_B|^2$ plays essentially the same role as $C^{-5}|G_N|^2$: they should be largely equivalent.

It is this decoupling that allows the simplicity of the Lighthill-based formalisms to be applied outside the 'cone of silence', since $|G_N|^2$ is near unity there. Another advantage is that this refractive $|G_N|^2$ is determined for a realistic spreading jet — either by calculation or experimentally — whereas $|G_B|^2$ has been evaluated only for idealized, infinite, nonspreading jets.

Issue of Shielding

Mani[11,12] and Balsa[15] have pointed to a 'shielding' role of the mean flow in reducing the convective amplification (a function of direction) at high frequencies. But comparisons[28,29] with the directivity measured by several investigators do not bear this out. The simple theoretical convective factor C^{-5} adequately predicts the curves up to M=0.9, source Strouhal no. 1.0 (observed Strouhal no. 1.33). This is a reduction in convective amplification compared with the classical factor $(1-M_c\cos\theta)^{-5}$, but *it is not an effect of flow shielding*. Instead, it results from retarded time variation across an 'eddy', not allowed for in the classical factor.

It would seem that the frequencies of jet noise are simply too low: for significant shielding the flow dimensions must be much larger than a typical wave length of the sound[55,56]. This is a requirement for geometric acoustics (ray acoustics) to apply. Schubert's calculations[8,9] show that jet noise is very far from that regime.

These remarks refer to a single round jet. The case of multitube jets (or equivalent corrugated nozzle jets) is another matter. Substantial shielding of the high frequency noise of the inner jets by a ring of the outermost jets is a demonstrated fact. Balsa[44], via a *moving source* Green's function, shows apparent agreement with measurements that he cites. No *stationary source* Green's function, as proposed herein, has as yet been evaluated for this scenario.

Range of Applicability

The results herein are for the *far field* only. Moreover, they are presumed to beapplicable primarily for *subsonic* jets. For supersonic jets additional noise sources come into play. Tam[60], in his review article, develops the case for instability waves, identified as 'large scale coherent structures', being a major source of noise. He evaluates the noise directly via a 'stochastic wave model' with very impressive agreement with experiment. But we note that these instability waves, to the extent that they coexist with the random turbulence, will contribute to the correlation function R_k used herein. But evaluation is another matter: the similarity laws for jet turbulence, which have been used with success for subsonic jets, would have to be reevaluated for the supersonic regime.

Perhaps more importantly, there will be a pattern of shock waves if the jet does not issue at the design speed from a properly contoured convergent-divergent nozzle. It was shown many years ago that shock-turbulence interaction would generate intense noise[57-59]. In recent years Tam (summarized in ref. 60) has attempted quantitative prediction of this shock-associated noise; he analyzed the interaction between instability waves and a 'wave guide' model for the shock structure. His near field patterns show a close match to measurements.

The results are further restricted to jets issuing into ambient fluid at rest: that is, static test conditions. The effects of forward flight on the jet noise are not considered. Michalke and Michel[29,49,50] have extended the Lighthill theory to provide a successful prediction of these effects. This takes the form of a scaling law that maps the intensity of a static jet at certain jet Mach number and direction into that for a moving jet at an altered Mach number, direction, and distance. (See also an approach via CFD methods[61].) Refraction, governing extension into the 'cone of silence', is not allowed for: this would involve a further development of the present stationary-source Green's function approach.

APPENDIX A: GENERALIZED CONVECTIVE WAVE MODIFICATIONS OF LIGHTHILL EQUATION

EXACT WAVE EQUATION

Expansion of Lighthill Source Term, Q_L

The restriction to a transversely sheared mean flow of uniform density, Eq. (8), is relaxed here: Lighthill's source term expression, which we shall call Q_L (the right-hand side of Eq. (5)), is expanded under the specifications

$$v_i = U_i + u_i; \qquad U_i = U_i(\mathbf{x}); \qquad \langle v_i \rangle_{av} = U_i \tag{A1}$$

$$\rho = \bar{\rho}(x_i) + \rho'; \qquad (\rho)_{av} = \bar{\rho}(x_i) \tag{A2}$$

By Csanady's[7] Eq. (3), Q_L expands as follows:

$$Q_L \equiv \frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j} + \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 \rho}{\partial t^2} = \rho\left(\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) + \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2}$$
$$- \frac{\partial^2 \rho}{\partial t^2} - 2v_i\frac{\partial^2 \rho}{\partial x_i \partial t} - v_i v_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} \tag{A3}$$

With this expansion it is easily shown that Eq. (5) of the main text is equivalent to Schubert's[8,9] *exact wave equation for an inviscid nonheatconducting fluid.* Inserting Eqs. (A1) into (A3) yields[7]

$$Q_L = \rho\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right) - \frac{\partial^2 \rho}{\partial t^2} - 2U_i\frac{\partial^2 \rho}{\partial x_i \partial t} - U_i U_j \frac{\partial^2 \rho}{\partial x_i \partial x_j}$$

$$+ \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} + \frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U_i}{\partial x_j}\frac{\partial \rho u_j}{\partial x_i} + 2\frac{\partial}{\partial x_j}\left(\rho u_j \frac{\partial U_i}{\partial x_i}\right) \tag{A4}$$

When $U_i = (U(x_2), 0, 0)$ this reduces to the right-hand side of Eq. (8) of the main text. With the further expansion

$$\frac{\partial^2 \rho u_i u_j}{\partial x_i \partial x_j} + 2\frac{\partial U_i}{\partial x_j}\frac{\partial \rho u_j}{\partial x_i} = \rho\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\rho\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + 2u_j\frac{\partial U_i}{\partial x_j}\frac{\partial \rho}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial \rho}{\partial x_i} + u_i u_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} \tag{A5}$$

and the definition

$$\frac{\hat{D}^2}{Dt^2} \equiv \frac{\partial^2}{\partial t^2} + 2U_i\frac{\partial^2}{\partial x_i \partial t} + U_i U_j \frac{\partial^2}{\partial x_i \partial x_j} \tag{A6}$$

and some rearrangement, Q_L becomes

$$Q_L = 2\rho \frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\hat{D}^2 p}{Dt^2} + \rho \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2u_j \frac{\partial U_i}{\partial x_j}\frac{\partial \rho}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial \rho}{\partial x_i}$$

$$+ u_i u_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2\frac{\partial}{\partial x_j}\left(\rho u_j \frac{\partial U_i}{\partial x_i}\right) + \rho\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right) \qquad \text{(A7)}$$

This expansion of the acoustic source terms, the right-hand side of the wave equation (5), is exact.

APPROXIMATE WAVE EQUATIONS

Incompressible Turbulence

We now introduce approximations in two stages. First, in all but the first three terms of Q_L, we neglect the compressibility of the turbulence in the application to subsonic flows. We argue that density perturbations ρ' in the nonexcepted terms account for scattering of sound by turbulence, and they may be neglected in dealing with generation of sound. Thus, *in these terms,* ρ *may be replaced by its local temporal mean* $\bar{\rho}(\mathbf{x})$. Consistently, it is implied that u_i, in all but the excepted terms, contains no compressible component. Despite this assumed incompressibility of the source terms, sound (pressure waves) will indeed be generated as Lighthill[1,2] showed. Thus compressibility has been retained where acoustically necessary: in the final left-hand-side wave operator.

The first excepted term is $2\rho(\partial U_i/\partial x_j)(\partial u_j/\partial x_i)$. By the argument in the main text above Eq. (11), the very small compressible, or *acoustic* part of this term, being *linear in* u_j, participates to the first order in wave propagation; thus we move it to the left-hand-side wave operator. The terms $c_0^{-2}\,\partial^2 p/\partial t^2$ and $-\hat{D}^2 p/Dt^2$ have also to do with wave propagation; we move them likewise to the left-hand side, approximating $\hat{D}^2 p/Dt^2$ by $\bar{c}^{-2}\hat{D}^2 p/Dt^2$ (Appendix B).

With these term shifts from the right-hand side (Q_L) and approximation of ρ by $\bar{\rho}$ therein, Eq. (5) may be rewritten. First we need an approximate equation for mean flow continuity. Taking the correlation $\overline{\rho' u_i}$ as negligible yields this as

$$\partial \bar{\rho} U_j / \partial x_j = 0 \qquad \text{(A8)}$$

By virtue of Eq. (A8), two terms of the approximate Q_L may be collapsed into one:

$$2\frac{\partial}{\partial x_j}\left(\bar{\rho} u_j \frac{\partial U_i}{\partial x_i}\right) + 2u_j \frac{\partial U_j}{\partial x_j}\frac{\partial \bar{\rho}}{\partial x_i} = -2U_i \frac{\partial}{\partial x_j}(u_j \frac{\partial \bar{\rho}}{\partial x_i}) \qquad \text{(A9)}$$

The modified Eq. (5) then reads

$$\frac{1}{\bar{c}^2}[\frac{\partial^2 p}{\partial t^2} + 2U_i\frac{\partial^2 p}{\partial x_i \partial t} + U_iU_j\frac{\partial^2 p}{\partial x_i \partial x_j}] - 2\bar{\rho}\frac{\partial U_i}{\partial x_j}\frac{\partial u_{jac}}{\partial x_i} - \nabla^2 p = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\bar{\rho}\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + 2\frac{\partial u_i u_j}{\partial x_j}\frac{\partial \bar{\rho}}{\partial x_i}$$

$$+ u_i u_j \frac{\partial^2 \bar{\rho}}{\partial x_i \partial x_j} - 2U_i \frac{\partial}{\partial x_j}(u_j \frac{\partial \bar{\rho}}{\partial x_i}) + \bar{\rho}\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_i}{\partial x_j} + \frac{\partial U_i}{\partial x_i}\frac{\partial U_j}{\partial x_j}\right) \qquad \text{(A10)}$$

Transversely Sheared Flow

Let us specialize now to a transversely sheared flow (Eq. (7) of the main text), with transverse density gradient as well:

$$U_i = (U(x_2), 0, 0); \qquad \bar{\rho} = \bar{\rho}(x_2) \qquad \text{(A11)}$$

The modification (A10) of Eq. (5) simplifies to

$$\frac{1}{\bar{c}^2}[\frac{\partial^2 p}{\partial t^2} + 2U\frac{\partial^2 p}{\partial x_1 \partial t} + U^2\frac{\partial^2 p}{\partial x_1^2}] - 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1}$$

$$+2\frac{\partial u_2 u_j}{\partial x_j}\frac{\partial \bar{\rho}}{\partial x_2} + u_2^2\frac{\partial^2 \bar{\rho}}{\partial x_2^2} \qquad \text{(A12)}$$

But it can be quickly verified by direct expansion that

$$\bar{\rho}\frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} = \bar{\rho}\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} + 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_2}{\partial x_1} \qquad \text{(A13)}$$

so that an alternative form is

$$\frac{1}{\bar{c}^2}[\frac{\partial^2 p}{\partial t^2} + 2U\frac{\partial^2 p}{\partial x_1 \partial t} + U^2\frac{\partial^2 p}{\partial x_1^2}] - 2\bar{\rho}\frac{\partial U}{\partial x_2}\frac{\partial u_{2ac}}{\partial x_1} - \nabla^2 p = \bar{\rho}\frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} + 2\frac{\partial u_2 u_j}{\partial x_j}\frac{\partial \bar{\rho}}{\partial x_2} + u_2^2\frac{\partial^2 \bar{\rho}}{\partial x_2^2}$$

(A14)

where it is recalled that v_i is the instantaneous resultant flow $U_i + u_i$, as defined in Eq. (A1). An equation of the form (A14) results also for a cylindrical jet $U_i = (U(r), 0,0)$, $\bar{\rho} = \bar{\rho}(r)$, if u_2, x_2 are replaced by u_r, r, respectively (see Eq.(A15). *Equations (A12) and (A14) are the key results of this Appendix; they generalize Eqs.(11) and (13) of the main text, respectively, to the case of flows of nonuniform mean density.* The applications and implications are discussed below.

Jet Flow

Following Schubert[8,9] we note that U_i in a jet is essentially unidirectional:

$$U_i = [U(r;x_1), 0, 0]; \qquad r = \sqrt{x_2^2 + x_3^2} \qquad \text{round jet} \qquad \text{(A15)}$$
$$= x_2 \qquad \text{two dimensional jet}$$

with a strong dependence on r, and a weak dependence on x_1. Further,

$$\bar{\rho} = \bar{\rho}\,(r;\, x_1) \qquad \text{(A16)}$$

with a similar dependence. Thus, for both U and $\bar{\rho}$, the gradients along x_1 are very much less than those along r. For the foregoing transversely sheared flow the x_1-gradients are identically zero; requiring this led to Eqs. (A12) to (A14). It follows that Eqs. (A12) to (A14) may be applied to jet flow as a close approximation, with x_2 replaced by r.

Density Scenarios: $\bar{\rho}$ = constant vs. $\bar{\rho} = \bar{\rho}$ (r)

Let us specialize further to a uniform mean density,

$$\bar{\rho} = \rho_o = \text{constant} \qquad \text{(A17)}$$

This, together with Eq. (A11), recovers the scenario of the main text. It is seen that the density gradient terms drop out, and Eqs. (A12) and (A14) reduce to Eqs. (11) and (13), respectively: *the modified Lighthill equation in the form of Eq.(13) is confirmed as a special case of the more general form of this Appendix, Eq.(A14).*

From the foregoing, it is clear that density gradients, via the additional source terms, cause more noise to be generated. This has been explored in the context of hot jets by Morfey[35], Mani[13], and by Michalke and Michel[49,50]. The source terms in (A12) and (A14) appear similar to those deduced by Mani. The term $2\,\frac{\partial u_2 u_j}{\partial x_j}\,\frac{\partial \bar{\rho}}{\partial x_2}$ is essentially of dipole form, $\partial Q_j/\partial x_j$, (treating $\partial\bar{\rho}/\partial x_2$ as a spatial constant): it would yield a factor $|\kappa|^2$ in place of $|\kappa|^4$ in an equation like (4). As a consequence of this, or by arguments given in the cited references, the corresponding radiated sound power would vary as U^6; they showed it could exceed the ordinary quadrupole-source jet noise, with its U^8 law, for sufficiently hot jets. The term $u_2^2\, \partial^2\bar{\rho}/\partial x_2^2$ is of monopole form, leading to a U^4 law ($|\kappa|^4$ factor replaced by unity). This would radiate very weakly, the curvature $\partial^2\bar{\rho}/\partial x_2^2$ being minimal in the zone of strongest turbulence, where the mean flow shear and $\partial\bar{\rho}/\partial x_2$ both maximize.

Over time "...dozens of equivalent (and nonequivalent) source term expansions have been published by flow noise researchers. *This multiplicity of competing source terms has* *been a major contributor to confusion...*" (Reference 29). In this author's view, the effective limit in the law of diminishing returns has been reached in the expansions of this Appendix: of the Lighthill wave operator (left hand side of Eq.(A10)) and of the Lighthill source term for an inviscid nonheatconducting fluid (exact, Eq.(A7); approximate, Eq.(A14)).

APPENDIX B: APPROXIMATION OF $\hat{D}^2\rho/Dt^2$ as $\bar{c}^{-2}\hat{D}^2p/Dt^2$

The exact relation for a moving fluid element of constant entropy

$$\frac{D\rho}{Dt} = \frac{1}{c^2}\frac{Dp}{Dt} \tag{B1}$$

implies

$$\frac{D^2\rho}{Dt^2} = \frac{1}{c^2}\frac{D^2p}{Dt^2} - \frac{\gamma-1}{\rho c^4}\left(\frac{Dp}{Dt}\right)^2 \tag{B2}$$

where D^2/Dt^2 is the convective second derivative

$$\frac{D^2}{Dt^2} = \frac{\partial^2}{\partial t^2} + 2v_i\frac{\partial^2}{\partial x_i \partial t} + v_i v_j \frac{\partial^2}{\partial x_i \partial x_j} + \frac{Dv_i}{Dt}\frac{\partial}{\partial x_i} \tag{B3}$$

in which

$$v_i = U_i + u_i \tag{B4}$$

is the mean velocity U_i plus a perturbation u_i. This may be compared with a *mean flow* convective derivative

$$\frac{\hat{D}^2}{Dt^2} = \frac{\partial^2}{\partial t^2} + 2U_i\frac{\partial^2}{\partial x_i \partial t} + U_i U_j \frac{\partial^2}{\partial x_i \partial x_j} \tag{B5}$$

It is argued, following Schubert[8,9], that replacement of (B3) by (B5) (and c^2 by $\bar{c}^2$) merely supresses scattering of sound by the jet turbulence; $(Dv_i/Dt)\partial/\partial x_i$ is neglected also, as yielding second order terms.

We illustrate this in terms of a plane sound wave in a reference frame following the mean flow:

$$p = \bar{p} + |p'|\, e^{i(k_i x_i - \omega t)}; \qquad \sqrt{k_1^2 + k_2^2 + k_3^2} = k = \omega/\bar{c} \tag{B6}$$

$$\rho = \bar{\rho} + |\rho'|\, e^{i(k_i x_i - \omega t)}$$

(Herein the contribution of turbulent perturbations to $\overline{c}\,^2$, $\bar{c}^2$, and $1/\overline{c}\,^{-2}$ are considered negligible, so the designated local time averages are taken to be equivalent. With this assumption , the averages involving soundspeed c in Eq.(B6) and later equations are compatible.) Using the momentum equation,

$$\frac{1}{\bar{c}^2}\frac{Dv_i}{Dt}\frac{\partial p}{\partial x_i} = -\frac{1}{\rho\bar{c}^2}\left(\frac{\partial p}{\partial x_i}\right)^2 \tag{B7}$$

And the amplitude is (with $\rho \sim \bar{\rho}$)

$$\left|\frac{1}{\bar{c}^2}\frac{Dv_i}{Dt}\frac{\partial p}{\partial x_i}\right| \approx \frac{k^2}{\bar{\rho}\bar{c}^2}|p'|^2 \approx \frac{k^2}{\gamma\bar{p}}|p'|^2 \tag{B8}$$

For comparison we evaluate $\bar{c}^{-2}\, D^2p/Dt^2$, which is $\bar{c}^{-2}\,\partial^2 p/\partial t^2$ in the moving frame, to a close approximation. By (B6) the amplitude is

$$\left|\frac{1}{\bar{c}^2}\frac{D^2p}{Dt^2}\right| = \frac{\omega^2}{\bar{c}^2}|p'| = k^2\,|p'| \tag{B9}$$

The term (B8), arising from the operator $(Dv_i/Dt)\partial/\partial x_i$, is seen to have an amplitude a factor $|p'|/\gamma\bar{p}$ smaller than the term (B9). This justifies neglect of that operator.

By a similar example we can show that the last term of Eq. (B2) is of higher order and may be dropped. Thus, on the basis of these order-of-magnitude estimates, we conclude that

$$\frac{\hat{D}^2\rho}{Dt^2} \approx \frac{1}{\bar{c}^2}\frac{\hat{D}^2p}{Dt^2} \tag{B10}$$

is a sufficient approximation for the modified Lighthill wave equation.

APPENDIX C: APPROXIMATION OF κ AS $\mathbf{k}$

For frequencies characteristic of jet noise, we examine the change in phase ψ in the *far field* when the source is displaced from $\mathbf{y}$ to $\mathbf{y'}$. Equation (25) gives a linear approximation as

$$[\psi' - \psi] \equiv [\psi(\mathbf{x},\mathbf{y'}|\omega) - \psi(\mathbf{x},\mathbf{y}|\omega)] = -\kappa\bullet(\mathbf{y'}-\mathbf{y}) \tag{C1}$$

where

$$\kappa \equiv -(\nabla_y\,\psi)_{x>>y} \tag{C2}$$

and wave convection by the jet flow is allowed for. In the absence of flow (or for a *virtual medium* at rest) this reduces to

$$\mathbf{k} = -[\nabla_y(kr)]_{x>>y} = [k\mathbf{r}/r]_{x>>y} \approx k\mathbf{x}/x \tag{C3}$$

with corresponding phase change

$$[\psi' - \psi]_{VM} = -\mathbf{k}\bullet(\mathbf{y'}-\mathbf{y}) \tag{C4}$$

This is an approximation to the exact relation

$$[\psi' - \psi]_{VM} = k(|\mathbf{x}-\mathbf{y'}| - |\mathbf{x}-\mathbf{y}|) \tag{C5}$$

The (negative) phase advance due to the flow is the left hand side of Eq. (C1) minus that of Eq. (C4); designating this as S, it is obtained as

$$S \equiv [\psi' - \psi] - [\psi' - \psi]_{VM} = (\mathbf{k}-\boldsymbol{\kappa})\bullet(\mathbf{y'}-\mathbf{y}) \qquad \text{(C6)}$$

Schubert[8], in his numerical computations for a point source in a jet flow, has evaluated this phase difference in a number of cases (he used the more precise relation (C5) in defining the left hand side as

$$S \equiv [\psi' - \psi] - k(|\mathbf{x}-\mathbf{y}'| - |\mathbf{x}-\mathbf{y}|) \qquad \text{(C7)}$$

effectively: for the large x/y ratio of our examples the difference is negligible). In his Figures 30 to 32 he plots S/π vs $(y'_1 - y_1)/D$; this corresponds to **y'** and **y** both directed along the jet axis; **x** is directed at angles $\vartheta = 0$, 6.8, and 16.2 degrees.

We take the simplest case, $\vartheta=0$. Now **k** is directed along **x** by Eq.(C3), and when **x** lies along the axis, so does **k**. By virtue of the flow symmetry about $\vartheta=0$, κ must also be directed along the axis. For this scenario Eq.(C6) becomes

$$S = (k-\kappa)(y'_1-y_1) \qquad \text{(C8)}$$

so that

$$\kappa/k = 1 - S/k(y'_1-y_1) = 1 - \varepsilon \qquad \text{(C9)}$$

The ratio κ/k is evaluated in several examples with data from Figs.30 and 32. The specifications are: $M_j = 0.3$, , $|\mathbf{x}-\mathbf{y}|/D=100$. The results are:

$W_0 \equiv kD=1.055$ and $(y'_1-y_1)/D = 1$: $\kappa/k = 1 - \pi(.005)/[1.055(1)] = 1 - .015$

$(y'_1-y_1)/D = 2$: $\kappa/k = 1 - \pi(.015)/[1.055(2)] = 1 - .022$

$W_0 \equiv kD= 2.46$ and $(y'_1-y_1)/D = 0.5$: $\kappa/k = 1 - \pi(.032)/[2.46(0.5)] = 1 - .08$

$(y'_1-y_1)/D = 1$: $\kappa/k = 1 - \pi(.101)/[2.46(1)] = 1 - .13$

It is seen that κ/k differs little from unity: the difference $|\varepsilon|$ increases with kD (proportional to frequency) and with $(y'_1-y_1)/D$. The values kD = 1.055 and 2.46 correspond to the upper part of the jet noise spectrum, which has a broad peak around kD= 0.3. Therefore, for the bulk of the spectrum the difference $|\varepsilon|$ will be less for a given $(y'_1-y_1)/D$. Furthermore, an appropriate source displacement (y'_1-y_1) must be less than the local macroscale L of the jet turbulence, a function of y_1. Estimates are L/D = .35, .15 for kD = 1.055, 2.46, respectively. Thus the values of $(y'_1-y_1)/D$ in the examples are unacceptably high, especially at the highest kD. It follows that appropriate values of $|\varepsilon|$ are substantially smaller than those of the examples: *κ/k differs from unity by not more than several percent at jet noise frequencies* for $M_j = 0.3$. The difference will, of course, increase with jet Mach number.

On going to $\vartheta > 0$, the curves of S/π vs $(y'_1-y_1)/D$ in Figs. 30 and 32 are not greatly different: the phase difference S between the flow and no flow cases remains very small as ϑ varies. This implies that the phase gradient $(\nabla_y \psi)_{x>>y}$ for the flow and no flow cases has nearly the same direction as well as magnitude. That is, the inference from the $\vartheta = 0$ examples above, with respect to magnitudes, that

$$\kappa \approx k \qquad \text{(C10)}$$

may be generalized to the directions as well; that is,

$$\boldsymbol{\kappa} = \mathbf{k} \qquad \text{(C11)}$$

to a close approximation for at least low speed jets at typical frequencies.

APPENDIX D: MOON-ZELAZNY MODEL OF THE TURBULENCE PARAMETERS

Moon and Zelazny[27] derived an equation roughly equivalent to the approximate Eq.(63) for the self noise. (They had another equation, more complex, for the shear noise.) These were based on a space-time (rather than wave number-frequency) domain solution of the Lighthill equation and involved various simplifying assumptions. In particular, the operation $\partial/\partial\tau$ was interpreted as effectively multiplying by a characteristic frequency (a function of axial position y_1 in the jet): ω_{sc} for the self noise and $(1/2)\omega_{se}$ for the shear noise. There was also an implicit heuristic assumption for the constant A of Eqs. (62)-(64). For implementation for the scenario of a round jet, they developed a model for determination of these and other needed turbulence parameters. A major feature (taken from a Ph.D. dissertation underlying ref. 27) was an eddy viscosity formalism for evaluation of the rms axial component of turbulence, u', as a function of r and y_1 throughout the jet. This led to evaluation of $(\overline{u_k^2})^2$ appearing in Eq.(63). Other heuristic features of their model are:

L_s = shear layer half-width	$L_1 = 0.358L_s$	$L_2 = L_3 = 0.179\, L_s$
$\omega_{se} = 0.3\, U(y_1)/L_2$	$\omega_{sh} = 0.3\, U(y_1)/L_1$	

The agreement with both turbulence and noise measurements was good. These were absolute levels: in particular, for the noise there was no shifting of the decibel scale for best match . This has to be impressive. On the other hand, the assumptions, e.g., of scales and frequencies, involves empiricism.

ACKNOWLEDGEMENTS

Support at NASA was provided by tenure part time at the Langley Research Center as a Distinguished Research Associate, and at the University of Toronto Institute for Aerospace Studies with funds from a grant from the Natural Sciences and Engineering Research Council of Canada.

REFERENCES AND NOTES

[1] Lighthill, M. J., "On Sound Generated Aerodynamically--I. General Theory," *Proceedings of the Royal Society of London,* Vol. 211, Ser. A, No. 1107, 1952, pp.564-587.

[2] Lighthill, M. J., "On Sound Generated Aerodynamically--II. Turbulence as a Source of Sound," *Proceedings of the Royal Society of London,* Vol. 222, Ser. A, 1954, pp. 1-32.

[3] Ribner, H. S., "Aerodynamic Sound from Fluid Dilatations: A Theory of Sound from Jets and Other Flows," Univ. of Toronto, Institute of Aerophysics (now Aerospace Studies), UTIA Rept. 86, AFOSR TN 3430, July 1962.

[4] Ribner, H. S., "The Generation of Sound by Turbulent Jets," *Advances in Applied Mechanics,* Vol. VIII, Academic Press, New York, 1964, pp. 103-182.

[5] Powell, A., "Survey of Experiments on Jet-Noise, *Aircraft Engineering*, Vol. 26, 1954, pp. 2-9.

[6] Phillips, O. M., "On the Generation of Sound by Supersonic Turbulent Shear Layers," *Journal of Fluid Mechanics,* Vol. 9, No. 1, 1960, pp. 1-28.

[7] Csanady, G. T., "The Effect of Mean Velocity Variations on Jet Noise," *Journal of Fluid Mechanics,* Vol. 26, Sept. 1966, pp. 183-197.

[8] Schubert, L. K., "Refraction of Sound by a Jet: a Numerical Study," Univ. of Toronto, Institute for Aerospace Studies, Rept. 144, Dec. 1969.

[9] Schubert, L. K., "Numerical Study of Sound Refraction by a Jet Flow II. Wave Acoustics," *Journal of the Acoustical Society of America,* Vol. 51, Feb. 1972, pp. 447-463.

[10] Lilley, G. M., "The Generation and Radiation of Supersonic Jet Noise. Vol. IV--Theory of Turbulence Generated Jet Noise, Noise Generation from Upstream Sources, and Combuistion Noise, Part II: Generation of Sound in a Mixing Region," Air Force Aero Propulsion Lab., AFAPL-TR-53, July 1972.

[11] Mani, R. "A Moving Source Problem Relevant to Jet Noise," *Journal of Sound and Vibration,* Vol. 25, No. 2, 1972, pp. 337-347.

[12] Mani, R. "The Influence of Jet Flow on Jet Noise. Part 1. The Noise of Unheated Jets," *Journal of Fluid Mechanics,* Vol. 73, Feb., 1976, pp. 753-758.

[13] Mani, R. "The Influence of Jet Flow on Jet Noise. Part 2. The Noise of Heated Jets," *Journal of Fluid Mechanics,* Vol. 73, part 4, 1976, pp. 779-793.

[14] Balsa, T. F., "The Far Field of High Frequency Convected Singularities in Sheared Flows, with an Application to Jet Noise Prediction," *Journal of Fluid Mechanics,* Vol. 74, No. 2, 1976, pp. 193-208.

[15] Balsa, T. F., "The Acoustic Field of Sources in Shear Flow with Application to Jet Noise: Convective Amplification," *Journal of Fluid Mechanics,* Vol. 79, No. 1, 1977, pp. 33-47.

[16] Tester, B. J., and Morfey, C. L., "Developments in Jet Noise Modelling--Theoretical Predictions and Comparisons with Measured Data," *Journal of Sound and Vibration,* Vol. 46, 1976, pp. 79-103.

[17] Morfey, C. L., Szewczyk, V. M., and Tester, B. J., "New Scaling Laws for Hot and Cold Jet Mixing Noise, Based on a Geometric Acoustics Model," *Journal of Sound and Vibration,* Vol. 61, No. 2, 1978, pp. 255-292.

[18] Balsa, T. F., and Gliebe, P. R., "Aerodynamics and Noise of Coaxial Jets," *AIAA Journal,* Vol. 15, No. 11, Nov. 1977, pp. 1550-1558.

[19] Balsa, T. F., Gliebe, P. R., Kantola, R. A., Mani, R., Stringas, E. J., and Wang, J. C. F., "High Velocity Jet Noise Source Location and Reduction. Task 2--Theoretical Developments and Basic Experiments," FAA-RD76-79, II, May 1978. (Available from DTIC as AD A094291.)

[20] Gliebe, P. R., and Balsa, T. F., "Aeroacoustics of Axisymmetric Single- and Dual-Flow Exhaust Nozzles," *Journal of Aircraft,* Vol. 15, No. 11, Nov. 1978, pp743-749.

[21] Gliebe, P. R., "Diagnostic Eval;uation of Jet Noise Suppression Mechanisms," *Journal of Aircraft,* Vol. 17, No. 12, Dec. 1980, pp. 837-842,

[22] Gliebe, P. R., Brausch, J. F., Majjigi, R. K., and Lee, R., "Jet Noise Suppression," in *Aeroacoustics of Flight Vehicles: Theory and Practice. Vol. 2: Noise Control,* NASA Reference Publication 1258, Vol. 2; WRDC Technical Rept. 90-3052, Aug. 1991, pp. 207-269.

[23] Ribner, H. S., "Quadrupole Correlations Governing the Pattern of Jet Noise," *Journal of Fluid Mechanics,* Vol. 38, No. 1, 1969, pp. 1-24.

[24] Pao, S. P., and Lowson, M. V., "Some Applications of Jet Noise Theory," *AIAA Paper 70-233,* New York, Jan. 1970.

[25] Krishnappa, G. "Estimation of the Intensity of Noise Radiated from a Subsonic Circular Jet," *Proceedings of the AFOSR/ Univ. of Toronto, Institute for Aerospace Studies, Symposium,* Toronto, 1968, Univ. of Toronto Press, Toronto, Canada.

[26] Krishnappa, G. and Csanady, G. T., "An Experimental Investigation of the Composition of Jet Noise," *Journal of Fluid Mechanics,* Vol. 37, June 1969, pp. 149-159.

[27] Moon, L. F. and Zelazny, S. W., "Experimental and Analytical Study of Jet Noise Modeling," *AIAA Journal,* Vol. 13, No. 3, March 1975, pp. 387-393.

[28] Ribner, H. S., "On the Role of the Shear Term in Jet Noise," *Journal of Sound and Vibration,* Vol. 52, No. 1, 1977, pp. 121-132.

[29] Ribner, H. S., "Perspectives on Jet Noise," Dryden Lecture, *AIAA Journal,* Vol. 19, No. 12, Dec. 1981 pp. 1513-1526.

[30] Chu, W. T., "Turbulence Measurements Relevant to Jet Noise," Univ. of Toronto, Institute for Aerospace Studies, UTIAS Rept. 119, Nov. 1966.

[31] Chu, W. T., "Moving Frame Analysis of Jet Noise," *Journal of the Acoustical Society of America,* Vol. 53, No. 5, 1973, pp. 1439-1440.

[32] Atvars, J., Schubert, L. K., Grande, E., and Ribner, H. S., "Refraction of Sound by Jet Flow or Jet Temperature," Univ. of Toronto, Institute for Aerospace Studies," UTIAS TN 109, May, 1965; NASA CR-494, May, 1966.

[33] Grande, E., "Refraction of Sound by Jet Flow and Jet Temperature II," Univ. of Toronto, Institute for Aerospace Studies, UTIAS TN 110, Dec. 1966; NASA CR-840, Aug. 1967.

[34] Schubert, L. K., "Numerical Study of Sound Refraction by a Jet Flow I. Ray Acoustics," *Journal of the Acoustical Society of America,* Vol. 51, Feb. 1972, pp. 439-446.

[35] Morfey, C. L., "Amplification of Aerodynamic Noise by Convected Flow Inhomogeneities," *Journal of Sound and Vibration,* Vol. 31, Dec. 1973, pp. 391-397.

[36] Doak, P. E., "Analysis of Internally Generated Sound in Continuous Materials: 2. A Critical Review of the Conceptual Adequacy and Physical Scope of Existing Theories of Aerodynamic Noise, with Special Reference to Supersonic Jet Noise," *Journal of Sound and Vibration,* Vol. 25, No. 2, Nov. 22, 1972, pp. 263-335.

[37] Goldstein, M. E., "Aeroacoustics," National Aeronautics and Space Administration, Lewis Research Ctr., NASA SP-346, 1974 (see pp. 389 and 391).

[38] Ffowcs Williams, J. E., "The Noise from Turbulence Convected at High Speed," *Philosophical Transactions of the Royal Society of London,* Series A, Vol. 255, 1963, pp. 469-503.

[39] Ffowcs Williams, J. E., "Some Thoughts on the Effects of Aircraft Motion and Eddy Convection on the Noise from Air Jets," Univ. of Southampton, Dept. of Aeronautics and Astronautics, USAA Rept. 155, 1960.

[40] Nossier, N.S.M., and Ribner, H. S., "Tests of a Theoretical Model of Jet Noise," AIAA Paper 75-436, March 1975.

[41] Maestrello, L., "Two-Point Correlations of Sound Pressure in the Far Field of a Jet: Experiment," NASA TM X-72835, 1976.

[42] Ribner, H. S., "Two Point Correlations of Jet Noise," *Journal of Sound and Vibration,* Vol. 56, No.1, pp.1-19.

[43] Richarz, W.G., "Theory of Cross-Spectral Densities of Jet Noise," *Mechanics of Sound Generation in Flows,* IUTAM/ICA/AIAA-Symposium Gottingen, Max-Planck-Institut fur Stromungsforschung, Ed. E.-A. Muller, Aug.28-31, 1979, pp.153-158.

[44] Balsa, T. F., "The Shielding of a Convected Source by an Annular Jet with an Application to the Performance of Multitube Suppressors," *Journal of Sound and Vibration,* Vol. 44, No. 2, 1976, pp. 179-189.

[45] Proudman, I., "The Generation of Noise by Isotropic Turbulence," *Proceedings of the Royal Society,* Ser. A, Vol. 214, 1952, pp.119-132.

[46] Kraichnan, R.H. "The Scattering of Sound in a Turbulent Medium," *Journal of the Acoustical Society of America,* Vol.25, 1953, pp. 1096-1104

[47] Mawardi, O.K. "On the Spectrum of Noise from Turbulence," *Journal of the Acoustical Society of America,* Vol. 27, 1955, pp.442-445.

[48] Lilley,G.M. "On the Noise from Air Jets," Aeronautical Research Council (Great Britain), ARC 20, 376-N40-FM2724, 1958

[49] Michalke, A., and Michel, U., "Prediction of Jet Noise in Flight from Static Tests, *Journal of Sound and Vibration,* Vol. 67, N0. 3, 1979, pp.347-367.

[50] Michalke, A. and Michel, U., "Prediction of Flyover Noise from Single and Coannular Jets," AIAA Paper 80-1031, June 1980.

[51] Musafir, R.E., Slama, J. G., Zindeluk, M., "Quadrupole Correlations and Jet Noise," **Inter-noise 84,** Honolulu, Dec.3-5, 1984, *"Physical Phenomena,"* pp.257-260.

[52] Musafir, R.E., "The Use of Polar Correlation in the Characterization of Multipolar Source Distributions," **Inter-noise 86**, Cambridge, Mass., July 21-23, 1986, *"Analysis,"* pp. 1335-1340.

[53]There is an apparent contradiction here with reference 29 on the connection between the self- and shear-noise terms: the following remarks are intended to resolve this. It is still maintained that, as asserted there (p. 1517), "the wave equation--either Lighthill or "Lilly"--in concert with the momentum equations governs the entire turbulent flow", etc.,

and also, "It follows that the wave and momentum equations serve to *correlate* the self- and shear-noise terms. Either the *equations* or *flow measurements* can provide the shear/self correlations." These remarks were made in reference to an equation like (11) with the *entire* shear term on the left hand side. But it is now realized that the employment of a Green's function technique for solution (as done by Lilley) will involve only the acoustic component of the shear term (the one with u_{2ac}) on the left hand side. That is to say, the far field Green's function will not compute the full shear term, and so will not serve to correlate it with the self-noise term. The technique is, of course, an artifice for predicting noise, the instantaneous turbulence being presumed known. (In applications, it is only space-time correlations that are needed.)

[54] Lush, P. A., "Measurements of Subsonic Jet Noise and Comparison with Experiment," *Journal of Fluid Mechanics*, Vol. 46, part 3, 1971, pp. 477-500.

[55] Powell, Alan, "Fundamental Notions Concerning Convection of Aerodynamic Noise Generators," Program, 59th Meeting of the Acoustical Society of America, Providence, R.I., June 9-11, 1960, Paper 05 (Abstract)

[56] Ribner, H. S., "Energy Flux from an Acoustic Source Contained in a Moving Fluid Element and Its Relation to Jet Noise," Journal of the Acoustical Society of America, Vol.32 (9), Sept. 1960, pp.1159-1160 (Letter).

[57] Lighthill, M.J., "On the Energy Scattered from the Interaction of Turbulence with Sound or Shock Waves," Proceedings of the Cambridge Philosophical Society, Vol. 49, Pt. 3, 1953, pp. 531-551.

[58] Ribner, H. S., "Convection of a Pattern of Vorticity Through a Shock Wave," NACA TN 2864, Jan. 1953 and NACA Rept. 1164, 1954.

[59] Ribner, H. S., "Shock-Turbulence Interaction and the Generation of Noise," NACA TN 3255, July 1954 and NACA Rept. 1233, 1955.

[60] Tam, C. K. W., "Jet Noise Generated by Large-Scale Coherent Motion," in Aeroacoustics of Flight Vehicles: Theory and Practice. Vol. 1: Noise Sources NASA Reference Publication 1258, Vol. 1; WRDC Technical Rept. 90-3052, Aug. 1991, pp. 311-390.

[61] Bayliss, A., Maestrello, L., McGreevy, J. L., Fenno, C. C. Jr., "Response of Multi-Panel Assembly to Noise from a Jet in Forward Motion," 1st Joint CEAS/AIAA Aeroacoustics Conference (16th AIAA Aeroacoustics Conference), Munich, 12-15 June 1995 (to be presented).

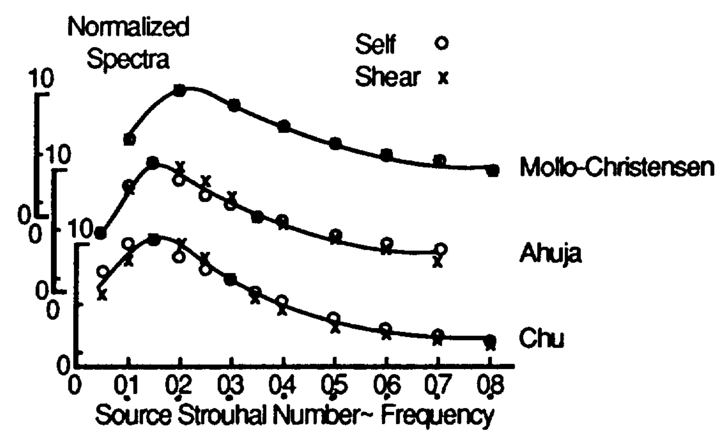

Fig.1. Shear term in source role. Match of experimentally derived shear noise and self noise spectra, normalized to same amplitude. The self noise has been downshifted one octave, in accordance with the theory. (After ref. 29, based on ref. 40.)

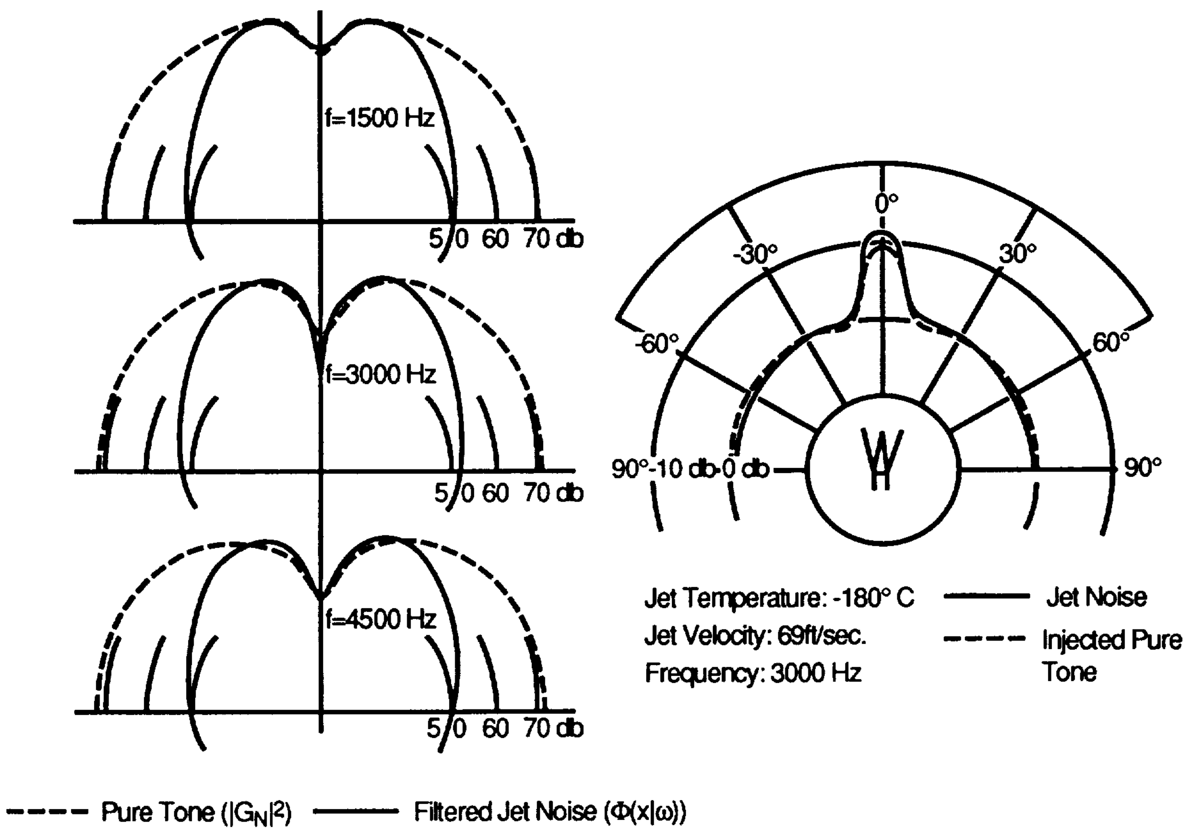

Fig. 2. Match of noise intensity patterns to indicate refraction-dominated zone. Left-hand side: room temperature jet (M=0.5) with reduced intensity 'cone of silence' (rays turn outward); right-hand side; very cold jet (-180°C) with enhanced intensity lobe (rays turn inward to a quasi-focus). (After ref. 29, based on refs. 32 and 33).

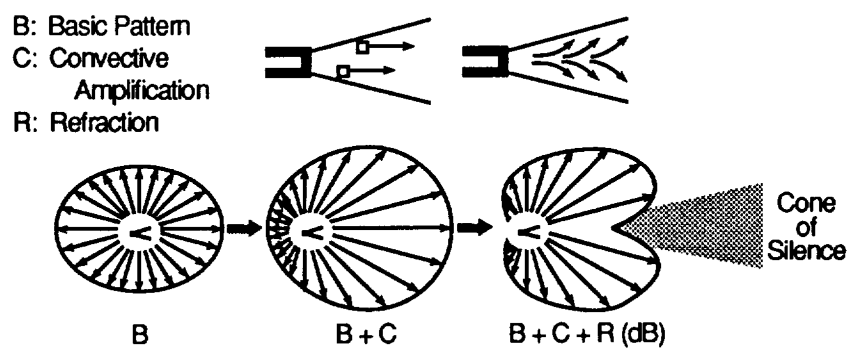

Fig.3. Convective amplification (C^{-5}) and refraction (normalized Green's function $|G_N|^2$) modify the basic pattern of intensity vs direction (in decibels). (After ref. 29, based on refs. 32 and 33).

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	May 1995	Technical Memorandum

4. TITLE AND SUBTITLE
An Extension of the Lighthill Theory of Jet Noise to Encompass Refraction and Shielding

5. FUNDING NUMBERS
505-59-52-01

6. AUTHOR(S)
Herbert S. Ribner

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, VA 23681-0001

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administreation
Washington, DC 20546-0001

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
NASA TM-110163

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Unclassified-Unlimited
Subject Category 71

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A formalism for jet noise prediction is derived that includes the refractive 'cone of silence' and other effects; outside the cone it approximates the simple Lighthill format. A key step is deferral of the simplifying assumption of uniform density in the dominant 'source' term. The result is conversion to a convected wave equation retaining the basic Lighthill source term. The main effect is to amend the Lighthill solution to allow for refraction by mean flow gradients, achieved via a frequency-dependent directional factor. A general formula for power spectral density emitted from unit volume is developed as the Lighthill-based value multiplied by a squared 'normalized' Green's function (the directional factor), referred to a stationary point source. The convective motion of the sources, with its powerful amplifying effect, also directional, is already accounted for in the Lighthill format: wave convection and source convection are decoupled. The normalized Green's function appears to be near unity outside the refraction dominated 'cone of silence', this validates our long term practice of using Lighthill-based approaches outside the cone, with extension inside via the Green's function. The function is obtained either experimentally (injected 'point' source) or numerically (computational aeroacoustics). Approximation by unity seems adequate except near the cone and except when there are shrouding jets: in that case the difference from unity quantities the shielding effect. Further extension yields dipole and monopole source terms (cf. Morfey, Mani, and others) when the mean flow possesses density gradients (e.g., hot jets).

14. SUBJECT TERMS
Jet noise; Jet flow refraction; Convected wave equation; Jet shielding

15. NUMBER OF PAGES
44

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified		