The nonlinear evolution of inviscid Görtler vortices in three–dimensional boundary layers

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Abstract

The nonlinear development of inviscid Görtler vortices in a three–dimensional boundary layer is considered. We do not follow the classical approach of weakly nonlinear stability problems and consider a mode which has just become unstable. Instead we extend the method of Blackaby, Dando & Hall (1993), which considered the closely related nonlinear development of disturbances in stratified shear flows. The Görtler modes we consider are initially fast growing and we assume, following others, that boundary–layer spreading results in them evolving in a linear fashion until they reach a stage where their amplitudes are large enough and their growth rates have diminished sufficiently so that amplitude equations can be derived using weakly nonlinear and non–equilibrium critical–layer theories. From the work of Blackaby, Dando & Hall (1993) it is apparent, given the range of parameters for the Görtler problem, that there are three possible nonlinear integro–differential evolution equations for the disturbance amplitude. These are a cubic due to viscous effects, a cubic which corresponds to the novel mechanism investigated in this previous paper and a quintic. In this paper we shall concentrate on the two cubic integro–differential equations and in particular on the one due to the novel mechanism as this will be the first to affect a disturbance. It is found that the consideration of a spatial evolution problem as opposed to temporal (as was considered in Blackaby, Dando & Hall 1993) causes a number of significant changes to the evolution equations.

1 Introduction

Since the work of Görtler (1940) there has been considerable interest in the boundary layer instability mechanism named after him. Much of the early work on Görtler vortices was shown to be flawed, by Hall (1982a,b), because it invoked the parallel flow approximation and thus ignored the effects of boundary–layer growth. Hall (1983) went on to show that for Görtler vortices of order–one wavenumber the ideas of unique neutral curves and growth rates are untenable. The stability properties of such modes depend upon the initial form and location of the disturbance.

Since the early 1980's there have been numerous theoretical studies on Görtler vortices (the reader is referred to the review papers by Hall 1990 and Saric 1994 for an overview of the subject). Most of these papers have concentrated on two–dimensional boundary layers but obviously many practical situations in which Görtler vortices arise will be three–dimensional in nature. One example of particular interest has been caused by the development of laminar flow control airfoils which have areas of concave curvature on the underside of the airfoil near the leading edge. When the wing is swept the flow becomes fully three–dimensional. Consequently a number of recent studies have considered the stability of Görtler vortices in three–dimensional boundary layers and this three–dimensionality has been shown to have an important effect upon the stability properties of these vortices.

Bassom & Hall (1991) looked at the viscous and inviscid stability problems for an incompressible boundary layer flow, which could support both Görtler and crossflow vortices, over an infinitely long swept cylinder. They found that, for sufficiently large values of the parameter representing the degree of three–dimensionality of the flow, there are no Görtler vortices present in a boundary layer which, in the zero crossflow case, is centrifugally unstable. The inviscid stability problem has been extended to compressible boundary layers by Dando (1992). Similar results to the incompressible case are found; three–dimensionality has a stabilizing effect on vortices of all wavelengths except for a band of small wavelengths where the vortices are dominated by crossflow effects and are in fact of the type considered by Gregory, Stuart & Walker (1955). The numerical results of Dando (1992) showed that for larger Mach numbers a larger crossflow was needed to completely stabilize Görtler vortices over a band of wavenumbers and this was confirmed by the asymptotic study of Fu & Hall (1994) who considered the hypersonic limit.

One of the most interesting points to emerge from the work of Bassom & Hall (1991) and Dando (1992) was that in the presence of a relatively weak crossflow, Görtler vortex disturbances of all wavelengths are stabilized such that the inviscid modes possess some of the largest growth rates whilst also being neutral at certain other wavenumbers. Furthermore their governing equation has many similarities to the Taylor–Goldstein equation which governs the linear stability of stratified shear flows. In fact Blackaby & Choudhari (1993) have illustrated the close connection between the two problems of inviscid Görtler modes in three–dimensional boundary layers and modes on unstable stratified shear layers, and proposed a definition of a generalized Richardson number (this is the parameter which characterizes the stratification of a shear flow) for such centrifugally–driven instabilities. The weakly nonlinear study of Bassom & Otto (1993) used a classical approach to consider the stability of $O(G^{\frac{1}{5}})$ wavenumber (here G is the Görtler number), essentially viscous, modes in three–dimensional boundary layers. In the present paper we restrict our attention to the $O(1)$ wavenumber inviscid modes.

It was this close connection between the two problems which encouraged the authors, in their desire to develop a theory describing the nonlinear evolution of these inviscid Görtler modes, to initially consider the nonlinear evolution of modes on stratified shear layers in Blackaby, Dando & Hall (1993). In order to place the latter paper and the current piece of work in context and understand the theory underpinning both it is necessary to review some of the recent contributions to nonlinear critical–layer theory (see the review papers by Stewartson 1981 and Maslowe 1986 for a more general and complete review of the theory). Over the last couple of decades much attention has been focussed on the nonlinear stability of non–stratified shear layers. In the case where there is no vertical density variation the linear stability of the flow is usually governed by the familiar Rayleigh equation (to which the Taylor–Goldstein equation reduces for zero Richardson number). Benney & Bergeron (1969) developed the so–called equilibrium critical layer theory: here the mode is treated as 'quasi–steady' inside the critical layer as well as outside it. Nonlinearity affects the jump imposed across the critical layer and hence leads to modified results for the neutral (equilibrium) modes. Haberman (1972) extended the theory to include critical layers where viscosity is also significant. Some of the early studies of non–equilibrium critical layers include the work of Brown & Stewartson (1978), Warn & Warn (1978) and Hickernell (1984). The key paper by Hickernell (1984) concerned a shear layer affected by Coriolis (rotational) effects; here the weakly nonlinear theory leads to an integro–differential equation rather than the (previously) more familiar Stuart–Watson–Landau equation with its 'polynomial' nonlinear terms. In fact such integro–differential equations result naturally from non–equilibrium nonlinear critical layer theories when the shear layer is coupled with other physical factors such as, for instance, Coriolis effects (eg. Hickernell 1984; Shukhman 1991); compressibility effects (eg. Goldstein & Leib 1989); three–dimensionality effects (eg. Goldstein & Choi 1989; Wu, Lee & Cowley 1993) and buoyancy effects (eg. Churilov & Shukhman 1988; Blackaby, Dando & Hall 1993). However the case of a 'simple' shear layer, not affected by any additional physical factors, is a special case in the sense that it does not lead to an integro–differential equation; instead, Goldstein & Leib (1988), found that the nonlinear evolution of a disturbance was governed by the full unsteady nonlinear critical–layer equations. This difference is due to the additional physical factors, of the former cases, resulting in stronger singularities of the inviscid disturbance quantities at the critical level.

At first sight, it appears that weakly nonlinear theories can only be usefully applied to marginally unstable flows; they rely on small growth rates and so the unstable disturbance of concern must be near to a neutral state. Thus it was believed that such theories are of no use in describing the initial evolution of 'far–from–neutral' unstable modes. However, a number of recent studies have derived integro–differential equations, using weakly nonlinear theories, to describe the nonlinear evolution of (general) unstable modes on a variety of shear layers (see the previous paragraph). These studies are based on the idea that, in actual physical flow situations, shear layer spreading or other external changes would result in the otherwise relatively unstable modes having their growth rates diminished in real terms, so that a weakly nonlinear critical–layer theory becomes appropriate. The work in this paper is based on the assumption that boundary layer growth acts in a similar manner to shear layer spreading. This theory is supported by the work of Michalke (1964), Crighton & Gaster (1976) and the excellent comparison with experiments recently achieved by Hultgren (1992). For further discussion of non–equilibrium critical layer theory the reader is directed to the reviews of Cowley & Wu (1993) and Goldstein (1994).

In this study we use weakly nonlinear and non–equilibrium critical–layer theories to describe the spatial, nonlinear development and evolution of inviscid, unstable Görtler modes in an incompressible, weakly three–dimensional boundary–layer. The theory of this paper is extendable to compressible boundary layers and also has obvious applications to inviscid modes in a flow above a heated plate, similar to those considered by Hall & Morris (1992).

Whilst different from the approach adopted in this study, there are alternate/complementary nonlinear theories that have been developed recently in which two or more of the flow disturbances mutually interact. Such theories generally require smaller disturbance amplitudes but may also need the disturbances to exist in specific 'configurations'. These other theories are generally referred to as wave/wave and vortex/wave interactions. For a discussion of wave/wave interactions and resonant triads the reader is directed to the book by Craik (1985). The ideas behind resonant triads and non–equilibrium critical layers have been combined in works by Goldstein & Lee (1992) and Wu (1992) which both consider resonant triad interactions where the growth rates of the disturbance are controlled by nonlinear interactions inside critical layers. Strongly nonlinear vortex/wave interactions were first looked at by Hall & Smith (1991) and their ideas were clarified and extended by Brown, Brown, Smith & Timoshin (1993) and Smith, Brown & Brown (1993). Recent work has in fact shown mathematical connections between these different nonlinear theories. Wu, Lee & Cowley (1993) in their non–equilibrium nonlinear critical layer study showed that the viscous limit of their amplitude equation is the same as the amplitude equation obtained by Brown, Brown, Smith & Timoshin (1993) in their vortex/wave interaction paper.

The rest of this paper is laid out as follows. In the next section we present some background details of the flow concerning us in this paper, namely inviscid Görtler vortices in three–dimensional boundary layers. In §3 the flow outside the critical layer is considered whilst §4 deals with the flow inside the critical layer, concentrating on deriving the non–viscous cubic amplitude equation. In §5 we consider briefly the viscous cubic amplitude equation. In §6 we look at some numerical solutions for the non–viscous cubic before finally drawing some conclusions in §7.

2 Inviscid Görtler Vortices in Three–Dimensional Boundary Layers

At this point it is helpful to recap the scalings and arguments that lead to the governing equation for inviscid Görtler vortices in three–dimensional boundary layers. In this paper we shall consider an incompressible flow. For a more detailed derivation of this governing equation see Bassom & Hall (1991) and for the compressible case see Dando (1992).

The boundary layer considered is that of a flow over the infinite cylinder $y = 0$, $-\infty < z < \infty$ so that the z–axis is a generator of the cylinder and y measures the distance normal to the surface. The x–coordinate measures distance along the curved surface, which has variable curvature $(1/m)K(x/l)$ where m and l are length scales. The Reynolds number, R, Görtler number, G, and curvature parameter, δ, are defined by

$$R = \frac{U_\infty l}{\kappa} \quad , \tag{2.1a}$$

$$G = 2R^{\frac{1}{2}}\delta \quad , \tag{2.1b}$$

$$\delta = \frac{l}{m} \quad , \tag{2.1c}$$

where U_∞ is a typical flow velocity in the streamwise direction and κ is the kinematic viscosity of the fluid.

The Reynolds number is assumed to be large, whilst δ is sufficiently small so that as $\delta \to 0$ the parameter G is fixed and of order one. The basic three–dimensional boundary layer is taken to be of the form

$$\underline{u} = U_\infty[\overline{u}(X,Y),\ R^{-\frac{1}{2}}\overline{v}(X,Y),\ R^{-\frac{1}{2}}\lambda^*\overline{w}(X,Y)].[1 + O(R^{-\frac{1}{2}})] \quad , \tag{2.2}$$

with

$$X = x/l \qquad Y = R^{\frac{1}{2}}y/l \quad , \tag{2.3a, b}$$

where the parameter λ^* is a measure of the relative strength of the crossflow present. The basic state is perturbed by writing

$$\underline{u} = U_\infty[\overline{u} + \epsilon\tilde{U}(X,Y)E,\ \overline{v}R^{-\frac{1}{2}} + \epsilon R^{-\frac{1}{2}}\tilde{V}(X,Y)E,$$

$$R^{-\frac{1}{2}}\lambda^*\overline{w} + \epsilon R^{-\frac{1}{2}}\tilde{W}(X,Y)E].[1 + O(R^{-\frac{1}{2}})] \quad , \tag{2.4a}$$

$$p = \overline{p}(X,Y) + R^{-1}\tilde{P}(X,Y)E \quad , \tag{2.4b}$$

where $\epsilon \ll 1$ and

$$E = \exp\{iaR^{\frac{1}{2}}z/l\} \quad . \tag{2.5}$$

We now consider the inviscid limit of the Görtler problem by introducing a scaled spatial growth rate, β, and the scalings

$$[\tilde{U}(X_0,Y),\ \tilde{V}(X_0,Y),\ \tilde{W}(X_0,Y),\ \tilde{P}(X_0,Y)] = [U(Y),$$

$$G^{\frac{1}{2}}V(Y),\ G^{\frac{1}{2}}W(Y),\ GP(Y)] \times \exp\left\{\int G^{\frac{1}{2}}\beta(X)dX\right\} \quad , \tag{2.6a}$$

$$\lambda^* = G^{\frac{1}{2}}\lambda \quad , \tag{2.6b}$$

which were obtained independently by Timoshin (1990) and Denier, Hall & Seddougui (1991) for two–dimensional problems. Here $X = X_0$ is the local streamwise location under consideration.

Upon substituting these expansions into the continuity and momentum equations and letting $G \to \infty$ we obtain

$$\beta U + \frac{dV}{dY} + iaW = 0 \quad , \tag{2.7a}$$

$$(\beta\overline{u} + ia\lambda\overline{w})U + \overline{u}_Y V = 0 \quad , \tag{2.7b}$$

$$(\beta\overline{u} + ia\lambda\overline{w})V + K\overline{u}U = -\frac{dP}{dY} \quad , \tag{2.7c}$$

$$(\beta\overline{u} + ia\lambda\overline{w})W + \lambda\overline{w}_Y V = -iaP \quad . \tag{2.7d}$$

From these it can easily be shown that

$$\frac{d^2V}{dY^2} + \left[-a^2 - \frac{(\beta\overline{u}_{YY} + ia\lambda\overline{w}_{YY})}{(\beta\overline{u} + ia\lambda\overline{w})} + \frac{a^2K\overline{u}\,\overline{u}_Y}{(\beta\overline{u} + ia\lambda\overline{w})^2}\right]V = 0 \quad , \tag{2.8}$$

and this is subject to the boundary conditions $V(0) = 0$ and $V \to 0$ as $Y \to \infty$. This is the equation that controls the inviscid growth of Görtler vortices in an incompressible, weakly three–dimensional boundary layer and, as noted earlier, it is very similar to the Taylor–Goldstein equation. We note that in Bassom & Hall (1991) and Dando (1992) it was chosen to scale the curvature, K, out of this equation, this was possible because of the local nature of the problem considered. At this point it is illustrative to reproduce three figures from Dando (1992) and consider some solutions of equation (2.8). The plots we show are of the scaled growth rate, $\tilde{\beta} = K^{-\frac{1}{2}}\beta$, against the spanwise wavenumber, a, for three values of the scaled crossflow, $\tilde{\lambda} = K^{-\frac{1}{2}}\lambda$. In Figure 2.1 we have no crossflow, in Figure 2.2 we have $\tilde{\lambda} = 5$ and for Figure 2.3 we have taken $\tilde{\lambda} = 10$. For further details the reader is referred to Bassom & Hall (1991) and Dando (1992) but these three diagrams illustrate clearly the effect that crossflow has on the stability of Görtler vortices. Thus in the presence of a relatively weak crossflow, Görtler vortices of almost all wavelengths are stabilized such that the inviscid modes possess some of the largest growth rates whilst also being neutral at certain other wavenumbers.

The close relationship between equation (2.8) and the Taylor–Goldstein equation was considered in detail by Blackaby & Choudhari (1993). In particular they proposed a generalized definition of the Richardson number for such vortex flows, namely

$$J = \frac{a^2K\overline{u}_c\overline{u}_{Yc}}{(\beta\overline{u}_{Yc} + ia\lambda\overline{w}_{Yc})^2} \quad , \tag{2.9}$$

where the subscript c's denote evaluation at the critical level, $Y = Y_c$, where $(\beta\overline{u} + ia\lambda\overline{w}) = 0$. It is possible to calculate neutral curves for the Görtler modes (see Figure 1 in Blackaby & Choudhari 1993 and §6 of this paper) and then using the same definition of ν as for the stratified shear layer problem,

$$\nu = \pm\frac{1}{2}(1 - 4J)^{\frac{1}{2}} \quad , \tag{2.10}$$

we can calculate numerically the values of ν on these neutral curves. It is found for the Görtler problem that $\nu < -\frac{1}{2}$ and so from Blackaby, Dando & Hall (1993) we know that for this range of the parameter ν there are three possible evolution equations; a viscous–cubic; a non–viscous–cubic (this is the equivalent of the novel non–viscous–cubic from this previous paper, it arose from a perturbation to the Richardson number and was denoted as the J_1–cubic) and a quintic. For our numerical calculations we shall consider a Falkner–Skan–Cooke profile for the base flow (see Bassom & Hall 1991 and Dando 1992). In order for a similarity variable to exist for the compressible problem (see Stewartson 1964 and Dando 1992) it is necessary to consider a model fluid and so for this earlier work we have taken a Prandtl number of unity. Here our consideration of an incompressible flow restricts us to effectively a Prandtl number of unity although we shall see later in §5 that this is a special case for the viscous–cubic. A similar definition of the Richardson number is available for the compressible Görtler problem where we again find that $\nu < -\frac{1}{2}$ (see Dando 1993).

3 Outside the Critical Layer

In order to derive the desired evolution equations a study of the fundamental and other higher harmonics is necessary both inside and outside of the critical layer. The details of this study are dependent on the flow under consideration but the method is quite general and can be applied to other flows (the reader will note many similarities between the next two sections and the corresponding work in Blackaby, Dando & Hall 1993 for the stratified shear layer problem where some aspects are discussed in more detail). In this section we consider the flow outside the critical layer whilst the following two sections are devoted to that inside the critical layer.

3.1 Scalings and Notation

It is wise, before embarking on the analysis of the critical layer, to consider for a moment the various x–scales that we have in this problem. In addition to the non–dimensional boundary layer variable, X (defined by equation 2.3a), we have to consider:
(i) the fast scale, $\hat{X} = G^{\frac{1}{2}}X$ say, of the disturbance which was implicit in the scalings (2.6a) and was necessary for the derivation of the governing equation (2.8);
(ii) the slower scale, $\tilde{X} = G^{\frac{1}{2}}\mu X$, over which the disturbance amplitude evolves (μ is a small parameter);
(iii) the slowest scale $x_1 = \mu^{-1}X$ (provided $G^{-\frac{1}{4}} \ll \mu$), over which the mean flow varies.

Due to the length of many of the equations in the critical–layer analysis we shall try to use as much abbreviated notation as possible. In particular the real functions $\overline{q}_0$, $\overline{q}_1$ and $\overline{q}_2$ will be defined by

$$i\overline{q}_0 = (\beta\overline{u}_0 + ia\lambda_0\overline{w}_0) \quad , \tag{3.1a}$$

$$i\overline{q}_1 = (\beta\overline{u}_1 + ia[\lambda_1\overline{w}_0 + \lambda_0\overline{w}_1]) \quad , \tag{3.1b}$$

$$i\overline{q}_2 = (\beta\overline{u}_2 + ia[\lambda_2\overline{w}_0 + \lambda_1\overline{w}_1 + \lambda_0\overline{w}_2]) \quad . \tag{3.1c}$$

Note that the critical layer $Y = Y_c$ occurs when $\overline{q}_0 = 0$. Also $\overline{u}_1$,$\overline{u}_2$, $\overline{w}_1$, $\overline{w}_2$, λ_1 and λ_2 are real functions of x_1 and for example

$$\overline{u}_1 = x_1\overline{u}_{0X}(X_0) \quad , \tag{3.2a}$$

$$\lambda_2 = \frac{1}{2} x_1^2 \lambda_{0XX}(X_0) \quad . \tag{3.2b}$$

Similarly the perturbation to the curvature, K_1, is also a real function of x_1. From this point onwards dashes on mean flow quantities will denote derivatives with respect to Y. It is also implicit that all of the mean flow quantities in the critical layer analysis (§3.3, §4, §5 and §6) are the values evaluated at the critical layer. We shall not explicitly write a subscript c to denote this.

3.2 The Solvability Condition

In order to derive the evolution equation we expand the fundamental as

$$V_1 = \epsilon V_1^{(1)} + \epsilon\mu V_1^{(2)} + \cdots \quad , \tag{3.3}$$

where ϵ is a small parameter characterizing the magnitude of the mode. Here $V_1^{(1)}$ is the neutral mode of the inviscid linear problem and $V_1^{(2)}$ takes into account the $\tilde{X}$–dependence of the solution. We note that

$$V_1^{(1)} = B_\pm(\tilde{X}) V(Y) \quad \text{and} \quad V(Y) \sim |Y - Y_c|^{\frac{1}{2}+\nu} \quad \text{as} \quad (Y - Y_c) \to 0 \quad , \tag{3.4}$$

where the $+$ and $-$ sign on the B denote, respectively, above and below the critical layer. The second term in the expansion of V_1 satisfies the equation

$$L V_1^{(2)} = Q_1 \quad , \tag{3.5}$$

subject to the boundary conditions $V_1^{(2)}(Y = 0) = 0$ and $V_1^{(2)} \to 0$ as $Y \to \infty$, where the operator L is given by

$$L = \left[\frac{\partial^2}{\partial Y^2} + \left(-a^2 - \frac{\overline{q}_0''}{\overline{q}_0} + \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) \right] \quad , \tag{3.6}$$

and the right hand side, Q_1, by

$$Q_1 = \left[-\frac{\overline{q}_1}{\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \left(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} \right) + \frac{(\overline{q}_1'' + a^2 \overline{q}_1)}{\overline{q}_0} \right] V_1^{(1)}$$
$$+ \left[-\frac{\overline{u}_0}{i\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) + \frac{(\overline{u}_0'' + a^2 \overline{u}_0)}{i\overline{q}_0} \right] \frac{\partial V_1^{(1)}}{\partial \tilde{X}} \quad . \tag{3.7}$$

The solution to this equation can be considered to be the sum, $V_1^{(2)} = V_{1PI}^{(2)} + V_{1CF}^{(2)}$, of a particular integral, $V_{1PI}^{(2)}$, and the complementary function, $V_{1CF}^{(2)}$. As $Y - Y_c \to 0$

$$V_{1CF}^{(2)} = B_\pm a_{1\pm}^{(2)} |Y - Y_c|^{\frac{1}{2}+\nu} (1 + O(|Y - Y_c|^{-1})) + B_\pm b_{1\pm}^{(2)} |Y - Y_c|^{\frac{1}{2}-\nu} (1 + O(|Y - Y_c|^{-1})) \quad , \tag{3.8}$$

where $a_{1\pm}^{(2)}$ and $b_{1\pm}^{(2)}$ are constants as yet undetermined. Note that if the Frobenius roots, $\frac{1}{2} \pm |\nu|$ differ by an integer then equation (3.8) is no longer appropriate (logarithms are needed). As such cases ($\nu = \frac{1}{2}m; \quad m$ an integer) are isolated, we choose not to concern ourselves with them (and their immediate neighborhood) in this paper.

A solvability condition for the above boundary–value problem is required. Note that: (i) the operator L is self–adjoint away from the critical level $Y = Y_c$ (where $\overline{q}_0 = 0$), and (ii) the right–hand side of (3.5) is singular at $Y = Y_c$.

Following the method of Hickernell (1984) we derive the solvability condition by multiplying both sides of equation (3.5) by $V_1^{(1)}$ and integrating over all Y, excluding the (sole) critical layer at $Y = Y_c$. After integrating by parts; imposing the boundary conditions at $Y = 0, \infty$; and the asymptotic forms of $V_1^{(1)}$ and $V_{1CF}^{(2)}$ as $Y \to Y_c$, we find that

$$\int_0^\infty V_1^{(1)} Q_1 = -[V_1^{(1)} V_{1CF}^{(2)'} - V_1^{(1)'} V_{1CF}^{(2)}]_{Y_c-}^{Y_c+} \equiv 2\nu [B_+^2 b_{1+}^{(2)} + B_-^2 b_{1-}^{(2)}]$$

$$\int_0^\infty V_1^{(1)} Q_1 = 2\nu B_+^2 (b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)}) \quad , \tag{3.9}$$

where the barred integral signs denote the finite parts of these integrals and we have used the relationships $B_- = i^{-1-2\nu} B_+$, $B_+ \equiv B$ (see §4.1). Manipulating the left–hand side of this equation we find that the solvability condition becomes

$$(I_3 - i^{-4\nu} I_1) \frac{\partial B}{\partial \tilde{X}} + (I_4 - i^{-4\nu} I_2) B = 2\nu B (b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)}) \quad , \tag{3.10}$$

where I_1, I_2, I_3 and I_4 are given by

$$I_1 = \int_0^{Y_c} V(Y) \left[-\frac{\overline{u}_0}{i\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) + \frac{(\overline{u}_0'' + a^2 \overline{u}_0)}{i\overline{q}_0} \right] V(Y) dY \quad , \tag{3.11a-d}$$

$$I_2 = \int_0^{Y_c} V(Y) \left[-\frac{\overline{q}_1}{\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \left(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} \right) + \frac{(\overline{q}_1'' + a^2 \overline{q}_1)}{\overline{q}_0} \right] V(Y) dY \quad ,$$

$$I_3 = \int_{Y_c}^\infty V(Y) \left[-\frac{\overline{u}_0}{i\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) + \frac{(\overline{u}_0'' + a^2 \overline{u}_0)}{i\overline{q}_0} \right] V(Y) dY \quad ,$$

$$I_4 = \int_{Y_c}^\infty V(Y) \left[-\frac{\overline{q}_1}{\overline{q}_0} \left(\frac{\partial^2}{\partial Y^2} - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \right) - \frac{a^2 K_0 \overline{u}_0 \overline{u}_0'}{(i\overline{q}_0)^2} \left(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0} \right) + \frac{(\overline{q}_1'' + a^2 \overline{q}_1)}{\overline{q}_0} \right] V(Y) dY \quad .$$

3.3 The Asymptotic Expansions as $(Y - Y_c) \to 0$

In terms of the critical layer variable $\eta = \mu^{-1}(Y - Y_c)$ we find that the asymptotes for V and Φ, as $(Y - Y_c) \to 0$, where the function Φ is given by

$$\Phi_l = V_l - \frac{l a^2}{(\frac{1}{2} - \nu) i \overline{q}_0'} P_l \quad , \tag{3.12}$$

(l denotes the harmonic), for the fundamental, zeroth and second harmonics are

$$V_1 = \epsilon\mu^{\frac{1}{2}+\nu}\left[B_\pm|\eta|^{\frac{1}{2}+\nu} + \cdots\right] + \epsilon\mu^{\frac{3}{2}+\nu}\left[B_\pm|\eta|^{\frac{3}{2}+\nu}\left(\frac{\overline{q}_0''}{\overline{q}_0'(1+2\nu)} + \frac{(1-2\nu)}{4}\left(\frac{\overline{q}_0''}{\overline{q}_0'} - \frac{\overline{u}_0''}{\overline{u}_0'} - \frac{\overline{u}_0'}{\overline{u}_0}\right)\right)\right.$$

$$\left. + B_\pm|\eta|^{\frac{1}{2}+\nu}a_{1\pm}^{(2)} + \cdots\right] + \epsilon\mu^{\frac{3}{2}-\nu}\left[B_\pm|\eta|^{\frac{1}{2}-\nu}b_{1\pm}^{(2)}\right] + \cdots \quad , \tag{3.13a}$$

$$\Phi_1 = \epsilon\mu^{\frac{1}{2}+\nu}[0] + \epsilon\mu^{\frac{3}{2}+\nu}\left[B_\pm|\eta|^{\frac{3}{2}+\nu}\left(\frac{\overline{q}_0''}{(1+2\nu)\overline{q}_0'} - \frac{\overline{u}_0''}{2\overline{u}_0'} - \frac{\overline{u}_0'}{2\overline{u}_0}\right)\right]$$

$$+ \epsilon\mu^{\frac{3}{2}-\nu}\left[|\eta|^{\frac{1}{2}-\nu}\frac{-2\nu b_{1\pm}^{(2)}B_\pm}{(\frac{1}{2}-\nu)}\right] + \cdots \quad , \tag{3.13b}$$

$$V_0 = \epsilon^2\mu^{-1+2\nu}\left[0\right] + \cdots \quad , \tag{3.14}$$

$$V_2 = \epsilon^2\mu^{-1+2\nu}\left[-\frac{(1+2\nu)}{(3-2\nu)i\overline{q}_0'}B_\pm^2|\eta|^{-1+2\nu} + \cdots\right] + \cdots \quad , \tag{3.15a}$$

$$\Phi_2 = \epsilon^2\mu^{-1+2\nu}\left[0\right] + \cdots \quad . \tag{3.15b}$$

4 The Critical Layer

The main purpose of this section is to derive a second relation between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ (the first being given by the solvability condition derived in the last section, 3.10) in order that we can obtain the desired nonlinear evolution equation. Denoting functions of η by a hat we find that the governing equations in the critical layer are

$$\hat{U}_X + \hat{V}_\eta + \hat{W}_Z = -\mu[\hat{U}_{\tilde{X}}] \quad , \tag{4.1}$$

$$\overline{u}_0\hat{U}_{\tilde{X}} + (\eta\overline{u}_0' + \overline{u}_1)\hat{U}_X + \overline{u}_0'\hat{V} + (\eta\lambda_0\overline{w}_0' + \lambda_0\overline{w}_1 + \lambda_1\overline{w}_0)\hat{U}_Z = -\mu^{-2}[\hat{U}\hat{U}_X + \mu\hat{U}\hat{U}_{\tilde{X}} + \hat{V}\hat{U}_\eta$$

$$+\hat{W}\hat{U}_Z] - \mu[(\eta\overline{u}_0' + \overline{u}_1)\hat{U}_{\tilde{X}} + (\frac{1}{2}\eta^2 i\overline{q}_0'' + \eta i\overline{q}_1' + i\overline{q}_2)l\hat{U} + (\eta\overline{u}_0'' + \overline{u}_1')\hat{V}] + \kappa\mu^{-3}[\hat{U}_{\eta\eta}] \quad , \tag{4.2}$$

$$K_0\overline{u}_0\hat{U} + \hat{P}_\eta = -\mu[(\eta K_0\overline{u}_0' + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}] - \mu^{-1}[\frac{1}{2}K_0\hat{U}^2] \quad , \tag{4.3}$$

$$\overline{u}_0\hat{W}_{\tilde{X}} + (\eta\overline{u}_0' + \overline{u}_1)\hat{W}_X + \lambda_0\overline{w}_0'\hat{V} + (\eta\lambda_0\overline{w}_0' + \lambda_0\overline{w}_1 + \lambda_1\overline{w}_0)\hat{W}_Z + \hat{P}_Z =$$

$$-\mu^{-2}[\hat{U}\hat{W}_X + \mu\hat{U}\hat{W}_{\tilde{X}} + \hat{V}\hat{W}_\eta + \hat{W}\hat{W}_Z] - \mu[(\eta\overline{u}_0' + \overline{u}_1)\hat{W}_{\tilde{X}} + (\frac{1}{2}\eta^2 i\overline{q}_0'' + \eta i\overline{q}_1' + i\overline{q}_2)l\hat{W}$$

$$+(\eta\lambda_0\overline{w}_0^{''} + \lambda_1\overline{w}_0^{'} + \lambda_0\overline{w}_1^{'})\hat{V}] + \kappa\mu^{-3}[\hat{W}_{\eta\eta}] \quad , \tag{4.4}$$

where we have retained on the right hand sides the leading order effects due to nonlinearity, viscosity and terms leading to a perturbation of the generalized Richardson number of Blackaby & Choudhari (1993), see equation (2.9).

In the rest of this section we shall solve the governing equations for the relevant higher order terms of the harmonics. As suggested by the work of Blackaby, Dando & Hall (1993) we consider in this initial work two evolution equations, a non–viscous–cubic and a viscous–cubic. In this section we shall concentrate on deriving the non–viscous–cubic which is the equivalent of the J_1–cubic from this previous paper. In order to do this it is helpful to introduce the operator

$$\hat{N}_{l,\chi} = \left[\overline{u}_0\frac{\partial}{\partial\hat{X}} + l(\eta i\overline{q}_0^{'} + i\overline{q}_1)\right]\frac{\partial}{\partial\eta} - l\chi i\overline{q}_0 \quad , \tag{4.5}$$

where l again denotes the harmonic. We have assumed that $\kappa\mu^{-3} \ll 1$, ie. viscous effects are not large enough to affect the operator, $\hat{N}_{l,\chi}$, at leading order.

The solution inside the critical layer is again constructed in the form of a Fourier series; we expand the fundamental, zeroth and second harmonics, respectively, as follows

$$\hat{V}_1 = \epsilon\mu^{\frac{1}{2}+\nu}\hat{V}_1^{(1)} + \cdots + \epsilon^3\mu^{-\frac{5}{2}+3\nu}\hat{V}_1^{(2)} + \cdots + \epsilon\mu^{\frac{3}{2}+\nu}\hat{V}_1^{(3a)} + \cdots$$

$$+\epsilon\mu^{\frac{3}{2}-\nu}\hat{V}_1^{(3b)} + \cdots + \epsilon^3\mu^{-\frac{3}{2}+3\nu}\hat{V}_1^{(4)} + \cdots \quad , \tag{4.6a}$$

$$\hat{V}_0 = \epsilon^2\mu^{-1+2\nu}\hat{V}_0^{(1)} + \cdots + \epsilon^2\mu^{2\nu}\hat{V}_0^{(2)} + \cdots \quad , \tag{4.6b}$$

$$\hat{V}_2 = \epsilon^2\mu^{-1+2\nu}\hat{V}_2^{(1)} + \cdots + \epsilon^2\mu^{2\nu}\hat{V}_2^{(2)} + \cdots \quad , \tag{4.6c}$$

and similarly for the $\hat{U}$'s, $\hat{W}$'s, $\hat{P}$'s and $\hat{\Phi}$'s. These expansions are not necessarily completely ordered (depending on the relative sizes of ϵ and μ) and moreover we have only retained the terms necessary for deriving the desired evolution equation. The scalings follow directly from the outer asymptotes and/or by considering the process of harmonic generation.

4.1 $\mathbf{O}(\epsilon\mu^{\frac{1}{2}+\nu})$ of the fundamental

At this order the governing equations give

$$\beta\hat{U}_1^{(1)} + \hat{V}_{1\eta}^{(1)} + ia\hat{W}_1^{(1)} = 0 \quad , \tag{4.7a}$$

$$\overline{u}_0\hat{U}_{1\hat{X}}^{(1)} + (\eta i\overline{q}_0^{'} + i\overline{q}_1)\hat{U}_1^{(1)} + \overline{u}_0^{'}\hat{V}_1^{(1)} = 0 \quad , \tag{4.7b}$$

$$K_0\overline{u}_0\hat{U}_1^{(1)} + \hat{P}_{1\eta}^{(1)} = 0 \quad , \tag{4.7c}$$

$$\overline{u}_0\hat{W}_{1\hat{X}}^{(1)} + (\eta i\overline{q}_0^{'} + i\overline{q}_1)\hat{W}_1^{(1)} + \lambda_0\overline{w}_0^{'}\hat{V}_1^{(1)} + ia\hat{P}_1^{(1)} = 0 \quad . \tag{4.7d}$$

From these four equations we find that

$$\hat{N}_{1,\frac{1}{2}+\nu}\hat{V}_1^{(1)} = \frac{1}{2}(1-2\nu)i\overline{q}_0'\hat{\Phi}_1^{(1)} \quad , \tag{4.8a}$$

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}_1^{(1)} = 0 \quad , \tag{4.8b}$$

and a solution of these two equations which matches to the outer solution is

$$\hat{V}_1^{(1)}(\tilde{X},\eta) = \frac{(1+2\nu)i^{\frac{3}{2}-\nu}}{2\Gamma(\frac{1}{2}-\nu)}\left(\frac{\overline{u}_0}{\overline{q}_0'}\right)^{\frac{1}{2}+\nu}\int_0^\infty dt B(\tilde{X}-t)t^{-\frac{3}{2}-\nu}\exp\left\{-it\left[\eta\frac{\overline{q}_0'}{\overline{u}_0}+\frac{\overline{q}_1}{\overline{u}_0}\right]\right\} \quad , \tag{4.9}$$

$$\hat{\Phi}_1^{(1)} = 0 \quad . \tag{4.10}$$

For a detailed discussion of the solution of equations of the type of (4.8a,b) the reader is directed to Churilov & Shukhman (1988) and Dando (1993). Note however, that for the Görtler problem because of the range of ν ($\nu < -\frac{1}{2}$) we do not have to consider the integral on the complex contour C as was required for the stratified shear flow problem. Matching with the asymptotes outside the critical layer yields

$$B_- = i^{-1-2\nu}B_+ \quad , \quad (B_+ \equiv B) \quad . \tag{4.11}$$

The function $\hat{V}_1^{(1)}(\tilde{X},\eta)$ has a single asymptotic representation in the lower–half plane ($-\pi \leq \arg\eta \leq 0$)

$$\hat{V}_1^{(1)}(\tilde{X},\eta) = B(\tilde{X})\eta^{\frac{1}{2}+\nu} + O(\eta^{-\frac{1}{2}+\nu}) \qquad \text{as} \qquad |\eta| \to \infty \quad . \tag{4.12}$$

Later we shall derive evolution equations for the amplitude, $B(\tilde{X})$, but for the moment we can regard it as an arbitrary function that satisfies the requirement $B(\tilde{X}) \to 0$ as $\tilde{X} \to -\infty$. This requirement is consistent with the initial condition used for our evolution equation (see §6), which itself is a result of insisting that the solution of the evolution equation matches to an 'earlier' linear stage.

4.2 $\mathbf{O}(\epsilon^2\mu^{-1+2\nu})$ of the second harmonic

Equations (4.1–4) give at this order

$$2\beta\hat{U}_2^{(1)} + \hat{V}_{2\eta}^{(1)} + 2ia\hat{W}_2^{(1)} = 0 \quad , \tag{4.13a}$$

$$\overline{u}_0\hat{U}_{2\tilde{X}}^{(1)} + 2(\eta i\overline{q}_0' + i\overline{q}_1)\hat{U}_2^{(1)} + \overline{u}_0'\hat{V}_2^{(1)} = -[\beta\hat{U}_1^{(1)}\hat{U}_1^{(1)} + \hat{V}_1^{(1)}\hat{U}_{1\eta}^{(1)} + ia\hat{W}_1^{(1)}\hat{U}_1^{(1)}] \quad , \tag{4.13b}$$

$$K_0\overline{u}_0\hat{U}_2^{(1)} + \hat{P}_{2\eta}^{(1)} = 0 \quad , \tag{4.13c}$$

$$\overline{u}_0\hat{W}_{2\tilde{X}}^{(1)} + 2(\eta i\overline{q}_0' + i\overline{q}_1)\hat{W}_2^{(1)} + \lambda_0\overline{w}_0'\hat{V}_2^{(1)} + ia\hat{P}_2^{(1)} =$$

$$-[\beta\hat{U}_1^{(1)}\hat{W}_1^{(1)}+\hat{V}_1^{(1)}\hat{W}_{1\eta}^{(1)}+ia\hat{W}_1^{(1)}\hat{W}_1^{(1)}] \quad , \qquad (4.13d)$$

from which we can obtain the two equations

$$\hat{N}_{2,\frac{1}{2}+\nu}\hat{V}_2^{(1)}=(1-2\nu)i\overline{q}_0'\hat{\Phi}_2^{(1)}+2(\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta}^{(1)}-\hat{V}_1^{(1)}\hat{V}_{1\eta\eta}^{(1)}) \quad , \qquad (4.14a)$$

$$\hat{N}_{2,\frac{1}{2}-\nu}\hat{\Phi}_2^{(1)}=0 \quad . \qquad (4.14b)$$

We note that the right–hand sides of these last two equations do not involve the conjugate of $\hat{V}_1^{(1)}$ and so $\hat{V}_2^{(1)}$ and $\hat{\Phi}_2^{(1)}$ have unique asymptotic representations as $|\eta|\to\infty$ (in the lower half plane of complex η). A solution of these equations which matches to the outside is

$$\hat{V}_2^{(1)}=\frac{i^{-2\nu}(1+2\nu)^2}{4\overline{u}_0\Gamma^2(\frac{1}{2}-\nu)}\left(\frac{\overline{u}_0}{\overline{q}_0'}\right)^{2\nu}\int_0^\infty dt\int_0^\infty dt_1\int_0^\infty dt_2 B(\tilde{X}-t-t_1)B(\tilde{X}-t-t_2)(t_1t_2)^{-\frac{3}{2}-\nu}\times$$

$$(t_1-t_2)^2(t_1+t_2)^{\frac{1}{2}+\nu}(2t+t_1+t_2)^{-\frac{3}{2}-\nu}\exp\left\{-i(2t+t_1+t_2)\left[\eta\frac{\overline{q}_0'}{\overline{u}_0}+\frac{\overline{q}_1}{\overline{u}_0}\right]\right\} \quad , \qquad (4.15a)$$

$$\hat{\Phi}_2^{(1)}=0 \quad , \qquad (4.15b)$$

(again see Churilov & Shukhman 1988 and Dando 1993 for a discussion of the solution of equations like 4.14a where there is a non–zero right hand side).

4.3 O($\epsilon^2\mu^{-1+2\nu}$) of the zeroth harmonic

Here we find the governing equations are

$$\hat{V}_{0\eta}^{(1)}=0 \quad , \qquad (4.16a)$$

$$\overline{u}_0\hat{U}_{0\tilde{X}}^{(1)}+\overline{u}_0'\hat{V}_0^{(1)}=-[\hat{V}_1^{(1)}\hat{U}_{-1\eta}^{(1)}+\hat{V}_{-1}^{(1)}\hat{U}_{1\eta}^{(1)}-ia\hat{W}_1^{(1)}\hat{U}_{-1}^{(1)}+ia\hat{W}_{-1}^{(1)}\hat{U}_1^{(1)}] \quad , \qquad (4.16b)$$

$$K_0\overline{u}_0\hat{U}_0^{(1)}+\hat{P}_{0\eta}^{(1)}=0 \quad , \qquad (4.16c)$$

$$\overline{u}_0\hat{W}_{0\tilde{X}}^{(1)}+\lambda_0\overline{w}_0'\hat{V}_0^{(1)}=-[-\beta\hat{U}_1^{(1)}\hat{W}_{-1}^{(1)}+\beta\hat{U}_{-1}^{(1)}\hat{W}_1^{(1)}+\hat{V}_1^{(1)}\hat{W}_{-1\eta}^{(1)}+\hat{V}_{-1}^{(1)}\hat{W}_{1\eta}^{(1)}] \quad , \qquad (4.16d)$$

where we have used the notation $\hat{V}_{-1}^{(1)}$ for example to denote the complex conjugate of $\hat{V}_1^{(1)}$. For our later working it is necessary to know $\hat{U}_0^{(1)}$, $\hat{V}_0^{(1)}$ and $\hat{W}_0^{(1)}$. From the above four equations and (4.7a-d) we find that

$$\hat{U}_0^{(1)}=-\frac{(1-2\nu)}{a^2K_0\overline{u}_0(1+2\nu)}\frac{\partial}{\partial\eta}(\hat{V}_{-1\eta}^{(1)}\hat{V}_{1\eta}^{(1)}) \quad , \qquad (4.17a)$$

$$\hat{V}_0^{(1)}=0 \quad , \qquad (4.17b)$$

$$\hat{W}_0^{(1)}=\left(-\frac{2}{ia(1+2\nu)i\overline{q}_0'}+\frac{\beta(1-2\nu)}{ia(1+2\nu)a^2K_0\overline{u}_0}\right)\frac{\partial}{\partial\eta}(\hat{V}_{-1\eta}^{(1)}\hat{V}_{1\eta}^{(1)}) \quad . \qquad (4.17c)$$

4.4 $\mathbf{O}(\epsilon^3\mu^{-\frac{5}{2}+3\nu})$ of the fundamental

This is the order of the largest nonlinear term and for many critical–layer problems we would expect to obtain our jump here. However, the four governing equations are

$$\beta\hat{U}_1^{(2)}+\hat{V}_{1\eta}^{(2)}+ia\hat{W}_1^{(2)}=0 \quad , \tag{4.18a}$$

$$\begin{aligned}\overline{u}_0\hat{U}_{1\tilde{X}}^{(2)}+(\eta i\overline{q}_0'+i\overline{q}_1)\hat{U}_1^{(2)}+\overline{u}_0'\hat{V}_1^{(2)}=-[\beta\hat{U}_2^{(1)}\hat{U}_{-1}^{(1)}+\beta\hat{U}_0^{(1)}\hat{U}_1^{(1)}+\hat{V}_2^{(1)}\hat{U}_{-1\eta}^{(1)}+\hat{V}_{-1}^{(1)}\hat{U}_{2\eta}^{(1)}+\hat{V}_1^{(1)}\hat{U}_{0\eta}^{(1)}\\+\hat{V}_0^{(1)}\hat{U}_{1\eta}^{(1)}-ia\hat{W}_2^{(1)}\hat{U}_{-1}^{(1)}+2ia\hat{W}_{-1}^{(1)}\hat{U}_2^{(1)}+ia\hat{W}_0^{(1)}\hat{U}_1^{(1)}] \quad ,\end{aligned} \tag{4.18b}$$

$$K_0\overline{u}_0\hat{U}_1^{(2)}+\hat{P}_{1\eta}^{(2)}=0 \quad , \tag{4.18c}$$

$$\begin{aligned}\overline{u}_0\hat{W}_{1\tilde{X}}^{(2)}+(\eta i\overline{q}_0'+i\overline{q}_1)\hat{W}_1^{(2)}+\lambda_0\overline{w}_0'\hat{V}_1^{(2)}+ia\hat{P}_1^{(2)}=-[-\beta\hat{U}_2^{(1)}\hat{W}_{-1}^{(1)}+2\beta\hat{U}_{-1}^{(1)}\hat{W}_2^{(1)}+\beta\hat{U}_0^{(1)}\hat{W}_1^{(1)}\\+\hat{V}_2^{(1)}\hat{W}_{-1\eta}^{(1)}+\hat{V}_{-1}^{(1)}\hat{W}_{2\eta}^{(1)}+\hat{V}_1^{(1)}\hat{W}_{0\eta}^{(1)}+\hat{V}_0^{(1)}\hat{W}_{1\eta}^{(1)}+ia\hat{W}_2^{(1)}\hat{W}_{-1}^{(1)}+ia\hat{W}_0^{(1)}\hat{W}_1^{(1)}] \quad ,\end{aligned} \tag{4.18d}$$

and from these we can derive the equation

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}_1^{(2)}=0 \quad . \tag{4.19}$$

The solution of this which matches to the outside is

$$\hat{\Phi}_1^{(2)}=0 \quad , \tag{4.20}$$

and so we do not obtain a jump from this largest nonlinear term (this is exactly as expected from the corresponding work on stratified shear layers of Churilov & Shukhman 1988 and Blackaby, Dando & Hall 1993).

4.5 $\mathbf{O}(\epsilon\mu^{\frac{3}{2}+\nu})$ of the fundamental

It is at this order that we first get terms on the right–hand sides of the governing equations due to the perturbation of the 'Richardson number'. However, the situation is much more complicated than for the stratified shear flow case as instead of just perturbing the Richardson number we now have to perturb the quantities in our generalized Richardson number, equation (2.9). We also have $\tilde{X}$–derivatives of previous critical–layer terms and higher order corrections to the base flow values appearing on the right–hand sides. Neither of these two effects were present for the stratified shear layer case considered in Blackaby, Dando & Hall (1993) and they are a result of considering a spatial as opposed to a temporal evolution problem. These effects combined with the need to perturb $\overline{u}$, $\overline{w}$, λ and K, cause most formulas from this point on to be considerably longer than their counterparts for the stratified shear flow problem. Specifically (4.1–4) give

$$\beta\hat{U}_1^{(3a)}+\hat{V}_{1\eta}^{(3a)}+ia\hat{W}_1^{(3a)}=-\hat{U}_{1\tilde{X}}^{(1)} \quad , \tag{4.21a}$$

$$\overline{u}_0\hat{U}^{(3a)}_{1\tilde{X}} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{U}^{(3a)}_1 + \overline{u}_0'\hat{V}^{(3a)}_1 =$$

$$-[(\eta\overline{u}_0' + \overline{u}_1)\hat{U}^{(1)}_{1\tilde{X}} + (\frac{1}{2}\eta^2 i\overline{q}_0'' + \eta i\overline{q}_1' + i\overline{q}_2)\hat{U}^{(1)}_1 + (\eta\overline{u}_0'' + \overline{u}_1')\hat{V}^{(1)}_1] \quad , \quad (4.21b)$$

$$K_0\overline{u}_0\hat{U}^{(3a)}_1 + \hat{P}^{(3a)}_{1\eta} = -[(\eta K_0\overline{u}_0' + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}^{(1)}_1] \quad , \quad (4.21c)$$

$$\overline{u}_0\hat{W}^{(3a)}_{1\tilde{X}} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{W}^{(3a)}_1 + \lambda_0\overline{w}_0'\hat{V}^{(3a)}_1 + ia\hat{P}^{(3a)}_1 =$$

$$-[(\eta\overline{u}_0' + \overline{u}_1)\hat{W}^{(1)}_{1\tilde{X}} + (\frac{1}{2}\eta^2 i\overline{q}_0'' + \eta i\overline{q}_1' + i\overline{q}_2)\hat{W}^{(1)}_1 + (\eta\lambda_0\overline{w}_0'' + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1')\hat{V}^{(1)}_1] \quad . \quad (4.21d)$$

From these four equations we can obtain the equation for $\hat{\Phi}^{(3a)}_1$, namely

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}^{(3a)}_1 = \overline{u}_0'\hat{V}^{(1)}_{1\tilde{X}} + i\overline{q}_0'\Bigg[\eta\Bigg\{\frac{\overline{q}_0''}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\Bigg(\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{\overline{u}_0'}{\overline{u}_0}\Bigg)\Bigg\}$$

$$+\Bigg\{\frac{\overline{q}_1'}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\Bigg(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0}\Bigg)\Bigg\}\Bigg]\hat{V}^{(1)}_1 \quad , \quad (4.22)$$

and instead of solving this explicitly we find it easier to write the solution in terms of previously calculated functions;

$$\hat{\Phi}^{(3a)}_1 = \Bigg[\eta\Bigg\{\Bigg(\frac{\overline{q}_0''}{(1+2\nu)\overline{q}_0'} - \frac{\overline{u}_0''}{2\overline{u}_0'} - \frac{\overline{u}_0'}{2\overline{u}_0}\Bigg)\Bigg\} + \Bigg\{\frac{1}{2\nu}\Bigg(-\frac{\overline{q}_1\overline{q}_0''}{(1+2\nu)\overline{q}_0'^2} + \frac{\overline{q}_1'}{\overline{q}_0'} + \frac{\overline{q}_1}{2\overline{q}_0'}\Bigg(\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{\overline{u}_0'}{\overline{u}_0}\Bigg)$$

$$-\frac{(1+2\nu)}{2}\Bigg(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0}\Bigg)\Bigg)\Bigg\}\Bigg]\hat{V}^{(1)}_1 + \Bigg[\frac{1}{2\nu i\overline{q}_0'}\Bigg(\frac{\overline{u}_0\overline{u}_0''}{2\overline{u}_0'} + \frac{3\overline{u}_0'}{2} - \frac{\overline{u}_0\overline{q}_0''}{(1+2\nu)\overline{q}_0'}\Bigg)\Bigg]\hat{V}^{(1)}_{1\tilde{X}} \quad . \quad (4.23)$$

4.6 O($\epsilon\mu^{\frac{3}{2}-\nu}$) of the fundamental

The governing equations give

$$\beta\hat{U}^{(3b)}_1 + \hat{V}^{(3b)}_{1\eta} + ia\hat{W}^{(3b)}_1 = 0 \quad , \quad (4.24a)$$

$$\overline{u}_0\hat{U}^{(3b)}_{1\tilde{X}} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{U}^{(3b)}_1 + \overline{u}_0'\hat{V}^{(3b)}_1 = 0 \quad , \quad (4.24b)$$

$$K_0\overline{u}_0\hat{U}^{(3b)}_1 + \hat{P}^{(3b)}_{1\eta} = 0 \quad , \quad (4.24c)$$

$$\overline{u}_0\hat{W}^{(3b)}_{1\tilde{X}} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{W}^{(3b)}_1 + \lambda_0\overline{w}_0'\hat{V}^{(3b)}_1 + ia\hat{P}^{(3b)}_1 = 0 \quad , \quad (4.24d)$$

and from these we obtain

$$\hat{N}_{1,\frac{1}{2}+\nu}\hat{V}^{(3b)}_1 = \frac{1}{2}(1-2\nu)i\overline{q}_0'\hat{\Phi}^{(3b)}_1 \quad , \quad (4.25a)$$

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}^{(3b)}_1 = 0 \quad . \quad (4.25b)$$

These two equations have solutions

$$\hat{V}_1^{(3b)} = \hat{\Phi}_1^{(3b)} = 0 \quad , \tag{4.26}$$

and this implies that there is no linear contribution to the evolution equation from inside the critical layer, instead it all comes from outside the critical layer. Later we shall balance our selected nonlinear term with this order and then derive our second relation involving $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ by matching with the outside asymptotes.

4.7 O($\epsilon^2\mu^{2\nu}$) of the second harmonic

At this order equations (4.1–4) give

$$2\beta\hat{U}_2^{(2)} + \hat{V}_{2\eta}^{(2)} + 2ia\hat{W}_2^{(2)} = -\hat{U}_{2\tilde{X}}^{(1)} \quad , \tag{4.27a}$$

$$\begin{aligned}\overline{u}_0\hat{U}_{2\tilde{X}}^{(2)} + 2(\eta i\overline{q}_0' + i\overline{q}_1)\hat{U}_2^{(2)} + \overline{u}_0'\hat{V}_2^{(2)} = \\ -\Big[2\beta\hat{U}_1^{(1)}\hat{U}_1^{(3a)} + \hat{U}_1^{(1)}\hat{U}_{1\tilde{X}}^{(1)} + \hat{V}_1^{(1)}\hat{U}_{1\eta}^{(3a)} + \hat{V}_1^{(3a)}\hat{U}_{1\eta}^{(1)} + ia\hat{W}_1^{(1)}\hat{U}_1^{(3a)} + ia\hat{W}_1^{(3a)}\hat{U}_1^{(1)}\Big] \\ -\Big[(\eta\overline{u}_0' + \overline{u}_1)\hat{U}_{2\tilde{X}}^{(1)} + (\eta^2 i\overline{q}_0'' + 2\eta i\overline{q}_1' + 2i\overline{q}_2)\hat{U}_2^{(1)} + (\eta\overline{u}_0'' + \overline{u}_1')\hat{V}_2^{(1)}\Big] \quad ,\end{aligned} \tag{4.27b}$$

$$K_0\overline{u}_0\hat{U}_2^{(2)} + \hat{P}_{2\eta}^{(2)} = -[\frac{1}{2}K_0\hat{U}_1^{(1)}\hat{U}_1^{(1)}] - [(\eta K_0\overline{u}_0' + K_1\overline{u}_0 + K_0\overline{u}_1)\hat{U}_2^{(1)}] \quad , \tag{4.27c}$$

$$\begin{aligned}\overline{u}_0\hat{W}_{2\tilde{X}}^{(2)} + 2(\eta i\overline{q}_0' + i\overline{q}_1)\hat{W}_2^{(2)} + \lambda_0\overline{w}_0'\hat{V}_2^{(2)} + 2ia\hat{P}_2^{(2)} = \\ -\Big[\beta\hat{U}_1^{(1)}\hat{W}_1^{(3a)} + \beta\hat{U}_1^{(3a)}\hat{W}_1^{(1)} + \hat{U}_1^{(1)}\hat{W}_{1\tilde{X}}^{(1)} + \hat{V}_1^{(1)}\hat{W}_{1\eta}^{(3a)} + \hat{V}_1^{(3a)}\hat{W}_{1\eta}^{(1)} + 2ia\hat{W}_1^{(1)}\hat{W}_1^{(3a)}\Big] \\ -\Big[(\eta\overline{u}_0' + \overline{u}_1)\hat{W}_{2\tilde{X}}^{(1)} + (\eta^2 i\overline{q}_0'' + 2\eta i\overline{q}_1' + 2i\overline{q}_2)\hat{W}_2^{(1)} + (\eta\lambda_0\overline{w}_0'' + \lambda_1\overline{w}_0' + \lambda_0\overline{w}_1')\hat{V}_2^{(1)}\Big] \quad .\end{aligned} \tag{4.27d}$$

It is only necessary to find $\hat{\Phi}_2^{(2)}$ to complete our working at the next order of the fundamental and from the above four equations we find that

$$\begin{aligned}\hat{N}_{2,\frac{1}{2}-\nu}\hat{\Phi}_2^{(2)} = \overline{u}_0'\hat{V}_{2\tilde{X}}^{(1)} + 2i\overline{q}_0'\Big[\eta\Big(\frac{\overline{q}_0''}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\Big(\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{\overline{u}_0'}{\overline{u}_0}\Big)\Big) + \Big(\frac{\overline{q}_1}{\overline{q}_0'} - \frac{(1+2\nu)}{2}\Big(\frac{\overline{u}_1'}{\overline{u}_0'} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0}\Big)\Big)\Big]\hat{V}_2^{(1)} \\ + \frac{(1-2\nu)i\overline{q}_0'}{2a^2K_0\overline{u}_0}\Big[2\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta\tilde{X}}^{(1)} - \hat{V}_{1\tilde{X}}^{(1)}\hat{V}_{1\eta\eta}^{(1)} + \hat{V}_1^{(1)}\hat{V}_{1\eta\eta\tilde{X}}^{(1)}\Big] - \frac{2\overline{u}_0'}{\overline{u}_0}\hat{V}_1^{(1)}\hat{V}_{1\eta}^{(1)} + \frac{(1-2\nu)i\overline{q}_0'}{a^2K_0\overline{u}_0^2}(\eta i\overline{q}_0' + i\overline{q}_1)\hat{V}_{1\eta}^{(1)}\hat{V}_{1\eta}^{(1)} \\ + \Big[2\hat{V}_{1\eta}^{(1)}\hat{\Phi}_{1\eta}^{(3a)} - 2\hat{V}_1^{(1)}\hat{\Phi}_{1\eta\eta}^{(3a)}\Big] \quad .\end{aligned} \tag{4.28}$$

We again choose not to solve this explicitly but instead write the solution as

$$\hat{\Phi}_2^{(2)} = \hat{V}_2^{(1)}\Big[\eta\Big(\frac{\overline{q}_0''}{(1+2\nu)\overline{q}_0'} - \frac{\overline{u}_0''}{2\overline{u}_0'} - \frac{\overline{u}_0'}{2\overline{u}_0}\Big) + \frac{1}{2\nu}\Big(-\frac{\overline{q}_1\overline{q}_0''}{(1+2\nu)\overline{q}_0'^2} + \frac{\overline{q}_1'}{\overline{q}_0'} + \frac{\overline{q}_1}{2\overline{q}_0'}\Big(\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{\overline{u}_0'}{\overline{u}_0}\Big) - \frac{(1+2\nu)}{2}\Big(\frac{\overline{u}_1'}{\overline{u}_0'}$$

$$+\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)\Big]+\hat{V}^{(1)}_{2\tilde{X}}\Big[\frac{1}{4\nu i\overline{q}_0'}\Big(\frac{\overline{u}_0\overline{u}_0''}{2\overline{u}_0'}+\frac{3\overline{u}_0'}{2}-\frac{\overline{u}_0\overline{q}_0'}{(1+2\nu)\overline{q}_0'}\Big)\Big]+\hat{V}^{(1)}_{1\eta}\hat{V}^{(1)}_{1\eta}\Big[\frac{1}{\nu(1+2\nu)i\overline{q}_0'}\Big(\eta\frac{\overline{u}_0'}{\overline{u}_0}+\frac{\overline{u}_0'\overline{q}_1}{\overline{u}_0\overline{q}_0'}\Big)\Big]$$

$$+\hat{V}^{(1)}_{1}\hat{V}^{(1)}_{1\eta}\Big[\frac{1}{(1-4\nu)i\overline{q}_0'}\Big(\frac{2\overline{q}_0''}{(1+2\nu)\overline{q}_0'}-\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{\overline{u}_0'}{\overline{u}_0}\Big)\Big]+\hat{V}^{(1)}_{1\eta}\hat{V}^{(1)}_{1\eta\tilde{X}}\Big[\frac{\overline{u}_0'}{\nu(1+2\nu)(i\overline{q}_0')^2}\Big]$$

$$+\Big[\frac{1}{4\nu 1(1-2\nu)(i\overline{q}_0')^2}\Big(\frac{\overline{u}_0\overline{u}_0''}{2\overline{u}_0'}+\frac{(3-2\nu)\overline{u}_0'}{2(1+2\nu)}-\frac{\overline{u}_0\overline{q}_0''}{(1+2\nu)\overline{q}_0'}\Big)\Big]\Big[\hat{V}^{(1)}_{1}\hat{V}^{(1)}_{1\eta\eta\tilde{X}}-\hat{V}^{(1)}_{1\tilde{X}}\hat{V}^{(1)}_{1\eta\eta}\Big] \quad . \qquad (4.29)$$

4.8 $\mathbf{O}(\epsilon^2\mu^{2\nu})$ of the zeroth harmonic

The governing equations give at this order

$$\hat{V}^{(2)}_{0\eta}=0 \quad , \qquad (4.30a)$$

$$\overline{u}_0\hat{U}^{(2)}_{0\tilde{X}}+\overline{u}_0'\hat{V}^{(2)}_0=-\Big[\frac{\partial}{\partial\eta}\Big(\hat{U}^{(3a)}_{-1}\hat{V}^{(1)}_1+\hat{U}^{(3a)}_1\hat{V}^{(1)}_{-1}+\hat{U}^{(1)}_{-1}\hat{V}^{(3a)}_1+\hat{U}^{(1)}_1\hat{V}^{(3a)}_{-1}\Big)+2\Big(\hat{U}^{(1)}_{-1}\hat{U}^{(1)}_{1\tilde{X}}$$

$$+\hat{U}^{(1)}_1\hat{U}^{(1)}_{-1\tilde{X}}\Big)\Big]-\Big[(\eta\overline{u}_0'+\overline{u}_1)\hat{U}^{(1)}_{0\tilde{X}}+(\eta\overline{u}_0''+\overline{u}_1')\hat{V}^{(1)}_0\Big] \quad , \qquad (4.30b)$$

$$K_0\overline{u}_0\hat{U}^{(2)}_0+\hat{P}^{(2)}_{0\eta}=-[K_0\hat{U}^{(1)}_1\hat{U}^{(1)}_{-1}]-[(\eta K_0\overline{u}_0'+K_0\overline{u}_1+K_1\overline{u}_0)\hat{U}^{(1)}_0] \quad , \qquad (4.30c)$$

$$\overline{u}_0\hat{W}^{(2)}_{0\tilde{X}}+\lambda_0\overline{w}_0'\hat{V}^{(2)}_0=-\Big[\frac{\partial}{\partial\eta}\Big(\hat{W}^{(3a)}_{-1}\hat{V}^{(1)}_1+\hat{W}^{(3a)}_1\hat{V}^{(1)}_{-1}+\hat{W}^{(1)}_{-1}\hat{V}^{(3a)}_1+\hat{W}^{(1)}_1\hat{V}^{(3a)}_{-1}\Big)+\Big(\hat{U}^{(1)}_1\hat{W}^{(1)}_{-1\tilde{X}}$$

$$+\hat{U}^{(1)}_{-1}\hat{W}^{(1)}_{1\tilde{X}}+\hat{W}^{(1)}_{-1}\hat{U}^{(1)}_{1\tilde{X}}+\hat{W}^{(1)}_1\hat{U}^{(1)}_{-1\tilde{X}}\Big)\Big]-\Big[(\eta\overline{u}_0'+\overline{u}_1)\hat{W}^{(1)}_{0\tilde{X}}+(\eta\lambda_0\overline{w}_0''+\lambda_1\overline{w}_0'+\lambda_0\overline{w}_1')\hat{V}^{(1)}_0\Big] \quad . \qquad (4.30d)$$

For our working in the next subsection, at order $(\epsilon^3\mu^{-\frac{3}{2}+3\nu})$ of the fundamental, we need to be able to substitute in for

$$\Big(\beta-\frac{2a^2K_0\overline{u}_0}{(1-2\nu)i\overline{q}_0'}\Big)\hat{U}^{(2)}_{0\eta}+ia\hat{W}^{(2)}_{0\eta} \quad , \qquad (4.31)$$

and from the above four equations we find that

$$\Big[\Big(\beta-\frac{2a^2K_0\overline{u}_0}{(1-2\nu)i\overline{q}_0'}\Big)\hat{U}^{(2)}_{0\eta}+ia\hat{W}^{(2)}_{0\eta}\Big]=-\frac{1}{\nu(1+2\nu)i\overline{q}_0'}\frac{\partial^2}{\partial\eta^2}\Big[\Big\{\eta\Big(\frac{\overline{q}_0''}{\overline{q}_0'}-\frac{(1+2\nu)\overline{u}_0''}{2\overline{u}_0'}-\frac{(1-2\nu)\overline{u}_0'}{2\overline{u}_0}$$

$$-\frac{(1+6\nu)\overline{u}_0'}{(1+2\nu)\overline{u}_0}\Big)+\Big(\frac{\overline{q}_1'}{\overline{q}_0'}-\frac{(1+6\nu)\overline{u}_0'\overline{q}_1}{(1+2\nu)\overline{u}_0\overline{q}_0'}-\frac{(1-2\nu)}{2}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)-\frac{(1+2\nu)\overline{u}_1'}{2\overline{u}_0'}\Big)\Big\}\hat{V}^{(1)}_{-1\eta}\hat{V}^{(1)}_{1\eta}\Big] \quad . \qquad (4.32)$$

4.9 $\quad$ $\mathbf{O(\epsilon^3\mu^{-\frac{3}{2}+3\nu})}$ of the fundamental

It is at this order that the non–viscous cubic jump will arise. The governing equations (4.1–4) eventually yield at this order

$$\hat{N}_{1,\frac{1}{2}+\nu}\hat{V}_1^{(4)} = \frac{(1-2\nu)}{2} i\overline{q}_0^{\,\prime}\hat{\Phi}_1^{(4)} + R_1^{(4)} \quad , \tag{4.33a}$$

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}_1^{(4)} = R_2^{(4)} \quad , \tag{4.33b}$$

where in particular

$$R_2^{(4)} = F_1^{(2)} + R_3^{(4)} \quad , \tag{4.34}$$

and $F_1^{(2)}$ is a function proportional to $\hat{V}_1^{(2)}$ whilst $R_3^{(4)}$ is given explicitly by

$$\begin{aligned} R_3^{(4)} = {} & \frac{(1-2\nu)(\eta i\overline{q}_0^{\,\prime} + i\overline{q}_1)}{(1+2\nu)a^2K_0\overline{u}_0^2}\Big[\hat{V}_{1\eta}^{(1)}\Big(\hat{V}_{-1\eta}^{(1)}\hat{V}_{1\eta\eta}^{(1)} + \hat{V}_{-1\eta\eta}^{(1)}\hat{V}_{1\eta}^{(1)}\Big)\Big] + \frac{(1-2\nu)i\overline{q}_0^{\,\prime}}{2a^2K_0\overline{u}_0^2}\Big[(\eta i\overline{q}_0^{\,\prime} + i\overline{q}_1)\Big(\frac{1}{4}\hat{V}_{2\eta}^{(1)}\hat{V}_{-1\eta}^{(1)} \\ & -\frac{1}{2}\hat{V}_{-1}^{(1)}\hat{V}_{2\eta\eta}^{(1)} + \hat{V}_2^{(1)}\hat{V}_{-1\eta\eta}^{(1)}\Big) - \frac{(3-2\nu)i\overline{q}_0^{\,\prime}}{8}\hat{V}_{-1}^{(1)}\hat{V}_{2\eta}^{(1)} + \frac{(1-2\nu)i\overline{q}_0^{\,\prime}}{2}\hat{V}_2^{(1)}\hat{V}_{-1\eta}^{(1)}\Big] + \Big(\eta\frac{\overline{u}_0^{\,\prime}}{\overline{u}_0} + \frac{\overline{u}_1}{\overline{u}_0} + \frac{K_1}{K_0}\Big)\times \\ & \Big[-\frac{1}{2}\hat{V}_{-1}^{(1)}\hat{V}_{2\eta\eta}^{(1)} + \hat{V}_2^{(1)}\hat{V}_{-1\eta\eta}^{(1)} - \frac{1}{2}\hat{V}_{2\eta}^{(1)}\hat{V}_{-1\eta}^{(1)} - \hat{V}_{1\eta}^{(1)}\Big(\beta\hat{U}_0^{(1)} + ia\hat{W}_0^{(1)}\Big)\Big] + \Big[\hat{V}_2^{(1)}\hat{\Phi}_{-1\eta\eta}^{(3a)} + \frac{1}{2}\hat{V}_{2\eta}^{(1)}\hat{\Phi}_{-1\eta}^{(3a)} \\ & -\hat{\Phi}_{1\eta}^{(3a)}(\beta\hat{U}_0^{(1)} + ia\hat{W}_0^{(1)}) - \hat{V}_{-1\eta}^{(1)}\hat{\Phi}_{2\eta}^{(2)} - \frac{1}{2}\hat{V}_{-1}^{(1)}\hat{\Phi}_{2\eta\eta}^{(2)} + \hat{V}_1^{(1)}\Big\{\Big(\beta - \frac{2a^2K_0\overline{u}_0}{(1-2\nu)i\overline{q}_0^{\,\prime}}\Big)\hat{U}_{0\eta}^{(2)} + ia\hat{W}_{0\eta}^{(2)}\Big\}\Big] \\ & + \frac{(1-2\nu)i\overline{q}_0^{\,\prime}}{2a^2K_0\overline{u}_0^2}\Big[\hat{V}_1^{(1)}\Big(\hat{V}_{1\eta}^{(1)}\hat{V}_{-1\eta\eta}^{(1)} - \frac{1}{2}\hat{V}_{-1}^{(1)}\hat{V}_{1\eta\eta\eta}^{(1)}\Big) + \hat{V}_{1\eta}^{(1)}\Big(\hat{V}_{-1\eta}^{(1)}\hat{V}_{1\eta}^{(1)} + \frac{1}{2}\hat{V}_{-1}^{(1)}\hat{V}_{1\eta\eta}^{(1)}\Big)\Big] \quad . \end{aligned} \tag{4.35}$$

To derive an evolution equation the asymptotic representation of $\hat{V}_1^{(4)}$ is needed. This is

$$\hat{V}_1^{(4)} = C_\pm^{(4)}\eta^{\frac{1}{2}+\nu} + D_\pm^{(4)}\eta^{\frac{1}{2}-\nu} + O(\eta^{-\frac{1}{2}+\nu}) \quad \text{as} \quad \eta \to \pm\infty, \tag{4.36}$$

and in particular $D_+^{(4)} - D_-^{(4)}$ is given by

$$\begin{aligned} D_+^{(4)} - D_-^{(4)} = {} & -\frac{i^{\frac{1}{2}-\nu}}{2\nu\overline{u}_0\Gamma(\frac{1}{2}-\nu)}\Big(\frac{\overline{q}_0^{\,\prime}}{\overline{u}_0}\Big)^{\frac{1}{2}-\nu} \quad \times \\ & \int_{-\infty}^{\infty} d\eta \int_0^\infty d\tilde{t}\; \tilde{t}^{\frac{1}{2}-\nu} R_3^{(4)}(\tilde{X} - \tilde{t}, \eta)\exp\Big\{-i\tilde{t}\Big[\eta\frac{\overline{q}_0^{\,\prime}}{\overline{u}_0} + \frac{\overline{q}_1}{\overline{u}_0}\Big]\Big\} \quad . \end{aligned} \tag{4.37}$$

After splitting $R_3^{(4)}$ into three parts: firstly one containing $\tilde{X}$ derivatives; secondly one involving terms of the second harmonic; and finally one that contains neither, we denote the pieces of the jump due to these parts by by subscript 1, 2 and 3 respectively so

$$(D_+^{(4)} - D_-^{(4)}) = (D_+^{(4)} - D_-^{(4)})_1 + (D_+^{(4)} - D_-^{(4)})_2 + (D_+^{(4)} - D_-^{(4)})_3 \quad , \tag{4.38}$$

where the terms on the right–hand side are given in Appendix A. Matching the inner asymptote, (4.36), with the outer asymptote, (3.13a), leads to the relations

$$D_{+}^{(4)} = b_{1+}^{(2)} B_{+} \quad , \quad D_{-}^{(4)} = i^{1-2\nu} b_{1-}^{(2)} B_{-} \quad , \tag{4.39a, b}$$

which can be combined to give

$$(D_{+}^{(4)} - D_{-}^{(4)}) = B(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)}) \quad , \tag{4.40}$$

and so we have found our second relation between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$.

5 The Viscous Cubic

In order to derive a viscous–cubic evolution equation it would be necessary to consider some additional critical layer terms and instead of the expansions (4.6) we would have to consider

$$\hat{V}_1 = \epsilon\mu^{\frac{1}{2}+\nu}\hat{V}_1^{(1)} + \cdots + \epsilon^3\mu^{-\frac{5}{2}+3\nu}\hat{V}_1^{(2)} + \cdots + \epsilon\mu^{\frac{3}{2}+\nu}\hat{V}_1^{(3a)} + \cdots + \epsilon\mu^{\frac{3}{2}-\nu}\hat{V}_1^{(3b)} + \cdots$$

$$+\epsilon^3\mu^{-\frac{3}{2}+3\nu}\hat{V}_1^{(2)} + \cdots + \epsilon\kappa\mu^{-\frac{5}{2}+\nu}\hat{V}_1^{(5)} + \cdots + \epsilon^3\kappa\mu^{-\frac{11}{2}+3\nu}\hat{V}_1^{(6)} + \cdots \quad , \tag{5.1a}$$

$$\hat{V}_0 = \epsilon^2\mu^{-1+2\nu}\hat{V}_0^{(1)} + \cdots + \epsilon^2\mu^{2\nu}\hat{V}_0^{(2)} + \cdots + \epsilon^2\kappa\mu^{-4+2\nu}\hat{V}_0^{(3)} + \cdots \quad , \tag{5.1b}$$

$$\hat{V}_2 = \epsilon^2\mu^{-1+2\nu}\hat{V}_2^{(1)} + \cdots + \epsilon^2\mu^{2\nu}\hat{V}_2^{(2)} + \cdots + \epsilon^2\kappa\mu^{-4+2\nu}\hat{V}_2^{(3)} + \cdots \quad , \tag{5.1c}$$

We would obtain the viscous–cubic by balancing the nonlinear term at $\mathrm{O}(\epsilon^3\kappa\mu^{-\frac{11}{2}+3\nu})$ with the term at $\mathrm{O}(\epsilon\mu^{\frac{3}{2}-\nu})$ and then matching with the outside asymptotes in order to get our second relationship between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$.

However, for our current problem this proves to be impossible. It is illustrative to consider the governing equations in the critical layer at $\mathrm{O}(\epsilon\kappa\mu^{-\frac{5}{2}+\nu})$ where viscous terms first appear on the right–hand sides

$$\beta\hat{U}_1^{(5)} + \hat{V}_{1\eta}^{(5)} + ia\hat{W}_1^{(5)} = 0 \quad , \tag{5.2a}$$

$$\overline{u}_0\hat{U}_{1\tilde{X}}^{(5)} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{U}_1^{(5)} + \overline{u}_0'\hat{V}_1^{(5)} = \hat{U}_{1\eta\eta}^{(1)} \quad , \tag{5.2b}$$

$$K_0\overline{u}_0\hat{U}_1^{(5)} + \hat{P}_{1\eta}^{(5)} = 0 \quad , \tag{5.2c}$$

$$\overline{u}_0\hat{W}_{1\tilde{X}}^{(5)} + (\eta i\overline{q}_0' + i\overline{q}_1)\hat{W}_1^{(5)} + \lambda_0\overline{w}_0'\hat{V}_1^{(5)} + ia\hat{P}_1^{(5)} = \hat{W}_{1\eta\eta}^{(1)} \quad . \tag{5.2d}$$

From these we obtain the equation

$$\hat{N}_{1,\frac{1}{2}-\nu}\hat{\Phi}_1^{(5)} = 0 \quad , \tag{5.3}$$

which has a solution

$$\hat{\Phi}_1^{(5)} = 0 \quad . \tag{5.4}$$

This result is actually implied by the corresponding work on stratified shear layers (see Blackaby, Dando & Hall 1993 for details) because there $\hat{\Phi}_1^{(5)}$ contains the factor $(Pr-1)$, where Pr is the Prandtl number. In this paper we have considered an incompressible fluid and hence we have effectively a Prandtl number of unity and as a consequence $\hat{\Phi}_1^{(5)} = 0$. In fact the expression for the whole viscous cubic jump (see equations 4.45a,b and 4.46 in Blackaby, Dando & Hall 1993) contains the factor $(Pr-1)$ and hence this viscous cubic jump does not occur for our present incompressible Görtler problem. Obviously this nonlinear term will be important if we take a more realistic value (eg. 0.72 for air) of the Prandtl number but, we note that for the corresponding compressible problem it is necessary to consider a model fluid in order for a similarity variable to exist for the base flow and this nonlinear term may prove hard to investigate (see Dando 1995 for a discussion of this nonlinear term for the stratified shear flow problem). We note that for the current problem viscosity still plays a role in the sense that the non–equilibrium critical–layer analysis of §4 is only valid when $\kappa\mu^{-3} \ll 1$.

6 The Evolution Equations

As the viscous–cubic is not present for our problem (because the Prandtl number is effectively equal to unity) we can write a composite evolution equation in the form

$$(I_3 - i^{-4\nu} I_1)\frac{\partial B}{\partial \tilde{X}} + (I_4 - i^{-4\nu} I_2)B =$$
$$2\nu B\left\{\epsilon^2\mu^{-3+4\nu}(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_{J_1} + \epsilon^4\mu^{-7+6\nu}(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_q\right\}, \qquad (6.1)$$

where we have derived $(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_{J_1}$ in §4 and $(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_q$ denotes the total part of the jump due to the quintic nonlinearity. We could derive $(b_{1+}^{(2)} - i^{-4\nu} b_{1-}^{(2)})_q$ explicitly in a similar manner to that in which we calculated the corresponding term in our study on stratified shear layers. However, this would be an extremely long and complicated task for the present problem and for the moment we shall concentrate instead on obtaining some numerical results for the non–viscous cubic evolution equation (this will determine how relevant the quintic nonlinearity is). This evolution equation is valid provided $\mu \gg G^{-\frac{1}{4}}$, otherwise the linear term will require modification (see for instance the amplitude equations 4.1a,b in Hall & Smith 1984).

In Figure 6.1 we show the regions of validity of the various amplitude equations. The non–viscous cubic becomes important when the amplitude of the disturbance, $A \sim \mu^{\frac{3}{2}-2\nu}$ (ie. when the cubic term becomes as large, $O(1)$, as the linear terms in the evolution equation, 6.1). The quintic term is of the same size as the cubic term when $A \sim \mu^{2-\nu}$ and so, if the cubic evolution equation results in a significant increase in the amplitude of the disturbance, the quintic nonlinear term will become the dominant term in the evolution equation.

6.1 Numerical results for the non–viscous cubic

We now concentrate on obtaining some numerical results for the non–viscous–cubic amplitude equation that we derived in §4. To ease numerical calculations the jump expression is turned into kernel form,

$$(D_+^{(4)} - D_-^{(4)}) = \frac{i^{1-2\nu}(1+2\nu)^2}{2^3\nu^2\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{1-4\nu}\int_0^\infty ds\; s^{2-4\nu}\int_0^1 d\sigma B(\tilde{X}-s)B(\tilde{X}-\sigma s)\times$$

$$\overline{B}(\tilde{X}-(1+\sigma)s)\Big[G_{11}(\sigma)+G_{12}(\sigma)\Big] \quad + \frac{i^{-2\nu}(1+2\nu)^2}{2^3\nu^2\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-4\nu}\int_0^\infty ds\; s^{1-4\nu}\int_0^1 d\sigma B(\tilde{X}-s)\times$$

$$B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)\Big[G_{21}(\sigma)+G_{22}(\sigma)\Big] \quad + \frac{i^{-2\nu}(1+2\nu)^3}{2^8\nu^2\overline{u}_0^2(1-2\nu)\Gamma^2(\frac{1}{2}-\nu)\Gamma(1-2\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-4\nu}\times$$

$$\int_0^\infty ds\; s^{2-4\nu}\int_0^1 d\sigma\Big[B(\tilde{X}-s)B_{\tilde{X}}(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)G_{31}(\sigma)$$

$$+B_{\tilde{X}}(\tilde{X}-s)B(\tilde{X}-\sigma s)\overline{B}(\tilde{X}-(1+\sigma)s)G_{32}(\sigma)\Big] \quad , \qquad (6.2)$$

where the kernels $G_{11}(\sigma)$, $G_{12}(\sigma)$, $G_{21}(\sigma)$, $G_{22}(\sigma)$, $G_{31}(\sigma)$, $G_{32}(\sigma)$ are all defined in Appendix B. An inspection of this kernel form shows that things are considerably more complicated than for the corresponding temporal stratified shear flow case. The jump now has real and imaginary parts and there are cubic terms in the evolution equation which contain $\tilde{X}$–derivatives. The inclusion of spatial derivatives of the amplitude in the nonlinear terms of the evolution equation is a relatively novel feature for non–equilibrium critical–layer studies. Previously streamwise derivatives have only been seen in the study of Churilov & Shukhman (1994) on the spatial evolution of helical disturbances to an axial jet and spanwise derivatives in the work of Gajjar (1995) on stationary crossflow vortices in fully three-dimensional boundary layers. As with these other works the inclusion of these nonlinear spatial derivative terms is wholly associated with the spatial formulation of the problem (note that there were no equivalent terms in the related temporal stability study for stratified shear layers). It is no longer possible to determine whether the 'kernel is positive or negative' and so we are unable to deduce the behavior of the disturbance amplitude, as was done by Churilov & Shukhman (1988) and Blackaby, Dando & Hall (1993). Instead we are forced to consider numerical solutions.

As mentioned earlier, in §2, it is possible to calculate neutral curves for the Görtler modes. In this paper we shall concentrate on obtaining numerical results from the amplitude equation for the first Görtler mode, as it becomes progressively harder to obtain neutral curves and ν–values for the higher modes. In Figure 6.2 we have plotted the neutral curve for this first mode. We note that the flow is stable to this mode inside the curve plotted and that the left–hand branch tends towards a constant wavenumber ($a = 1.305$) as the value of the crossflow parameter tends to infinity. This limit corresponds to the neutral crossflow vortices that are associated with the inflection point of the directional profile, and were first investigated by Gregory, Stuart & Walker (1955). In Figure 6.3 we plot the values of ν, as determined from (2.9) and (2.10), on the left–hand branch of this neutral curve for the first Görtler mode. We concentrate our attention on this area of the neutral curve as the numerical computations become more complicated (in particular calculating the integrals I_1, I_2, I_3 and I_4; see equations 3.11a-d) as ν becomes increasingly negative.

Substituting (6.2) into equation (4.41) and then matching the two relationships we have derived between $b_{1+}^{(2)}$ and $b_{1-}^{(2)}$ (equations 3.10 and 4.41) leads to the evolution equation

$$\frac{\partial B}{\partial \tilde{X}} = \gamma_1 B + \gamma_2 \Big\{ \gamma_3 \int_0^\infty ds\ s^{2-4\nu} \int_0^1 d\sigma B(\tilde{X}-s) B(\tilde{X}-\sigma s) \overline{B}(\tilde{X}-(1+\sigma)s) \Big[G_{11}(\sigma) + G_{12}(\sigma) \Big]$$

$$+\gamma_4 \int_0^\infty ds\ s^{1-4\nu} \int_0^1 d\sigma B(\tilde{X}-s) B(\tilde{X}-\sigma s) \overline{B}(\tilde{X}-(1+\sigma)s) \Big[G_{21}(\sigma) + G_{22}(\sigma) \Big]$$

$$+\gamma_5 \int_0^\infty ds\ s^{2-4\nu} \int_0^1 d\sigma \Big[B(\tilde{X}-s) B_{\tilde{X}}(\tilde{X}-\sigma s) \overline{B}(\tilde{X}-(1+\sigma)s) G_{31}(\sigma)$$

$$+B_{\tilde{X}}(\tilde{X}-s) B(\tilde{X}-\sigma s) \overline{B}(\tilde{X}-(1+\sigma)s) G_{32}(\sigma) \Big] \Big\} \quad , \tag{6.3}$$

where the constants γ_1, γ_2, γ_3, γ_4 and γ_5 are given by

$$\gamma_1 = \frac{(i^{-4\nu} I_2 - I_4)}{(I_3 - i^{-4\nu} I_1)} \quad , \tag{6.4a}$$

$$\gamma_2 = \frac{i^{-2\nu}(1+2\nu)^2}{(I_3 - i^{-4\nu} I_1) 4\nu^2 \overline{u}_0^2 \Gamma^2(\frac{1}{2}-\nu)} \left(\frac{\overline{q}_0'}{\overline{u}_0} \right)^{-4\nu} \quad , \tag{6.4b}$$

$$\gamma_3 = \frac{i}{\Gamma^2(\frac{1}{2}-\nu)} \left(\frac{\overline{q}_0'}{\overline{u}_0} \right) \quad , \tag{6.4c}$$

$$\gamma_4 = \frac{1}{\Gamma^2(\frac{1}{2}-\nu)} \quad , \tag{6.4d}$$

$$\gamma_5 = \frac{(1+2\nu)}{32(1-2\nu)\Gamma(1-2\nu)} \quad . \tag{6.4e}$$

For our numerical calculations it is convenient to introduce the variable

$$\tilde{x} = \exp\{2\gamma_{1r}\tilde{X}\} \quad , \tag{6.5}$$

with γ_{1r} denoting the real part of γ_1 (this is similar to the 'logarithmic time' variable which was introduced by Churilov & Shukhman 1988). We also set

$$B(\tilde{X}) = b(\tilde{X}) \exp\{\gamma_{1r}\tilde{X}\} \quad , \tag{6.6}$$

and assume that $b(\tilde{X}) \to 1$ as $\tilde{X} \to \infty$, note that the requirement $B(\tilde{X}) \to 0$ as $\tilde{X} \to -\infty$ is satisfied as γ_{1r} is negative. After this we find that the evolution equation reduces to

$$\frac{\partial b}{\partial \tilde{x}} = \int_0^1 d\sigma G_1(\sigma) \int_0^\infty dr r^{2-4\nu} e^{-2\gamma_{1r} r} b(\tilde{x} e^{-r/(1+\sigma)}) b(\tilde{x} e^{-\sigma r/(1+\sigma)}) \overline{b}(\tilde{x} e^{-r})$$

$$+\int_0^1 d\sigma G_2(\sigma)\int_0^\infty dr r^{1-4\nu}e^{-2\gamma_{1r}r}b(\tilde{x}e^{-r/(1+\sigma)})b(\tilde{x}e^{-\sigma r/(1+\sigma)})\overline{b}(\tilde{x}e^{-r})$$

$$+\tilde{x}\int_0^1 d\sigma G_{3A}(\sigma)\int_0^\infty dr r^{2-4\nu}e^{-2\gamma_{1r}r}b(\tilde{x}e^{-r/(1+\sigma)})\frac{\partial b(\tilde{x}e^{-\sigma r/(1+\sigma)})}{\partial\tilde{x}}\overline{b}(\tilde{x}e^{-r})$$

$$+\tilde{x}\int_0^1 d\sigma G_{3B}(\sigma)\int_0^\infty dr r^{2-4\nu}e^{-2\gamma_{1r}r}\frac{\partial b(\tilde{x}e^{-r/(1+\sigma)})}{\partial\tilde{x}}b(\tilde{x}e^{-\sigma r/(1+\sigma)})\overline{b}(\tilde{x}e^{-r}) \quad , \qquad (6.7)$$

with the new kernel functions being defined as

$$G_1 = \frac{\gamma_2}{2\gamma_{1r}(1+\sigma)^{2-4\nu}}[\gamma_3(G_{11}+G_{12})+\gamma_1\gamma_5(G_{31}+G_{32})] \quad , \qquad (6.8a)$$

$$G_2 = \frac{\gamma_2\gamma_4}{2\gamma_{1r}(1+\sigma)^{1-4\nu}}[G_{21}+G_{22}] \quad , \qquad (6.8b)$$

$$G_{3A} = \frac{\gamma_2\gamma_5}{(1+\sigma)^{2-4\nu}}G_{31} \quad , \qquad (6.8c)$$

$$G_{3B} = \frac{\gamma_2\gamma_5}{(1+\sigma)^{2-4\nu}}G_{32} \quad . \qquad (6.8d)$$

Finally we present some numerical solutions for the evolution equation (6.7). We note that we have not eliminated the parameter x_1 from the equation (see 3.11b,d and 3.2) as it does not occur in all terms (cf. Blackaby, Dando & Hall 1993 where the parameter J_1 was eliminated from the final evolution equation) and so for these numerical solutions we have taken the representative value $x_1 = 1.0$. We consider the effect of a small change in the crossflow so we have also taken $\lambda_1 = 1.0$ and $K_1 = 0$. In Figures 6.4a-d we show the evolution of the amplitude $b(\tilde{x})$ for the case when $\lambda_0 = 10.0$ (ie. close to the neutral point $a = 1.371$). We have plotted the real part of $b(\tilde{x})$ on the x–axis and the imaginary part on the y–axis and from these Figures we can see that the disturbance evolves in a spiral (in the complex plane of b) with rapidly increasing amplitude. In fact the amplitude increases so rapidly that the structure of the spiral is somewhat lost and this is why we show four plots of the evolution (with different scales). In Figures 6.5a-b we consider the evolution of $b(\tilde{x})$ when $\lambda_0 = 25.0$, this is close to the neutral point $a = 1.316$. In Figure 6.5a we have plotted the argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) and in Figure 6.5b we show the amplitude. The evolution again takes the form of a spiral in the complex plane of $b(\tilde{x})$ with a large increase in amplitude but this does not occur as quickly as for the previous case considered. We note that the rapid increase in the disturbance amplitude for these two cases means that the disturbance will enter region IIIb in Figure 6.1 where the dominant nonlinear term in the evolution equation is the quintic. This means that it may be necessary in the future to derive the quintic nonlinear term via a critical layer analysis similar to that of §4 and obtain numerical solutions for this new evolution equation. However, if we increase the crossflow further we find that this rapid increase in the disturbance amplitude does not occur. In Figures 6.6a-b we plot the argument and amplitude of $b(\tilde{x})$ for $\lambda_0 = 55.0$ and we can clearly see that the amplitude increase has been greatly slowed. If we further increase the crossflow to a value of $\lambda_0 = 75.0$ in Figures 6.7a-b we can see that there is hardly any increase in the disturbance amplitude. Consequently the disturbance will not enter region IIIb in Figure 6.1 where the evolution equation would be dominated by the quintic nonlinearity.

We have shown that, at least for the nonlinear evolution equation with cubic nonlinearity due to the novel mechanism, crossflow has the same effect as with a linear stability study: it has a stabilizing influence on inviscid Görtler vortices. As mentioned earlier we have looked at a flow with a Prandtl number of unity, this was done because we wish to extend this work to the compressible regime where it is necessary to consider a model fluid in order for a similarity variable to exist for the base flow. However, this particular choice of Prandtl number turns out to be a mathematical anomaly as the viscous–cubic jump expression contains the factor $(\eta - 1)$ and so is not existent for this problem. However, if we considered a more general Prandtl number then the viscous–cubic would be important because, if we consider Figure 6.8, which shows the regions of validity of the various amplitude equations when the viscous cubic is present, we see that our above results suggest that for larger values of the crossflow parameter the disturbance will pass from region IIIa to region IIIc where the dominant nonlinear term is the viscous cubic. Furthermore, for the related stratified shear flow problem (Dando 1995) it is known that the viscous–cubic jump results in an explosive burst–like growth. So it would appear to be necessary next to consider the viscous–cubic, with $\eta = 0.72$, for the Görtler problem and determine whether a similar explosive growth occurs (so a disturbance with a reasonable crossflow moves from region IIIa to region IIIc and then onto IIIb) or if sufficient crossflow can prevent this and hence for large values of λ_0 (which still corresponds to a small amount of three–dimensionality) inviscid Görtler vortices will be nonlinearly stable.

7 Conclusion

In this paper we have considered the nonlinear development of inviscid, incompressible Görtler vortices in a three–dimensional boundary layer using non–equilibrium critical–layer theory. From the earlier work of Blackaby, Dando & Hall (1993) we knew that it was necessary to consider three different amplitude equations, a viscous cubic, a non–viscous cubic and a quintic. In this study we have concentrated our attention on the two cubic evolution equations (the non–viscous cubic in particular) as these will be the first to affect the disturbance amplitude. In §5 we showed that for the problem we have considered, with a Prandtl number of unity (a necessary choice in order for a similarity variable to exist for a base flow which is compressible; we wish to extend this work to the compressible problem), the viscous–cubic jump is zero. Even if this were not the case we note that, based on our assumptions concerning viscous–spreading effects resulting in an unstable linear disturbance mode approaching a later neutral state, initially our disturbance will lie in the bottom right–hand corner of Figure 6.1 or 6.8 and this indicates that the base evolution equation with the non–viscous cubic nonlinearity deserves the first attention. Consequently we have concentrated in this paper on obtaining numerical solutions of the non–viscous cubic evolution equation.

Our numerical solutions show that for small values of the crossflow parameter the disturbance amplitude evolves by describing a spiral, in the complex plane, of rapidly increasing amplitude. This large increase in the amplitude of the disturbance means that the evolution process will soon move on to a stage where the evolution of the mode is governed by an integro–differential amplitude equation with a quintic nonlinearity. In the related study of Blackaby, Dando & Hall (1993) the integro–differential equation with a quintic nonlinearity led to continued growth of the disturbance amplitude which resulted in the effects of nonlinearity spreading to outside the critical level, by which time the flow has become fully nonlinear. Although we have pointed out the many similarities between the problems of the nonlinear evolution of modes on unstable stratified shear layers and the nonlinear evolution of inviscid Görtler vortices in three–dimensional boundary layers we are unable to deduce the behavior of the quintic integro–differential equation for the Görtler problem from the earlier results of Blackaby, Dando & Hall (1993) because of the major differences in the kernels for the two problems. A large number of these differences (including the novel feature of streamwise derivatives in the nonlinear terms) are due to the spatial formulation of the Görtler problem as opposed to the temporal formulation of the problem considered in Blackaby, Dando & Hall (1993). Consequently the daunting task of obtaining numerical solutions for the quintic integro–differential equation for the Görtler problem will need to be undertaken in order to clarify further the nonlinear stability of inviscid Görtler vortices in flows that are only very slightly three–dimensional.

However, for larger values of the crossflow we find that the disturbance amplitude still evolves by describing a spiral in the complex plane but its amplitude increases only very slowly. Consequently when the Prandtl number is not unity the disturbance will pass from region IIIa in Figure 6.8 to region IIIc (as opposed to IIIb when the crossflow is small) and will be controlled by an integro differential equation with a cubic nonlinearity due to viscous effects. In an investigation of the corresponding stratified shear flow problem (Dando 1995) it was found that the viscous–cubic resulted in large amplitude growth (of either a burst–like nature or by oscillating). So it would be desirable to investigate the viscous cubic amplitude equation for the Görtler problem (with a Prandtl number not equal to unity) and determine whether the same thing happens or if this amplitude growth is prevented by a sufficiently large crossflow (as we have found for the novel cubic amplitude equation in this paper). This may however, prove difficult because for the compressible problem as we need to consider a model fluid (Prandtl number of unity) in order for a similarity variable to exist for the base flow.

As we have already mentioned we wish to extend the work in this paper to a compressible flow. Linear stability studies have showed that a larger crossflow is needed to stabilize the Görtler vortices when the flow is compressible and it would be interesting to see what effect compressibility has on the results presented in this paper. We also note that the theory developed here can be applied to a nonlinear study of the inviscid vortex instabilities in the three–dimensional flow above a heated plate.

Acknowledgements

The authors are extremely grateful to Professor S.M. Churilov for his detailed correspondence regarding Churilov & Shukhman (1988). The second author is grateful to colleagues at Imperial

College who gave him advice and encouragement regarding the numerical computations when he presented a seminar there.

Appendix A

The three pieces of the non–viscous cubic jump (see equation 4.38) are given by

$$(D_+^{(4)}-D_-^{(4)})_1 = \frac{i^{-2\nu}(1+2\nu)^3}{512\nu^2\overline{u}_0^2(1-2\nu)\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-4\nu}\int_0^\infty dt_1\int_0^\infty dt_2\int_0^\infty dt_3\overline{B}(\tilde{X}+t_1+t_2-2t_3) \quad\times$$
$$(t_1t_2t_3)^{-\frac{3}{2}-\nu}(t_3-t_1-t_2)^{\frac{1}{2}-\nu}\Big\{B(\tilde{X}+t_2-t_3)\frac{\partial}{\partial\tilde{X}}B(\tilde{X}+t_1-t_3)t_1^3(t_1-2t_3)$$
$$+B(\tilde{X}+t_1-t_3)\frac{\partial}{\partial\tilde{X}}B(\tilde{X}+t_2-t_3)t_2^3(t_2-2t_3)\Big\}H(t_3-t_1-t_2) \quad , \qquad (A.1)$$

$$(D_+^{(4)}-D_-^{(4)})_2 = \frac{i^{1-2\nu}(1+2\nu)^3}{32\nu^2\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-4\nu}\int_0^\infty dt\int_0^\infty dt_1\int_0^\infty dt_2\int_0^\infty dt_3B(\tilde{X}+t+t_1-t_3) \quad\times$$
$$B(\tilde{X}+t+t_2-t_3)\overline{B}(\tilde{X}+2t+t_1+t_2-2t_3)(t_1t_2t_3)^{-\frac{3}{2}-\nu}(t_1-t_2)^2(t_1+t_2)^{\frac{1}{2}+\nu}(2t+t_1+t_2)^{-\frac{3}{2}-\nu} \quad\times$$
$$(t_3-t_1-t_2-2t)^{\frac{1}{2}-\nu}\Bigg\{\frac{\overline{q}_0'}{\overline{u}_0}\Bigg[-t_3^2\left(\frac{\overline{q}_1'}{\overline{q}_0'}-\frac{(1+2\nu)}{2}\frac{\overline{u}_1'}{\overline{u}_0'}-\frac{(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{(1-2\nu)}{2}\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)+t_3(2t$$
$$+t_1+t_2)\left(-\frac{1}{2}\frac{\overline{q}_1'}{\overline{q}_0'}+\frac{(1+2\nu)}{4}\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{(1+4\nu)}{2(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+\frac{(1-2\nu)}{4}\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)-(2t+t_1+t_2)^2\left(-\frac{1}{2}\frac{\overline{q}_1'}{\overline{q}_0'}\right.$$
$$\left.+\frac{(1+2\nu)}{4}\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{(1-2\nu)}{2(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+\frac{(1-2\nu)}{4}\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)\Bigg]+i\Bigg[t_3\left(\frac{(1-2\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{3(1+4\nu)(1-\nu)}{4(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\right.$$
$$\left.-\frac{(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}\right)-(2t+t_1+t_2)\left(\frac{(1+2\nu)}{4}\frac{\overline{u}_0''}{\overline{u}_0'}-\frac{1}{2}\frac{\overline{q}_0''}{\overline{q}_0'}+\left(\frac{3}{4}-\frac{\nu(1-2\nu)}{(1+2\nu)}\right)\frac{\overline{u}_0'}{\overline{u}_0}\right)\Bigg]\Bigg\}H(t_3-t_1-t_2-2t) \quad ,$$
$$(A.2)$$

$$(D_+^{(4)}-D_-^{(4)}))_3 = -\frac{i^{1-2\nu}(1+2\nu)^3}{32\nu^2\overline{u}_0^2\Gamma^4(\frac{1}{2}-\nu)}\left(\frac{\overline{q}_0'}{\overline{u}_0}\right)^{-4\nu}\int_0^\infty dt_1\int_0^\infty dt_2\int_0^\infty dt_3B(\tilde{X}+t_1-t_3)B(\tilde{X}+t_2-t_3)$$
$$\overline{B}(\tilde{X}+t_1+t_2-2t_3)(t_1t_2t_3)^{-\frac{3}{2}-\nu}(t_3-t_1-t_2)^{\frac{1}{2}-\nu}\Bigg\{\frac{\overline{q}_0'}{\overline{u}_0}\Bigg[-\frac{t_3^3(t_1+t_2)}{2(1+2\nu)}\left(-2\frac{\overline{q}_1'}{\overline{q}_0'}+(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{2(1+6\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}\right.$$
$$\left.+(1-2\nu)\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)+\frac{t_1t_2t_3^2}{(1+2\nu)}\left(2\frac{\overline{q}_1'}{\overline{q}_0'}-(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}-2(1-2\nu)\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-(1-2\nu)\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)$$
$$+\frac{t_3^2(t_1^2+t_2^2)}{(1+2\nu)}\left(-2\frac{\overline{q}_1'}{\overline{q}_0'}+(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{2(1+6\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+(1-2\nu)\left(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\right)\right)-\frac{t_1t_2t_3(t_1+t_2)}{2(1+2\nu)}\left(2\frac{\overline{q}_1'}{\overline{q}_0'}\right.$$

$$-(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}-\frac{2(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-(1-2\nu)\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)-\frac{t_3(t_1^3+t_2^3)}{2}\Big(-\frac{2}{(1+2\nu)}\frac{\overline{q}_1'}{\overline{q}_0'}+\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{1}{4(1-2\nu)}\frac{\overline{q}_1\overline{u}_0''}{\overline{q}_0'\overline{u}_0'}$$

$$+\frac{(11+36\nu-100\nu^2)}{4(1-2\nu)(1+2\nu)^2}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{1}{2(1-2\nu)(1+2\nu)}\frac{\overline{q}_1\overline{q}_0''}{\overline{q}_0'}+\frac{(1-2\nu)}{(1+2\nu)}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)+\frac{(t_1^4+t_2^4)}{16(1-2\nu)}\Big(\frac{\overline{q}_1\overline{u}_0''}{\overline{q}_0'\overline{u}_0'}$$

$$+\frac{(3-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{2}{(1+2\nu)}\frac{\overline{q}_1\overline{q}_0''}{\overline{q}_0'}\Big)\Big]+i\Big[\frac{t_3^2(t_1+t_2)}{2(1+2\nu)}\Big(-\frac{(5+6\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}+\frac{(5+6\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{(4\nu^2+64\nu+15)}{2(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}$$

$$\Big)-\frac{t_1t_2t_3}{4(1-4\nu)}\Big(\frac{(3-28\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}-\frac{(3-28\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{(8\nu^2+42\nu-17)}{2(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big)-\frac{t_3(t_1^2+t_2^2)}{2(1-2\nu)(1+2\nu)^2(1-4\nu)}$$

$$\Big(-2(48\nu^3-8\nu^2-26\nu+7)\frac{\overline{q}_0''}{\overline{q}_0'}+(1+2\nu)(48\nu^3-8\nu^2-26\nu+7)\frac{\overline{u}_0''}{\overline{u}_0'}-(96\nu^4-288\nu^3+180\nu^2+36\nu-15)\frac{\overline{u}_0'}{\overline{u}_0}\Big)$$

$$+\frac{t_1t_2(t_1+t_2)}{16(1-2\nu)(1-4\nu)}\Big(\frac{2(56\nu^2-22\nu-1)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}-(56\nu^2-22\nu-1)\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{(-80\nu^3+28\nu^2+16\nu-9)}{(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big)$$

$$+\frac{(t_1^3+t_2^3)}{16(1-2\nu)(1-4\nu)}\Big(\frac{2(8\nu^2+30\nu-9)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}-(8\nu^2+30\nu-9)\frac{\overline{u}_0''}{\overline{u}_0'}+\frac{(16\nu^3+68\nu^2-116\nu+23)}{(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big)$$

$$\Big]\Big\}H(t_3-t_1-t_2)\quad, \tag{A.3}$$

where H denotes the Heaviside function.

Appendix B

For our numerical calculations it is necessary to convert the jump expression into kernel form, (6.2). The kernels in this equation are then given by

$$G_{11}(\sigma)=\sigma^{1-2\nu}(1-\sigma)^{-2\nu}\int_0^1 dt\frac{t^{-\frac{1}{2}-\nu}(1-t)^{\frac{1}{2}-\nu}}{(1-\sigma t)^{\frac{3}{2}-3\nu}(1-\sigma^2t)^{\frac{3}{2}+\nu}(1+\sigma t)^{\frac{3}{2}+\nu}}\Big\{(1-\sigma^2t)^2\Big[\frac{\overline{q}_1'}{\overline{q}_0'}-\frac{(1+2\nu)}{2}\frac{\overline{u}_1'}{\overline{u}_0'}$$

$$-\frac{(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{(1-2\nu)}{2}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big]-(1-\sigma)(1-\sigma^2t)(1+\sigma t)\Big[-\frac{1}{2}\frac{\overline{q}_1'}{\overline{q}_0'}+\frac{(1+2\nu)}{4}\frac{\overline{u}_1'}{\overline{u}_0'}$$

$$+\frac{(1+4\nu)}{2(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+\frac{(1-2\nu)}{4}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big]+(1-\sigma)^2(1+\sigma t)^2\Big[-\frac{1}{2}\frac{\overline{q}_1'}{\overline{q}_0'}+\frac{(1+2\nu)}{4}\frac{\overline{u}_1'}{\overline{u}_0'}$$

$$+\frac{(1-2\nu)}{2(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+\frac{(1-2\nu)}{4}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big]\Big\}F\Big(-\frac{1}{2}-\nu,-\frac{1}{4}-\frac{\nu}{2};\frac{3}{4}-\frac{\nu}{2};\sigma^2t^2\Big)\quad, \tag{B.1}$$

$$G_{12}(\sigma)=-\frac{(1+2\nu)\Gamma^2(\frac{1}{2}-\nu)}{2\Gamma(1-2\nu)}\sigma^{-2\nu}(1+\sigma)^{-\frac{3}{2}+\nu}\Big[\Big\{\frac{(1-2\nu)}{2(1+2\nu)^2}F_1\Big(\frac{3}{2}-\nu,\frac{1}{2}+\nu,-\frac{3}{2}+\nu,1-2\nu;\sigma;$$

$$\frac{\sigma}{(1+\sigma)}\Big) - \frac{\sigma}{4(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{3}{2}+\nu,-\frac{3}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\Big\}\Big(-2\frac{\overline{q}_1'}{\overline{q}_0'}+(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}$$

$$+\frac{2(1+6\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+(1-2\nu)\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)+\frac{\sigma(1+\sigma)}{2(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{1}{2}+\nu,-\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\times$$

$$\Big(2\frac{\overline{q}_1'}{\overline{q}_0'}-(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}-2(1-2\nu)\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-(1-2\nu)\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)+\Big\{-\frac{(1+\sigma)(1-2\nu)}{(1+2\nu)^2}F_1\Big(\frac{3}{2}-\nu,-\frac{1}{2}+\nu,$$

$$-\frac{1}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)+\frac{\sigma^2(1+\sigma)(1-2\nu)}{8(1-\nu)(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{3}{2}+\nu,-\frac{1}{2}+\nu,3-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\Big\}\times$$

$$\Big(-2\frac{\overline{q}_1'}{\overline{q}_0'}+(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{2(1+6\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}+(1-2\nu)\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)-\Big\{\frac{\sigma(1+\sigma)^2}{4(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,-\frac{1}{2}+\nu,$$

$$\frac{1}{2}+\nu,2-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)+\frac{\sigma^2(1+\sigma)^2(1-2\nu)}{16(1-\nu)(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{1}{2}+\nu,\frac{1}{2}+\nu,3-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\Big\}\Big(2\frac{\overline{q}_1'}{\overline{q}_0'}$$

$$-(1+2\nu)\frac{\overline{u}_1'}{\overline{u}_0'}-\frac{2(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-(1-2\nu)\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)+\Big\{\frac{(1+\sigma)^2(1-2\nu)}{2(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,-\frac{3}{2}+\nu,$$

$$\frac{1}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)-\frac{\sigma^3(1+\sigma)^2(1-2\nu)}{32(1-\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{1}{2}+\nu,4-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\Big\}$$

$$\Big(-\frac{2}{(1+2\nu)}\frac{\overline{q}_1'}{\overline{q}_0'}+\frac{\overline{u}_1'}{\overline{u}_0'}+\frac{1}{4(1-2\nu)}\frac{\overline{q}_1\overline{u}_0''}{\overline{q}_0'\overline{u}_0'}+\frac{(11+36\nu-100\nu^2)}{4(1-2\nu)(1+2\nu)^2}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{1}{2(1-2\nu)(1+2\nu)}\frac{\overline{q}_1\overline{q}_0''}{\overline{q}_0'}$$

$$+\frac{(1-2\nu)}{(1+2\nu)}\Big(\frac{\overline{u}_1}{\overline{u}_0}+\frac{K_1}{K_0}\Big)\Big)+\Big\{-\frac{(1+\sigma)^3}{16(1+2\nu)}F_1\Big(\frac{3}{2}-\nu,-\frac{5}{2}+\nu,\frac{3}{2}+\nu,1-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)$$

$$+\frac{\sigma^4(1+\sigma)^3(5-2\nu)}{256(4-2\nu)(2-2\nu)}F_1\Big(\frac{3}{2}-\nu,\frac{3}{2}+\nu,\frac{3}{2}+\nu,5-2\nu;\sigma;\frac{\sigma}{(1+\sigma)}\Big)\Big\}\quad\times$$

$$\Big(\frac{\overline{q}_1\overline{u}_0''}{\overline{q}_0'\overline{u}_0'}+\frac{(3-2\nu)}{(1+2\nu)}\frac{\overline{q}_1\overline{u}_0'}{\overline{q}_0'\overline{u}_0}-\frac{2}{(1+2\nu)}\frac{\overline{q}_1\overline{q}_0''}{\overline{q}_0'}\Big)\Big]\quad, \qquad (B.2)$$

$$G_{21}(\sigma)=\sigma^{1-2\nu}(1-\sigma)^{-2\nu}\int_0^1 dt\frac{t^{-\frac{1}{2}-\nu}(1-t)^{\frac{1}{2}-\nu}}{(1-\sigma t)^{\frac{1}{2}-3\nu}(1-\sigma^2 t)^{\frac{3}{2}+\nu}(1+\sigma t)^{\frac{3}{2}+\nu}}\Big\{(1-\sigma^2 t)\Big[\frac{(1-2\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'}$$

$$+\frac{3(1+4\nu)(1-\nu)}{4(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}-\frac{(1-2\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}\Big]-(1-\sigma)(1+\sigma t)\Big[\frac{(1+2\nu)}{4}\frac{\overline{u}_0''}{\overline{u}_0'}+\Big(\frac{3}{4}-\frac{\nu(1-2\nu)}{(1+2\nu)}\Big)\frac{\overline{u}_0'}{\overline{u}_0}-\frac{1}{2}\frac{\overline{q}_0''}{\overline{q}_0'}\Big]\Big\}$$

$$F\Big(-\frac{1}{2}-\nu,-\frac{1}{4}-\frac{\nu}{2};\frac{3}{4}-\frac{\nu}{2};\sigma^2t^2\Big)\quad, \qquad (B.3)$$

$$G_{22}(\sigma)=\frac{(1+2\nu)\Gamma^2(\frac{1}{2}-\nu)}{2\Gamma(1-2\nu)}\sigma^{-2\nu}(1+\sigma)^{-\frac{1}{2}+\nu}\Big[\Big\{-\frac{(1-2\nu)}{2(1+2\nu)^2}F_1\Big(\frac{3}{2}-\nu,\frac{1}{2}+\nu,-\frac{1}{2}+\nu,1-2\nu;\sigma;$$

$$\frac{\sigma}{(1+\sigma)}\Big) + \frac{\sigma}{4(1+2\nu)} F_1\Big(\frac{3}{2}-\nu, \frac{3}{2}+\nu, -\frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big)\Big\}\Big(-\frac{(5+6\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'} + \frac{(5+6\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'}$$

$$+\frac{(4\nu^2+64\nu+15)}{2(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big) - \frac{\sigma(1+\sigma)}{8(1-4\nu)} F_1\Big(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{1}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big)\Big(\frac{(3-28\nu)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'}$$

$$-\frac{(3-28\nu)}{2}\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{(8\nu^2+42\nu-17)}{2(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big) + \Big\{\frac{(1+\sigma)}{2(1+2\nu)^3(1-4\nu)} F_1\Big(\frac{3}{2}-\nu, -\frac{1}{2}+\nu, \frac{1}{2}+\nu, 1-2\nu; \sigma;$$

$$\frac{\sigma}{(1+\sigma)}\Big) - \frac{\sigma^2(1+\sigma)}{16(1+2\nu)^2(1-4\nu)(1-\nu)} F_1\Big(\frac{3}{2}-\nu, \frac{3}{2}+\nu, \frac{1}{2}+\nu, 3-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big)\Big\}\Big(-2(48\nu^3-8\nu^2$$

$$-26\nu+7)\frac{\overline{q}_0''}{\overline{q}_0'} + (1+2\nu)(48\nu^3-8\nu^2-26\nu+7)\frac{\overline{u}_0''}{\overline{u}_0'} - (96\nu^4-288\nu^3+180\nu^2+36\nu-15)\frac{\overline{u}_0'}{\overline{u}_0}\Big)$$

$$+\Big\{\frac{\sigma(1+\sigma)^2}{32(1-2\nu)(1-4\nu)} F_1\Big(\frac{3}{2}-\nu, -\frac{1}{2}+\nu, \frac{3}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big) + \frac{\sigma^2(1+\sigma)^2}{64(1-\nu)(1-4\nu)} F_1\Big(\frac{3}{2}-\nu,$$

$$\frac{1}{2}+\nu, \frac{3}{2}+\nu, 3-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big)\Big\}\Big(\frac{2(56\nu^2-22\nu-1)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'} - (56\nu^2-22\nu-1)\frac{\overline{u}_0''}{\overline{u}_0'}$$

$$+\frac{(-80\nu^3+28\nu^2+16\nu-9)}{(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big) + \Big\{-\frac{(1+\sigma)^2}{16(1+2\nu)(1-4\nu)} F_1\Big(\frac{3}{2}-\nu, -\frac{3}{2}+\nu, \frac{3}{2}+\nu, 1-2\nu; \sigma;$$

$$\frac{\sigma}{(1+\sigma)}\Big) + \frac{\sigma^3(1+\sigma)^2}{256(1-\nu)(1-4\nu)} \quad F_1\Big(\frac{3}{2}-\nu, \frac{3}{2}+\nu, \frac{3}{2}+\nu, 4-2\nu; \sigma; \frac{\sigma}{(1+\sigma)}\Big)\Big\}\times$$

$$\Big(\frac{2(8\nu^2+30\nu-9)}{(1+2\nu)}\frac{\overline{q}_0''}{\overline{q}_0'} - (8\nu^2+30\nu-9)\frac{\overline{u}_0''}{\overline{u}_0'} + \frac{(16\nu^3+68\nu^2-116\nu+23)}{(1+2\nu)}\frac{\overline{u}_0'}{\overline{u}_0}\Big)\Big] \quad , \qquad (B.4)$$

$$G_{31}(\sigma) = \sigma^{-2\nu}(1+\sigma)^{\frac{3}{2}+\nu}\Big\{\frac{(1-2\nu)}{(1+2\nu)} F_1\Big(\frac{3}{2}-\nu, -\frac{3}{2}+\nu, \frac{3}{2}+\nu, 1-2\nu; \sigma; \frac{\sigma}{1+\sigma}\Big)$$

$$-\frac{\sigma}{2} F_1\Big(\frac{3}{2}-\nu, -\frac{3}{2}+\nu, \frac{3}{2}+\nu, 2-2\nu; \sigma; \frac{\sigma}{1+\sigma}\Big)\Big\} \quad , \qquad (B.5)$$

$$G_{32}(\sigma) = -\frac{\sigma^{3-2\nu}(1+\sigma)^{\frac{3}{2}+\nu}}{2^4}\Big\{(1+\sigma) F_1\Big(\frac{3}{2}-\nu, \frac{3}{2}+\nu, \frac{3}{2}+\nu, 4-2\nu; \sigma; \frac{\sigma}{1+\sigma}\Big)$$

$$+F_1\Big(\frac{3}{2}-\nu, \frac{1}{2}+\nu, \frac{3}{2}+\nu, 4-2\nu; \sigma; \frac{\sigma}{1+\sigma}\Big)\Big\} \quad , \qquad (B.6)$$

where F is the hypergeometric function of one variable and F_1 the hypergeometric function of two variables (see Erdélyi 1953 and Abramowitz & Stegun 1964 for details).

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Figure 2.1. Real parts of the scaled growth rate, $\tilde{\beta}$, when there is no crossflow.

Figure 2.2. Real parts of the scaled growth rate, $\tilde{\beta}$, when the scaled crossflow, $\tilde{\lambda}$ has a value of 5.

Figure 2.3. Real parts of the scaled growth rate, $\tilde{\beta}$, when the scaled crossflow, $\tilde{\lambda}$ has a value of 10.

Figure 6.1. A diagram of the various regimes of the critical layer when $\eta = 1$.
I: viscous, steady critical layer; Landau–Stuart–Watson equation.
II: strongly nonlinear, equilibrium critical layer; Benney & Bergeron theory.
$IIIa$: unsteady critical layer; largest term in (6.1) is cubic and due to non–viscous effects.
$IIIb$: unsteady critical layer; largest term in (6.1) is quintic.
Solid lines represent boundaries with areas where the critical layers are not unsteady, dashed lines represent boundaries between different base evolution equations from (6.1) and the dotted lines indicate the threshold values at which the nonlinear evolution equations become valid. The thick line on the diagram indicates the evolutionary path of the disturbance for small values of the crossflow parameter.

A $\quad$ $A \sim \mu^{\frac{3}{2}-\nu}$ $\quad$ $A \sim \mu^{2-\nu}$

II $\quad$ IIIb

$\kappa^{\frac{1}{2}-\frac{1}{3}\nu}$ $\quad$ IIIa $\quad$ $A \sim \mu^{\frac{3}{2}-2\nu}$

$\kappa^{\frac{2}{3}-\frac{1}{3}\nu}$ $\quad$ I

$\kappa^{\frac{1}{3}}$ $\quad$ μ

Figure 6.2. The neutral curve of the first Görtler mode.

Figure 6.3. The values of ν for the left–hand branch of the neutral curve of the first Görtler mode.

Figure 6.4a. The real and imaginary parts of $b(\tilde{x})$, showing its evolution for $\lambda_0 = 10$, part a.

Figure 6.4b. The real and imaginary parts of $b(\tilde{x})$, showing its evolution for $\lambda_0 = 10$, part b.

Figure 6.4c. The real and imaginary parts of $b(\tilde{x})$, showing its evolution for $\lambda_0 = 10$, part c.

Figure 6.4d. The real and imaginary parts of $b(\tilde{x})$, showing its evolution for $\lambda_0 = 10$, part d.

Figure 6.5a. The argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) for $\lambda_0 = 25$.

ARG(b)

$\tilde{x}$

Figure 6.5b. The amplitude of $b(\tilde{x})$ for $\lambda_0 = 25$.

$\tilde{x}$

Figure 6.6a. The argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) for $\lambda_0 = 55$.

ARG(b)

$\tilde{x}$

Figure 6.6b. The amplitude of $b(\tilde{x})$ for $\lambda_0 = 55$.

$\tilde{x}$

Figure 6.7a. The argument of $b(\tilde{x})$ (in the range $-\pi$ to $+\pi$) for $\lambda_0 = 75$.

ARG(b)

$\tilde{x}$

Figure 6.7b. The amplitude of $b(\tilde{x})$ for $\lambda_0 = 75$.

$\tilde{x}$

Figure 6.8. A diagram of the various regimes of the critical layer when $\eta \neq 1$.
I: viscous, steady critical layer; Landau–Stuart–Watson equation.
II: strongly nonlinear, equilibrium critical layer; Benney & Bergeron theory.
$IIIa$: unsteady critical layer; largest term in (6.1) is cubic and due to non–viscous effects.
$IIIb$: unsteady critical layer; largest term in (6.1) is quintic.
$IIIc$: unsteady critical layer; largest term in (6.1) would be cubic and due to viscosity. Solid lines represent boundaries with areas where the critical layers are not unsteady, dashed lines represent boundaries between different base evolution equations from (6.1) and the dotted lines indicate the threshold values at which the nonlinear evolution equations become valid. The thick line on the diagram indicates the evolutionary path of the disturbance for sufficiently large values of the crossflow parameter.

A $\quad A \sim \mu^{\frac{3}{2}-\nu}$ $\quad A \sim \mu^{2-\nu}$

II $\quad IIIb$

$\kappa^{\frac{1}{2}-\frac{1}{3}\nu}$ $\quad IIIa$ $\quad A \sim \mu^{\frac{3}{2}-2\nu}$

$A \sim \kappa^{\frac{1}{2}}\mu^{-\nu}$

I $\quad IIIc$

$A \sim \kappa^{-\frac{1}{2}}\mu^{\frac{7}{2}-2\nu}$

0 $\quad \kappa^{\frac{1}{3}}$ $\quad \kappa^{\frac{1}{4}}$ $\quad \mu$