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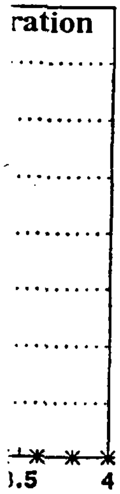

Optimal Control of Thermally Coupled Navier Stokes Equations

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The optimal boundary temperature control of the stationary thermally coupled incompressible Navier-Stokes equation is considered. Well-posedness and existence of the optimal control and a necessary optimality condition are obtained. Optimization algorithms based on the augmented Lagrangian method with second order update are discussed. A test example motivated by control of transport process in the high pressure vapor transport (HPVT) reactor is presented to demonstrate the applicability of our theoretical results and proposed algorithm.

1. Introduction

In this paper we discuss the optimal control problem of the stationary thermally coupled incompressible Navier-Stokes equations. Consider the following optimal control problem

$$\text{minimize} \quad J(g) = \varphi(u, T - T_0) + \frac{\beta}{2} |g - T_0|^2_{L^2(\Gamma_1)}, \quad g \in \mathcal{C} \tag{1}$$

subject to

$$\begin{gathered} u \cdot \nabla u + \nabla p = \nu \, \Delta u + \gamma \, (T - T_0) \, e_d + f \,, \\ \nabla \cdot u = 0, \quad u|_\Gamma = 0 \,, \\ u \cdot \nabla T = \nabla \cdot (\kappa \, \nabla T) \,, \\ T = T_0 \ \text{ on } \Gamma_0 \ \text{ and } \ n \cdot \nabla T = H \, (g - T) \ \text{ on } \Gamma_1 \,, \end{gathered} \tag{2}$$

where $f \in L^2(\Omega)^d$ is a source field, u, p, T stand for the nondimensionalized velocity vector in R^d with $d = 2, 3$, pressure, and temperature, respectively and $\mathcal{C}$ is the closed convex set in $L^2(\Gamma_1)$ such that

$$\bar{T}_1 \le g \le \bar{T}_2 \quad \text{on } \Gamma_1 \tag{3}$$

Here, $\bar{T}_1 \le T_0 \le \bar{T}_2$ and Γ_i, $i = 0, 1$ are disjoint open sets in Γ such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. The constant γ is given by $\gamma = \dfrac{\bar{g}}{T_0}$ where $T_0 > 0$ is a constant reference temperature and $\bar{g}$ is the gravitational constant, and e_d denotes the d−th unit vector of R^d. Throughout this paper we assume that Ω is sufficiently smooth and ν, κ and H are positive constants.

This control problem is motivated by control of transport and growth processes in the high pressure vapor transport (HPVT) reactor [13]. For example, we may

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consider the Scholz geometry (Figure 1 in [13]). The source material and the growing crystal are sealed in a fused silica ampoule that is heated by a furnace liner at its outer cylindrical surface. The substrate Γ_0 (the single crystal) is located on a fused silica window (the bottom of the ampoule) which is cooled by a jet of helium gas from the outer surface. HPVT processes are based on physical vapor transport and can be described very roughly as proceeding via evaporation at the polycrystalline source and condensation at the surface of the cooler substrate. The system (1)-(2) is called the Boussinesq equations, where we assume that the flow is incompressible and the transport phenomena of a single (carrier) gas is modeled. At the wall we assume Newton's law of cooling holds.

The cost-functional can be of tracking type

$$\varphi(u, T - T_0) = \int_\Omega |u - u_d|^2 + |T - T_d|^2 \, dx$$

where (u_d, T_d) is the desired state, or minimization of friction force of flow in a subregion Ω_1 of Ω

$$\varphi(u, T - T_0) = \int_{\Omega_1} |\nabla \times u|^2 \, dx \, .$$

2. Well-posedness

In this section, we discuss existence and uniqueness of solutions to (2). Let $U = L^2(\Gamma_1)$ be the control space, $V = V_0 \times V_1$ where V_0 is the divergence-free subspace of $(H_0^1(\Omega))^d$ [7] and

$$V_1 = H^1_{\Gamma_0} = \{\phi \in H^1(\Omega) : \phi|_{\Gamma_0} = 0\}$$

and set $H = H_0 \times L^2(\Omega)$. H_0 is the closure of V_0 with respect to the $L^2(\Omega)^d$-norm, and is defined by

$$H_0 = \{\phi \in L^2(\Omega)^d : \ \nabla \cdot \phi = 0 \text{ and } n \cdot \phi = 0 \text{ on } \Gamma\}.$$

H is equipped with the natural L^2-norm and V is equipped with the norm

$$|(\phi, \chi)|_V^2 = |\phi|_{V_0}^2 + |\chi|_{V_1}^2$$

where

$$|\phi|_{V_0}^2 = |\nabla \phi|_{L^2(\Omega)}^2 \quad \text{and} \quad |\chi|_{V_1}^2 = |\nabla \chi|_{L^2(\Omega)}^2 + H\, |\chi|_{L^2(\Gamma_1)}^2$$

for $\psi = (\phi, \chi) \in V$. Define the trilinear form b on $H^1(\Omega)^d$ by

$$b(\phi_1, \phi_2, \phi_3) = \langle \phi_1 \cdot \nabla \phi_2, \, \phi_3 \rangle \tag{4}$$

for $\phi_i \in H^1(\Omega)^d$, $i = 1, 2, 3$. Then we have

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Lemma 1 *The trilinear form b satisfies*

(a) $|b(\phi_1, \phi_2, \phi_3)| \le |\phi_1|_{L^4} |\nabla \phi_2|_{L^2} |\phi_3|_{L^4} \le M_1\, |\phi_1|_{H^1} |\phi_2|_{H^1} |\phi_3|_{H^1}$

(b) $b(\phi_1, \phi_2, \phi_3) + b(\phi_1, \phi_3, \phi_2) = 0$
provided that $\nabla \cdot \phi_1 = 0$ *and* $(n \cdot \phi_1)(\phi_2 \cdot \phi_3) = 0$ *on* Γ

(c) $|b(\phi_1, \phi_2, \phi_3)| \le M_2 |\phi_1|^{1/2}_{L^2} |\phi_1|^{1/2}_{H^1} |\phi_3|^{1/2}_{L^2} |\phi_3|^{1/2}_{H^1} |\phi_2|_{H^1}$ *for* $d = 2$

(d) $|b(\phi_1, \phi_2, \phi_3)| \le M_2 |\phi_1|^{1/4}_{L^2} |\phi_1|^{3/4}_{H^1} |\phi_3|^{1/4}_{L^2} |\phi_3|^{3/4}_{H^1} |\phi_2|_{H^1}$ *for* $d = 3$

for $\phi_i \in H^1(\Omega)^d,\ i = 1, 2, 3$

Proof: By Green's formula

$$b(\phi_1, \phi_2, \phi_3) + b(\phi_1, \phi_3, \phi_2) = (\phi_1, \nabla(\phi_2 \cdot \phi_3)) = (n \cdot \phi_1, \phi_1 \cdot \phi_3)_\Gamma$$

for $\phi_2 \in C^1(\Omega)$ and $\nabla \cdot \phi_1 = 0$. Hence (b) follows from the continuity of b. The last two assertions follow from the fact that $|\psi|_{L^4} \le c\, |\psi|^{1/2}_{L^2} |\psi|^{1/2}_{H^1}$ for $d = 2$ and $|\psi|_{L^4} \le c\, |\psi|^{3/4}_{L^2} |\psi|^{1/4}_{H^1}$ for $\psi \in H^1(\Omega)$ and some constant c. □

In particular, Lemma 1 implies that

$$b(u, \phi, \phi) = 0 \quad \text{for } u,\ \phi \in V_0\,. \tag{5}$$

The weak or variational form of (2) is given by

$$\nu\, (\nabla u, \nabla \phi) + b(u, u, \phi) = (f + \gamma\, (T - T_0) e_d, \phi) \tag{6}$$

for all $\phi \in V_0$ and

$$\begin{aligned} \kappa\, (\nabla(T - T_0), \nabla \chi) \quad &+ \quad \langle u \cdot \nabla(T - T_0), \chi \rangle_{V_1^* \times V_1} \\ &+ \quad \kappa\, (H\, (T - T_0), \chi)_{\Gamma_1} = \kappa\, (H\, (g - T_0), \chi)_{\Gamma_1}. \end{aligned} \tag{7}$$

for all $\chi \in V_1$. The pair $(u, T - T_0) \in V$ is said to be a weak solution of (2) if (6), (7) holds for all $\psi = (\phi, \chi) \in V$. Then we have the following theorem.

Theorem 2 *Given* $g \in L^2(\Gamma_1)$ *there exists a weak solution* $(u, T - T_0) \in V$ *to (2) and*

$$|(u, T - T_0)|_V \le const\, (|f|_{L^2} + |g|_{L^2(\Gamma_1)})\,.$$

Moreover, if $\bar{T}_1 \le g(x) \le \bar{T}_2$ *a.e. in* $x \in \Gamma_1$ *then* $\bar{T}_1 \le T(x) \le \bar{T}_2$ *a.e in* Ω *for every solution* $(u, T - T_0) \in V$.

Step 1: (Existence) We show that (6), (7) has a solution $z = (u, T - T_0) \in V$. Given $\bar{u} \in V_0$, we consider the linear equation

$$\nu\, (\nabla u, \nabla \phi) + b(\bar{u}, u, \phi) = (\gamma\, (T - T_0) e_d + f, \phi) \quad \text{for } \phi \in V_0 \tag{8}$$

$$\kappa\,(\nabla(T-T_0),\,\nabla\chi)+\langle\bar{u}\cdot\nabla(T-T_0),\,\chi\rangle \;+\; \kappa\,(H\,(T-T_0),\,\chi) \;=\; \kappa\,(H\,(g-T_0),\,\chi)_{\Gamma_1}\,, \qquad (9)$$

for $\chi \in V_1$. First, we show that (8),(9) has the unique solution $(u,\,T-T_0) \in V$. Then, we show that the solution map S on V_0 defined by $S(\bar{u}) = u$ where $(u,\,T-T_0) \in V$ is the unique solution to (8),(9) has a fixed point by Schauder fixed point theorem. The fixed point $u \in V_0$ and the corresponding solution $T-T_0 \in V_1$ define a solution to (6), (7). By Green's formula

$$\langle\bar{u}\cdot\nabla\theta,\,\chi\rangle+\langle\bar{u}\cdot\nabla\chi,\,\theta\rangle = 0 \quad \text{and in particular} \quad \langle\bar{u}\cdot\chi,\,\chi\rangle = 0 \qquad (10)$$

for $\bar{u} \in V_0$, and $\theta,\ \chi \in H^1(\Omega)$. Hence, from Lemma 1 the sesquilinear form

$$\kappa\,(\nabla\chi_1,\,\nabla\chi_2)+\langle\bar{u}\cdot\nabla\chi_1,\,\chi_2\rangle+(\kappa H\,\chi_1,\,\chi_2)_{\Gamma_1} = 0$$

on $V_1 \times V_1$ is bounded and V_1−coercive. It thus follows from the Lax-Milgram theorem that equation (9) has a unique solution $T-T_0 \in V_1$. Choosing $\chi = T-T_0$ in (9), we have (independent of $\bar{u} \in V_0$)

$$|T-T_0|^2_{V_1} \le H\,|g-T_0|^2_{L^2(\Gamma_1)}\,. \qquad (11)$$

Next, the sesquilinear form on $V_0 \times V_0$ defined by

$$\nu\,(\nabla\phi_1,\,\nabla\phi_2)+b(\bar{u},\,\phi_1,\,\phi_2)$$

is bounded and V_0−coercive from Lemma 1 and (5). Thus, by the Lax-Milgram theorem, equation (8) has a unique solution $u \in V_0$, and we have

$$|u|_{V_0} \le \frac{M_3}{\nu}\,(|f|_{L^2}+\gamma\,|T-T_0|_{L^2}) \qquad (12)$$

where $|\phi|_{H_0} \le M_3\,|\phi|_{V_0}$, $\phi \in V_0$. Let C be a closed convex subspace of V_0, defined by

$$C = \{\phi \in V_0 : |\phi|_{V_0} \le \frac{M_3}{\nu}\,(|f|_{L^2}+\gamma\,M_4 H\,|g-T_0|_{L^2(\Gamma_1)}\}\,,$$

where $|\chi|_{L^2} \le M_4\,|\chi|_{V_1}$ for $\chi \in V_1$. Then it follows from (11)-(12) that S maps from C into C. Moreover, the solution map S is compact. In fact, if $\bar{u}_k$ converges weakly to $\bar{u}$ in V_0 then $|\bar{u}_k-\bar{u}|_{L^4} \to 0$ since $H^1(\Omega)$ is compactly embedded into $L^4(\Omega)$. Let $(u_k,\,T_k-T_0) \in V$ and $(u,\,T-T_0) \in V$ be the corresponding solution of (8),(9), respectively to $\bar{u}_k \in V_0$ and $\bar{u} \in V_0$. Then we have

$$\kappa\,(T_k-T,\,\chi)_{V_1}+\langle(\bar{u}_k-\bar{u})\cdot\nabla(T-T_0)+\bar{u}_k\cdot\nabla(T_k-T),\,\chi\rangle = 0$$

for $\chi \in V_1$ from Lemma 1 and (10)

$$\kappa\,|T_k-T|_{V_1} \le M_1\,|\bar{u}_k-\bar{u}|_{L^4}|T-T_0|_{V_1}$$

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which implies $|T_k - T|_{V_1} \to 0$. Similarly, we have

$$\nu\, |u_k - u|_{V_0} \leq M_1\, |\bar{u}_k - \bar{u}|_{L^4} |u|_{V_0} + \gamma M_3\, |T_k - T|_{L^2}$$

and thus $|u_k - u|_{V_0} \to 0$. Now, by the Schauder fixed point theorem (e.g, see [19]) there exists at least one solution to (6), (7).

Step 2: (L^∞ estimate) We show that if $\bar{T}_1 \leq g \leq \bar{T}_2$ then

$$\bar{T}_1 \leq T \leq \bar{T}_2 \quad \text{a.e. } x \in \Omega\,.$$

for all solutions $(u, T - T_0) \in V$ to (6), (7). In fact, let $\chi = \inf(T, \bar{T}_1)$. Then $\chi \in V_1$ [19] and we have from (6), (7)

$$\kappa\, (\nabla T,\ \nabla\chi) + \langle u \cdot \nabla T,\ \chi\rangle + (\kappa H\, (T - g),\ \chi)_{\Gamma_1} = 0\,.$$

Since from (10) $\langle u \cdot \nabla T,\ \chi\rangle = 0$ we have

$$\kappa\, (\nabla\chi,\ \nabla\chi) + (\kappa H\, (T - g),\ \chi)_{\Gamma_1} = 0$$

where

$$(T - g)\chi = (T - \bar{T}_1 - (g - \bar{T}_1))\,\chi \geq |\chi|^2 \ \text{ on } \Gamma_1\,.$$

Thus, we obtain $|\chi|_{V_1}^2 = 0$ which implies $\chi = 0$ and hence $T \geq \bar{T}_1$. Similarly, one can prove that $T \leq \bar{T}_2$, choosing the test function $\chi = \sup(T, \bar{T}_2)$.

We have also the uniqueness of solutions under the smallness assumption on f and $g - T_0$.

Theorem 3 *If $|f|_{L^2}$ and $\frac{1}{T_0}|g - T_0|_{L^2(\Gamma_1)}$ are sufficiently small then (6), (7) has a unique solution in V.*

Proof: Suppose $(u_i, T_i - T_0) \in V,\ i = 1, 2$ are two solutions to (6), (7). Then we have

$$\begin{aligned} &\nu\, (\nabla\tilde{u},\ \nabla\phi) + \langle u_1 \cdot \nabla\tilde{u} + \tilde{u} \cdot \nabla u_2,\ \phi\rangle = \gamma\, (\tilde{T}\, e_d,\ \phi) \\ &\kappa\, (\nabla\tilde{T},\ \nabla\chi) + \langle u_1 \cdot \nabla\tilde{T} + \tilde{u} \cdot \nabla(T_2 - T_0),\ \chi\rangle + (\kappa H\, \tilde{T},\ \chi)_{\Gamma_1} = 0 \end{aligned}$$

for $\phi \in V_0$ and $\chi \in V_1$, where $\tilde{u} = u_1 - u_2 \in V_0$ and $\tilde{T} = T_1 - T_2 \in V_1$. Setting $\phi = \tilde{u}$ and $\chi = \tilde{T}$ we obtain from (5) and (10)

$$\begin{aligned} &\nu\, |\tilde{u}|_{V_0}^2 \leq M_1\, |u_2|_{V_0} |\tilde{u}|_{V_0}^2 + \gamma M_3 M_4\, |\tilde{T}|_{V_1} |\tilde{u}|_{V_0} \\ &\kappa\, (|\nabla\tilde{T}|^2 + H\, |\tilde{T}|_{\Gamma_1}^2) \leq M_1 |T_2 - T_0|_{V_1} |\tilde{u}|_{V_0} |\tilde{T}|_{V_1}\,. \end{aligned}$$

If we set $X = |\tilde{u}|_{V_0}$ and $Y = |\tilde{T}|_{V_1}$ then this implies

$$(\nu - M_1\, |u_2|_{V_0})\, X^2 \leq \gamma M_3 M_4\, XY \quad \text{and} \quad \kappa\, Y^2 \leq M_1 |T_2 - T_0|_{V_1} XY\,.$$

Hence if $\kappa(\nu - M_1\,|u_2|_{V_0}) - \gamma M_1 M_3 M_4\,|T_2 - T_0|_{V_1} > 0$ then $X = Y = 0$ and thus $(u_1, T_1) = (u_2, T_2)$. From (11)-(12) we have

$$\begin{aligned} |T_2 - T_0|_{V_1} &\le H\,|g - T_0|_{L^2(\Gamma_1)} \\ |u_2|_{V_0} &\le \frac{M_3}{\nu}\,(|f|_{L_2} + \gamma M_4 H\,|g - T_0|_{L^2(\Gamma_1)}). \end{aligned} \tag{13}$$

Thus, if $|f|_{L^2}$ and $\frac{1}{T_0}|g - T_0|_{L^2(\Gamma_1)}$ are sufficiently small then (6), (7) has a unique solution. □

Moreover, we can make the following demonstration of the regularity of the solution $(u, T - T_0) \in V$. Define the Stokes operator A on H_0 by

$$(Au,\ \phi)_{H_0} = (\nabla u,\ \nabla\phi) \quad \text{for } \phi \in V_0 \tag{14}$$

with domain

$$\mathrm{dom}(A) = \{u \in V_0 :\ |(\nabla u,\ \nabla\phi)| \le c\,|\phi|_{H_0} \text{ for all } \phi \in V_0\}. \tag{15}$$

Then it is known [18] that A is a positive self-adjoint operator on H_0, $\mathrm{dom}(A) \subset H^2(\Omega)^3$ and $V_0 = \mathrm{dom}(A^{1/2}) = [H_0, \mathrm{dom}(A)]_{1/2}$. Let $d = 3$. Since

$$|(u \cdot \nabla u,\ \phi)| \le |u|_{L^6} |\nabla u|_{L^2} |\phi|_{L^3}$$

for $u,\ \phi \in V_0$ and

$$H^1(\Omega) \subset L^6(\Omega) \quad \text{and} \quad V_{1/2} \subset L^3(\Omega),$$

where $V_{1/2} = [V_0, H_0]_{1/2}$, $u\nabla u \in V_{-1/2} = \mathrm{dom}(A^{-1/4})$ for $u \in V_0$. Thus,

$$u = A^{-1}(\rho(T - T_0)\,e_3 + f - u\cdot\nabla u) \in \mathrm{dom}(A^{3/4}) = [V, \mathrm{dom}(A)]_{1/2} \subset H^{3/2}(\Omega)^3.$$

Hence, $u \cdot \nabla u \in L^2(\Omega)^d$ and $u \in \mathrm{dom}(A) \subset H^2(\Omega)^3$.

3. Necessary optimality condition

We now show the existence of solutions to the optimal control problem (1)-(2). Let us denote by $S(g)$, the solution set of (6), (7) for $g \in L^2(\Gamma_1)$.

Theorem 4 *Consider the minimization problem (1)-(2):*

$$\begin{aligned} \textit{minimize}\quad & J(g) = \varphi(u, T - T_0) + \tfrac{\beta}{2}\,|g - T_0|^2_{L^2(\Gamma_1)} \\ & \textit{over } (u, T - T_0) \in S(g) \textit{ and } g \in \mathcal{C}. \end{aligned} \tag{16}$$

Assume that φ satisfies

$$\begin{aligned} &\varphi(z) :\ z = (u, T - T_0) \in V \to R^+ \textit{ is convex and lower semicontinous} \\ &\textit{and } \varphi(z) \le b_1\,|z|_V^2 + b_2 \textit{ for } b_1,\ b_2 \in R^+. \end{aligned}$$

Then Problem (1)-(2) has a solution.

Proof: Let $((u_k, T_k - T_0), g_k) \in S(g_k) \times \mathcal{C}$ be a minimizing sequence. Since $\beta > 0$, $|g_k - T_0|_{L^2(\Gamma_1)}$ is uniformly bounded in k and thus from (11)-(12) so is $|(u_k, T_k - T_0)|_V$. Hence there exists a subsequence of $\{k\}$, which will be denoted by the same index, such that $(u_k, T_k - T_0, g_k)$ converges weakly to $(u, T - T_0, g) \in V \times \mathcal{C}$ since $V \times L^2(\Gamma_1)$ is a Hilbert space and $\mathcal{C}$ is closed and convex. Define the trilinear form $\tilde{b}$ on V^3 by

$$\tilde{b}(z_1, z_2, \psi) = b(u_1, u_2, \phi) + (u_1 \cdot \nabla\theta_2, \chi)$$

for $z_i = (u_i, \theta_i)$, $i = 1, 2$, $\psi = (\phi, \chi) \in V$. Since $H^1(\Omega)$ is compactly embedded into $L^4(\Omega)$ it follows from Lemma 1 that $\tilde{b}(z_k, z_k, \psi) \to \tilde{b}(z, z, \psi)$ for $\psi \in V$, where $z_k = (u_k, T_k - T_0)$ and $z = (u, T - T_0)$. Hence, for $\psi = (\phi, \chi) \in V$ the limit (z, g) satisfies (6), (7) and thus $z \in S(g)$. Now, since φ is convex and lower semicontinuous, it follows from [5] that (z, g) minimizes (16). □

Problem (16) is equivalently written as a constrained minimization on $x = ((u, T - T_0), g) \in X = V \times L^2(\Gamma_1)$ with

$$\begin{aligned} &\text{minimize} \quad J(x) = \varphi(u, T - T_0) + \tfrac{\beta}{2}\,|g - T_0|^2_{L^2(\Gamma_1)} \quad \text{over } x \in X \\ &\qquad\qquad \text{subject to } e(x) = 0 \text{ and } g \in \mathcal{C} \end{aligned} \tag{17}$$

where the equality constraint $e : X \to Y = V^*$ is defined by (6), (7); i.e.,

$$\langle e(x), \psi\rangle = a(z, \psi) + \tilde{b}(z, z, \psi) - \kappa\,(H(g - T_0), \chi)_{\Gamma_1} - (f, \phi) \tag{18}$$

for $\psi = (\phi, \chi) \in V$, where

$$a(z, \psi) = \nu\,(\nabla u, \nabla\phi) - \gamma\,((T - T_0)\,e_d, \phi) + \kappa\,(\nabla(T - T_0), \nabla\chi) + \kappa H\,(T - T_0, \chi)_{\Gamma_1}$$

for $z = (u, T - T_0)$, $\psi = (\phi, \chi) \in V$. Assume that $x^* = (z^* = (u^*, T^* - T_0), g^*)$ denotes the optimal pair of (16). Then we have

Theorem 5 *Assume that x^* is a regular point in the sense [14] that*

$$0 \in \mathit{int}\,\{e'(x^*)(v, h - g^*) : v \in V \text{ and } h \in \mathcal{C}\}. \tag{19}$$

Then there exists a Lagrange multiplier $\lambda^* \in V$ such that

$$a(\lambda^*, \psi) + \tilde{b}(\psi, z^*, \lambda^*) + \tilde{b}(z^*, \psi, \lambda^*) + \langle\varphi'(z^*), \psi\rangle = 0 \tag{20}$$

for $\psi \in V$ and

$$(\beta\, g^* - \kappa H\, \lambda_2^*,\ h - g^*)_{\Gamma_1} \ge 0 \quad \text{for all } h \in \mathcal{C}. \quad \square \tag{21}$$

Proof: It follows from [14] that if (19) is satisfied, then there exists a Lagrange multiplier $\lambda^* = (\lambda_1^*, \lambda_2^*) \in V$ such that

$$\langle\varphi'(z^*), \psi\rangle + \beta\,(g^*, h - g^*)_{\Gamma_1} + e'(x^*)(\psi, h - g^*) \ge 0$$

for all $\psi \in V$ and $h \in \mathcal{C}$, that is

$$\begin{aligned}\langle \varphi'(z^*), \psi\rangle + \beta\,(g^*, h - g^*)_{\Gamma_1} + a(\lambda^*, \psi)\\ +\tilde{b}(\psi, z^*, \lambda^*) + \tilde{b}(z^*, \psi, \lambda^*) - \kappa H\,(h - g^*, \lambda_2^*)_{\Gamma_1} \geq 0\end{aligned} \tag{22}$$

for all $\psi \in V$ and $h \in K$. Setting $\psi = 0$, we obtain (21). Next, setting $h = g^*$ in (22), we obtain (20). □

Concerning the regular point condition (19), we have

Lemma 6 *If $g^* \in int\,(\mathcal{C})$ then the regular point condition (19) is equivalent to the condition that for $v = (v_1, v_2) \in V$*

$$\begin{aligned}a(v, \psi) + \tilde{b}(\psi, z^*, v) + \tilde{b}(z^*, \psi, v) = 0\\ \textit{for all } \psi \in V \quad \textit{and} \quad v_2 = 0 \ \textit{on} \ \Gamma_1\end{aligned} \tag{23}$$

implies $v = 0$.

Proof: If $g^* \in \mathrm{int}\,(\mathcal{C})$ then (19) is equivalent to the condition that $G = e'(x^*)$ is surjective. Define the linear map $C \in \mathcal{L}(X, V)$ by $C(v, h) = \xi$ where $\xi \in V$ is the unique solution to

$$a(\xi, \psi) + \tilde{b}(v, z^*, \psi) + \tilde{b}(z^*, v, \psi) - \kappa H\,(h, \chi)_{\Gamma_1} = 0 \quad \text{for } \psi = (\phi, \chi) \in V.$$

Then, since $H^1(\Omega)$ is embedded compactly to $L^4(\Omega)$, Lemma 1 implies that C is compact. Thus, by the Banach closed range theorem and the Riesz-Schauder theorem, $e'(x^*)(v, h)$ is surjective if and only if $ker(G^*) = \{0\}$ [4], which is equivalent to (21). □

4. Augmented Lagrangian method

In this section we discuss applications of the augmented Lagrangian method for the constrained minimization problem (17). The augmented Lagrangian method [8], [15] is based on an equivalent formulation of (17):

$$\text{minimize} \quad J(x) + \frac{c}{2}\,|e(x)|_Y^2 \quad \text{over } x \in X \text{ and } g \in \mathcal{C}. \tag{24}$$

subject to $e(x) = 0$, where $x = ((u, T - T_0), g) \in X = V \times L^2(\Gamma_1)$ and $c > 0$ is the penalty parameter. The augmented Lagrangian algorithm [8], [15] is the multiplier method applied to (17), i.e., it involves a sequence of minimizations of the functional

$$\begin{aligned}L_{c_k}(x; \lambda^k) = J(x) + \langle \lambda^k, e(x)\rangle + \frac{c}{2}\,|e(x)|_Y^2\\ \text{subject to } g \in \mathcal{C},\end{aligned} \tag{25}$$

where the multiplier sequence $\{\lambda^k\}$ in Y^* is generated by the first order update

$$\lambda^{k+1} = \lambda^k + c_k\, e(x_k)\,. \tag{26}$$

for $k \geq 1$, where x_k is a minimizer of $L_{c_k}(\cdot, \lambda^k)$ and we assumed that $Y^* = Y$, otherwise each element in Y^* has its Riesz representation. To carry out this iteration, a sequence of monotonically nondecreasing, positive real numbers $\{c_k\}$, $c_1 > c_0 \geq 0$ and a start up value λ^1 for the Lagrange multiplier for the equality constraint $e(x) = 0$ need to be chosen. The convergence results of the augmented Lagrangian method for the infinite dimensional optimization problem are established, for example, in [9], [16]. The augmented Lagrangian method is a hybrid of the penalty method (i.e., $\lambda^k = 0$) and the Lagrange multiplier method (i.e., $c_k = 0$) and combines good properties of both. It overcomes the difficulty of the penalty method which requires a large value of c_k. The augmented functional $L_{c_k}(x, \lambda^k)$ is locally strictly convex provided that λ^k is sufficiently close to λ^* and the second order optimality condition

$$\begin{gathered} L_0''(x^*, \lambda^*)((v, h), (v, h)) \geq \sigma\, (|v|_V^2 + |h|_U^2) \\ \text{for all } (v, h) \in X \text{ satisfying } e'(x^*)(v, h) = 0\,, \end{gathered} \tag{27}$$

for some $\sigma > 0$, is satisfied. Here, $L_0''(x^*, \lambda^*)$ denotes the bilinear form that characterizes the second derivative of $L_0(x, \lambda) = J(x) + \langle \lambda,\ e(x) \rangle$ with respect to x at x^*. That is, it is not necessary that the cost functional J be (locally) convex, which is required for convergence of the multiplier method. The algorithm (25)-(26) has been successfully applied to parameter estimation problems in elliptic PDEs [10], [12] and optimal control problems for 2-D incompressible Navier-Stokes [4]. The first order update (26) provides Q-linear convergence of the iterates x_k in X. In [11] we have investigated a second order update scheme for the augmented Lagrangian method. In what follows we assume that $g^* \in \text{int}\,(C)$. Thus, (19) reduces to $e'(x^*)$ is surjective (see Lemma 6). Hence the necessary condition implies that

$$L_c'(x^*, \lambda^*) = 0 \quad \text{and} \quad e(x^*) = 0, \tag{28}$$

for all $c \geq 0$. An algorithm proposed in [11] applies Newton's method to (28). The resulting algorithm is stated as: given a current iterate (x, λ) the next iterate (x_+, λ_+) satisfies

$$\begin{pmatrix} L_c''(x, \lambda) & e'(x)^* \\ e'(x) & 0 \end{pmatrix} \begin{pmatrix} x_+ - x \\ \lambda_+ - \lambda \end{pmatrix} = - \begin{pmatrix} L_c'(x, \lambda) \\ e(x) \end{pmatrix}. \tag{29}$$

Note that

$$L_c'(x, \lambda) = L_0'(x, \lambda + c\, e(x))$$

and

$$L_c''(x, \lambda) = L_0''(x, \lambda + c\, e(x)) + c\, \langle e'(x)(\cdot),\ e'(x)(\cdot) \rangle\,. \tag{30}$$

Consequently, suppose $|(x,\lambda)-(x^*,\lambda^*)|$ is sufficiently small. Then it follows from (27) [11] that $L_c''(x,\lambda)$ is coercive on $X\times Y$. Thus equation (29) can be regarded as a general Stokes equation. Following an argument due to Bertsekes [2], we can avoid forming L_c'' during the iteration. From the second equation of (29) we have $e'(x)(x_+-x)=-e(x)$. Thus the first equation can be written as

$$L_0''(x,\lambda+c\,e(x))(x_+-x)+e'(x)^*(\lambda_+-(\lambda+c\,e(x)))=-L_0'(x,\lambda+c\,e(x))$$

and hence (29) is equivalent to

$$\begin{pmatrix} L_0''(x,\hat\lambda) & e'(x)^* \\ e'(x) & 0 \end{pmatrix}\begin{pmatrix} x_+-x \\ \lambda_+-\hat\lambda \end{pmatrix}=-\begin{pmatrix} L_0'(x,\hat\lambda) \\ e(x) \end{pmatrix} \tag{31}$$

where $\hat\lambda=\lambda+c\,e(x)$.

Note that $\hat\lambda$ is nothing but the first order update of the Lagrange multiplier if the current iterate x minimizes $L_c(x,\lambda)$. Equation (31) is more advantageous than (29) since the squaring term $c\,e'(x)^*e'(x)$ is absorbed and less calculation is involved. If we define a matrix operator S on $X\times Y$ by

$$S(x,\lambda)=\begin{pmatrix} L_0''(x,\lambda) & e'(x)^* \\ e'(x) & 0 \end{pmatrix},$$

then it follows from (27) that $S(x^*,\lambda^*)$ is boundedly invertible. Thus, if (x,λ) is sufficiently close to (x^*,λ^*), then equation (31) has a unique solution. We summarize our discussions as

Algorithm 1

(1) Choose $\lambda^1\in Y$, $c>\bar c\geq 0$, and set $\hat c=c-\bar c$, $k=1$.

(2) Determine $x=((u,T-T_0),g)\in V\times C$ such that

$$L_c(x,\lambda^k)\leq L_c(x^*,\lambda^k)=f(x^*).$$

(3) Set $\hat\lambda=\lambda^k+\hat c\,e(x)$.

(4) Solve for $(x_+,\lambda_+)\in X\times Y$:

$$S(x,\hat\lambda)\begin{pmatrix} x_+-x \\ \lambda_+-\hat\lambda \end{pmatrix}=-\begin{pmatrix} L_0'(x,\hat\lambda) \\ e(x) \end{pmatrix}.$$

(5) Set $x_{k+1}=x_+$ and $\lambda^{k+1}=\lambda_+$. If the convergence criterion is not satisfied then set $k=k+1$ and go to (2).

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Remark. A variant of Algorithm 1 is obtained by skipping step (2). Then it is reduced to the Newton method applied to equation (28). Step (2) implies a sufficient reduction of the merit functional (the augmented Lagrange functional). For example, if $x = (z, g)$ minimizes $L_c(\cdot, \lambda^k)$ over $V \times U$ then step (2) is completed. Assume that (19) and (27) hold. It is proved in [11] that if $|\lambda^1 - \lambda^*|_Y$ is sufficiently small, then Algorithm 1 is well-posed and (x^k, λ^k) converges to (x^*, λ^*) Q-quadratically.

5. Hybrid method and test example

In this section we present an example to demonstrate the applicability of our theoretical results and proposed algorithm. We consider the optimal control of the two dimensional stationary thermally driven cavity flow

$$\begin{aligned} \text{minimize} \quad & J(g) = \tfrac{1}{2}\int_\Omega |u - u_d|^2 + |T - T_d|^2\, dx + \tfrac{\beta}{2}|g - T_0|^2_{L^2(\Gamma_1)} \\ & \text{over } g \in L^2(\Gamma_1) \end{aligned} \tag{32}$$

subject to (2), where $\Omega = (0, L)^2$, $\Gamma_1 = (x_1, 1)$, $0 < x_1 < L$ and Γ_0 is the relative interior of $\Gamma - \Gamma_1$. On Γ_0 we have the Dirichlet boundary condition:

$$\begin{aligned} & T(0, x_2) = T(L, x_2) = T_0, \quad 0 < x_2 < L \\ & T(x_1, 0) = T_0 - 50\min(1, \tfrac{2}{3}(2.5 - |\tfrac{5}{L}x_1 - 2.5|), \quad 0 < x_1 < L\,. \end{aligned} \tag{33}$$

with a reference temperature $T_0 = 1350^o K$, and the temperature on the substrate $(x_1, 0)$, $\frac{1.5}{5}L \le x_1 \le \frac{3.5}{5}L$ is $1300^o K$. The desired state (u_d, T_d) appearing in (32) is chosen as follows. $u_d = L\, u^0$ where (u^0, T^0) is the solution pair for the flow with $L = 1$ and $g = T_0$ and $T_d = T^1$ where (u^1, T^1) is the solution pair for the flow with $L = 5$ and $g = T_0$. Here, our numerical calculation strongly indicates that the solution to (2) is unique and (dynamically) stable for the range of physical parameters that we chosen for our calculation.

The problem is scaled so that the velocity field u has dimension cm/sec and T has dimension K^o. The constants ν, κ are chosen for P_2 at $3atm$ pressure and are given by $\nu = .155$ and $\kappa = .110$. H, β and L are set to be $H = 100$, $\beta = \frac{1000}{1350^2} = 5.5 \times 10^{-5}$ and $L = 5$ respectively. Figures 1 and 2 show the vector field of u^0 and u^1, respectively. It can be observed that u^0 has more vertical transport than u^1 does and u^1 is confined in the two bottom corners. Hence the cost-functional (24) is formulated so that the thermal control g on the top Γ_1 increases the vertical transport of flow with $L = 5$ while retaining the temperature distribution T^1 at the reference temperature $g = T_0$.

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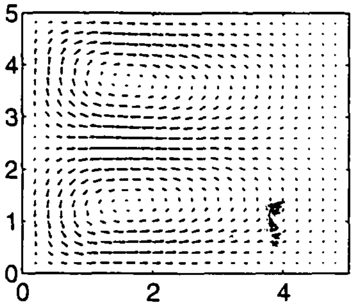

Figure 1. The desired flow.

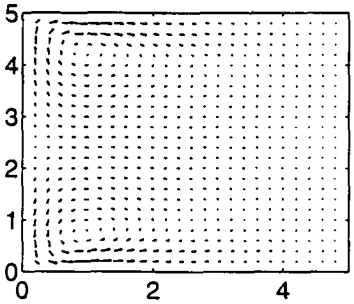

Figure 2. The uncontrolled flow.

The second order augmented Lagrangian method (Algorithm 1) described in Section 4 is used to solve problem (32). To obtain a good starting value λ^1 for the Lagrange multiplier in Algorithm 1, we employed a few steps of the gradient method. The gradient of the cost functional can be calculated by the adjoint equation as follows. Consider the cost functional $J(z, g)$ subject to $E(z, g) = 0$ where $z \in V$ and $g \in U$. Assume that V and U are Hilbert spaces, $J(z, g) : X = V \times U \to R$ is Fréchet differentiable and $E(z, g) : X \to Y$ is continuously Fréchet differentiable in a neighborhood of $x_0 = (z_0, g_0)$. Suppose that $E_z(x_0) \in \mathcal{L}(V, Y)$, the F−derivative of E with respect to z at x_0, has a bounded inverse. Then by the implicit function theorem, there exists a unique C^1 mapping Ψ defined in a neighborhood N of g_0 in U, $\Psi : U \to V$ such that $\Psi(g_0) = z_0$ and $E(\Psi(g), g) = 0$ for $g \in N$. Then the F−derivative d of $J(\Psi(g), g)$ with respect to g in N exists and is given by

$$d = J_g + E_g^* \psi \tag{34}$$

where $\psi \in Y$ satisfies the adjoint equation

$$E_z^* \psi + J_z = 0\,. \tag{35}$$

Here we assume that V, U and Y are identified with their dual spaces. In fact, if $v = \Psi'(h)$, $h \in U$ then

$$(d, h)_U = (J_g, h)_U + (J_z, v)_V$$

and $v \in V$ satisfies

$$E_z v + E_g h = 0\,.$$

Hence, for $h \in U$

$$(d, h)_U = (J_g, h)_U - (E_z^* \psi, v)_V = (J_g, h)_U + (\lambda, E_z h)_V = (J_g + E_z^* \psi, h)$$

which implies (34). The projected gradient method can be written as:

Algorithm 2

(1) Choose $g_1 \in U$ and set $k = 1$.

(2) Let z_k be a solution of $E(z, g_k) = 0$, ψ_k be the solution of $E_z(z_k, g_k)^* \psi_k + J_z(z_k, g_k) = 0$ and set $d_k = J_g(z_k, g_k) + E_g(z_k, g_k)^* \psi_k$.

(3) Set $d_k = J_g(z_k, g_k)$ and determine $\alpha_k \geq 0$ such that $J(\Psi(g_\alpha), g_\alpha)$ is minimized where $g_\alpha = Proj_C(g_k - \alpha\, d_k)$.

(4) Set $g_{k+1} = g_k - \alpha_k\, d_k$. If the convergence criterion is not satisfied, then set $k = k + 1$ and go to Step (2).

In our specific example equation (35) is written as

$$\begin{gathered} -\nu\, \Delta\lambda - u \cdot \nabla\lambda + \lambda^t \nabla u - (T - T_0)\, \nabla\mu + \nabla q + u - u_d = 0 \\ \nabla \cdot \lambda = 0 \quad \text{and} \quad \lambda|_\Gamma = 0 \\ -\kappa\, \Delta\mu - u \cdot \nabla\mu - \frac{g}{T_0}\, \lambda_2 + T - T_d = 0 \\ \mu = 0 \text{ on } \Gamma_0 \quad \text{and} \quad n \cdot \nabla\mu + \kappa H\, \mu = 0 \text{ on } \Gamma_1\,, \end{gathered} \tag{36}$$

where $\psi = (\lambda, \mu) \in V = V_0 \times V_1$, $q \in L^2(\Omega)$ and

$$E_z^* \psi = -\kappa H\, \mu \quad \text{on } \Gamma_1\,. \tag{37}$$

We used the mixed-finite element method [7] based on the Legendre polynomials to approximate problem (1)-(2) numerically. Detailed discussions about the method are given in [13]. In our implementation we calculated the adjoint system for the approximated problem and solved equations (2) and (36) using GMRES [3]. The specific implementation of GMRES applied to our example is described in [13] in detail. Concerning the divergence-free constraint $\nabla \cdot u = 0$, we employed the feasible method, projecting the first equation onto the divergence-free space V_0 as in [4], [6], [13]. The line search in Step 3 of Algorithm 2 was performed by the linearization of the constraint $E(z, g) = 0$ at (z_k, g_k) since the cost functional is quadratic. That is, if v_k is the solution to $E_z(z_k, g_k)v_k + E_g(z_k, g_k)d_k = 0$ then $\alpha > 0$ is chosen so that $J(z_k - \alpha\, v_k, g_k - \alpha\, d_k)$ is minimized.

For this specific example, three steps of Algorithm 2 were performed. We then set x^1 and λ^1 as $x^1 = (z_4, g_4)$ and $\lambda^1 = \psi_3$ for Algorithm 1 without Step 2. The matrix operator $S(x, \lambda)$ was calculated for the approximated system and the resulting linear equation (31) was again solved by GMRES.

The calculations were performed using a 20×20 Cartesian product of Legendre polynomials, choosing $c = 1$ and $g_1 = 0$. Algorithm 1 was terminated after three iterates, since the necessary and sufficient optimality condition (28) was satisfied within a residual norm of 1×10^{-7}.

We may compare this rapid convergence of the hybrid method with the results of using either algorithm by itself. Algorithm 2 did not fully converge after 50 iterates. Algorithm 1, with the start-up $x^1 = ((u^1, T^1), 0) \in X$ and $\lambda^1 = 0$, also failed to converge. Thus, the use of the hybrid method combining the gradient method and the second-order augmented Lagrangian was essential for the success of our numerical calculations. Figure 3 shows the iterates g_k (the first three curves from top to middle) for Algorithm 1 and the (calculated) optimal control g^* (the lowest curve). The iterates for Algorithm 1 are not shown because they coincide with g^* within the accuracy of the plotting. Figure 4 shows the resulting vector field u^* which corresponds to g^*. It shows clearly that the vertical transport of flow is increased.

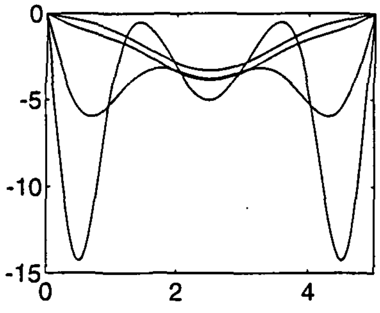

Figure 3. Optimal control iterates.

References

[1] F. Abergel
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[2] D. Bertseka
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[3] P. Brown an
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[4] M. Desai ar
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[5] I. Ekeland a
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[6] R. Glowins
Springer, B
[7] V. Girault ar
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[8] M.R.Hester
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[9] K. Ito and
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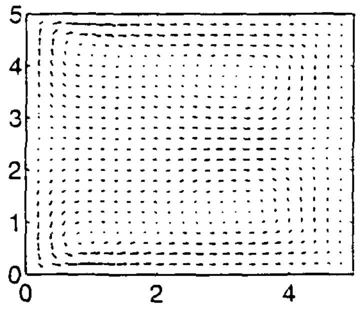

Figure 4. The controlled flow.

References

[1] F. Abergel and R. Temam; On some control problems in fluid mechanics, *Theor. Comp. Fluid Mech.* 1, 1990, 303–325.

[2] D. Bertsekas; *Constraint Optimization an Lagrange Multiplier Methods*, Academic, Paris, 1982.

[3] P. Brown and Y. Saad; Hybrid Krylov methods for nonlinear systems of equations, *SIAM J. Sci. Statist. Comput.* 11, 1990, 450–481.

[4] M. Desai and K. Ito; Optimal control of Navier-Stokes equations, *SIAM J. Cont.*, to appear.

[5] I. Ekeland and R. Temam; *Convex Analysis and Variational Problems*, North Holland, Amsterdam, 1976.

[6] R. Glowinski; *Numerical Methods for Nonlinear Variational Problems*, Springer, Berlin, 1984.

[7] V. Girault and P.-A. Raviart; *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1984.

[8] M.R.Hestenes; Multiplier and gradient methods, *J. Opt. Theory Appl.* 4, 1968, 303–320.

[9] K. Ito and K. Kunisch; The augmented Lagrangian method for equality and inequality constraints in Hilbert spaces, *Math. Prog.* 46 ,1990, 341–360.

[10] K. Ito and K. Kunisch; The augmented Lagrangian method for parameter estimation in elliptic systems, *SIAM J. Cont. & Optm.* 28, 1990, 113–136.

[11] K. Ito and K. Kunisch; The augmented Lagrangian method with second order update and its application to parameter estimation in elliptic systems, to appear.

[12] K. Ito, M. Kroller, and K. Kunisch; A numerical study of an augmented Lagrangian method for the estimation of parameters in elliptic systems, *SIAM J. Sci. Stat. Comput.* **12**, 1991, 884–910.

[13] K. Ito, J. Scroggs. and H. Tran; Mathematical issues in optimal design of a vapor transport reactor, *Flow Control* (Ed. by M. Gunzburger), Springer, 1995, to appear.

[14] H. Maurer and J. Zowe; First and second-order necessary and sufficient optimality conditions for infinite-dimensional programming problems, *Math. Prog.* **16**, 1979, 98–110.

[15] M. Powell; A method for nonlinear constraints in minimization problems, in *Optimization* (Ed. by R. Fletcher), Academic, New York, 1969.

[16] V. Polyak and Tret'yakov; The method of penalty estimates for conditional extremum problems, *Z. Vychisl. Mat. i Mat. Fiz.* **13** 1973, 34–46.

[17] R. Temam; *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.

[18] R. Temam; *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1988.

[19] G. Troianiello; *Elliptic Differential Equations and Obstacle Problems*, Plenum, New York, 1987.