hp-ADAPTIVITY AND ERROR ESTIMATION FOR HYPERBOLIC CONSERVATION LAWS

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SUMMARY

This paper presents an *hp*-adaptive discontinuous Galerkin method for linear hyperbolic conservation laws. *A priori* and *a posteriori* error estimates are derived in mesh-dependent norms which reflect the dependence of the approximate solution on the element size (h) and the degree (p) of the local polynomial approximation. The *a posteriori* error estimate, based on the element residual method, provides bounds on the actual global error in the approximate solution. The adaptive strategy is designed to deliver an approximate solution with the specified level of error in three steps. The *a posteriori* estimate is used to assess the accuracy of a given approximate solution and the *a priori* estimate is used to predict the mesh refinements and polynomial enrichment needed to deliver the desired solution. Numerical examples demonstrate the reliability of the *a posteriori* error estimates and the effectiveness of the *hp*-adaptive strategy.

INTRODUCTION

Adaptive methods, based on successive refinement of an existing mesh or a complete re-meshing of the computational domain, have become invaluable tools in computational fluid dynamics. The amount of refinement or the clustering of grid points is often determined by an element refinement indicator, θ_K, of the form

$$\theta_K = h^{\alpha}|D^n u| \tag{1}$$

where h is a measure of the element size, α is an exponent, and $D^n u$ represents some higher-order derivative of a key variable. Since these indicators are based on interpolation or truncation error estimates, they are applicable to a large class of problems and are independent of the numerical method used to obtain an approximate solution. Although these indicators detect certain flow features, they may not relate to the actual error in the solution. Often, these indicators only provide a relative measure of the error and do not provide a criteria for stopping the adaptive process.

This paper summarizes some of the work presented in [4] aimed at developing *hp*-adaptive strategies based on reliable error estimates for hyperbolic conservation laws. The work focuses on the discontinuous Galerkin method applied to a model class of linear hyperbolic conservation laws for which it is possible to develop mathematically rigorous *a priori* and *a posteriori* error estimates.

The notion of discontinuous Galerkin methods for hyperbolic problems originated in the classical work of Lesaint and Raviart [1] over two decades ago. Johnson and Pitkaranta [2] generalized the theory of discontinuous Galerkin methods by introducing mesh-dependent norms and were able to derive *a priori* error estimates in such norms for linear hyperbolic problems. Discontinuous Galerkin methods were extended to nonlinear hyperbolic conservation laws by Cockburn, Hou, and Shu [3] who developed a local projection strategy to provide nonlinear stability.

The local nature of the discontinuous Galerkin method makes it ideally suited for adaptive strategies which combine local mesh refinement (h) with local enrichment of the polynomial approximation (p) in an element to improve solution accuracy. The theory of discontinuous Galerkin methods was extended to *hp*-finite element approximations by Bey in [4] for a class of linear hyperbolic conservation laws. In [4], very high accuracies and convergence rates were observed in applying discontinuous Galerkin methods to representative test problems.

The *a posteriori* error estimates developed in [4] are based on the element residual method and provide bounds on the global error. Error estimates are combined with an *hp*-adaptive strategy that predicts the mesh required to deliver a solution with the specified level of error.

The theoretical developments of [4] are summarized in this paper. The discontinuous Galerkin method, the *a priori* error estimate which establishes the accuracy and convergence of the method, and the *a posteriori* error estimate used to assess the accuracy of the numerical solution are presented. The reliability of the *a posteriori* error estimate is assessed by solving two examples problems with known discontinuous solutions. The effectiveness of the adaptive strategy at delivering a solution with the specified level of error is also demonstrated using the numerical examples.

THE DISCONTINUOUS GALERKIN METHOD

Consider the following hyperbolic conservation law

$$\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u + au = f \quad \text{in } \Omega \subset \mathcal{R}^2 \tag{2}$$

$$\boldsymbol{\beta} \cdot \mathbf{n}\, u = \boldsymbol{\beta} \cdot \mathbf{n}\, g \quad \text{on } \Gamma_- \tag{3}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ denotes a constant unit velocity vector, $\mathbf{n}$ denotes the unit normal vector pointing outward to the domain boundary $\partial\Omega$, $\Gamma_- = \{\mathbf{x} \in \partial\Omega \mid \boldsymbol{\beta} \cdot \mathbf{n}(x) < 0\}$ denotes the inflow boundary, $a = a(\mathbf{x})$ is a bounded measurable function on Ω such that $0 < a_0 \leq a(\mathbf{x})$, $f \in L^2(\Omega)$, and $g \in L^2(\Gamma_-)$. While this is the simplest of hyperbolic conservation laws, solutions to (2) may contain discontinuities along characteristic lines $\mathbf{x}(s)$ defined by $\frac{\partial \mathbf{x}}{\partial s} = \boldsymbol{\beta}$. Solutions to (2) belong to the space of functions $V(\Omega) = \{v \in L^2(\Omega) \mid v_\beta \in L^2(\Omega)\}$ where $v_\beta = \boldsymbol{\beta} \cdot \boldsymbol{\nabla} v$.

The starting point for the discontinuous Galerkin methods is to develop an appropriate weak formulation of (2) defined on a partition of Ω into elements, denoted by $\mathcal{P}_h$. Here the elements $K \in \mathcal{P}_h$ are general quadrilaterals of diameter h_K with outward unit normals $\mathbf{n}_K$. The element boundaries ∂K have an inflow boundary $\partial K_- = \{x \in \partial K : \boldsymbol{\beta} \cdot \mathbf{n}_K < 0\}$ and an outflow boundary $\partial K_+ = \partial K \setminus \partial K_-$. The space of admissible solutions is extended to the partition using the broken space $V(\mathcal{P}_h) = \Pi_{K \in \mathcal{P}_h} V(K)$. The standard conventions in finite element meshing are assumed to be in force: $\mathcal{P}_h$ is a family of partitions $\mathcal{F}_h$ and each element K of $\mathcal{P}_h$ is the image of an invertible map F_K of a master element $\hat{K} = [-1,1]^2$. The partitions $\mathcal{P}_h \in \mathcal{F}_h$ are regular and, in the present study, it is sufficient to take F_K as affine maps. For each partition $\mathcal{P}_h$, approximate solutions are sought in the subspace $V_p(\mathcal{P}_h) = \{v \in L^2(\Omega) \mid v|_K \circ F_K^{-1} \in Q^{p_K}(\hat{K})\}$ where $Q^{p_K}(\hat{K})$ denotes the space of functions formed by tensor products of Legendre polynomials of degree p_K on the master element $\hat{K}$. Note that the polynomial degree, p_K, may vary over different elements in the mesh and that functions $v_h^p \in V_p(\mathcal{P}_h)$ are discontinuous across element interfaces. The approximation properties of such spaces are typified by local interpolation estimates of the following type (see [5]): if $u \in H^s(K)$, there exists a constant C, independent of $h_K = \mathrm{diam}(K)$ and p_K (the minimal order of the polynomial shape functions for K), and a polynomial w of degree p_K, such that

$$||u - w||_{r,K} \le C \frac{h_K^{\min(p_K+1,s)}}{p_K^{s-r}} ||u||_{s,K} \quad ; \quad r = 0,1 \tag{4}$$

where $|| \cdot ||_{r,K}$ denotes the usual Sobolev norm.

The following notation is used for functions $v \in V(\mathcal{P}_h)$:

$$\left.\begin{array}{rcl} v^{\pm} & = & \lim_{\epsilon \to 0} v(\mathbf{x} \pm \epsilon \boldsymbol{\beta}) \\ v^{\mathrm{int}\ K} & = & v|_K(x), \quad x \in \partial K \\ v^{\mathrm{ext}\ K} & = & v|_L(x), \quad x \in \partial K \cap \partial L \\ (v,w)_K & = & \int_K v\ w\ d\mathbf{x} \quad ; \quad ||v||_K = \sqrt{(v,v)_K} \\ \langle v,w \rangle_{\partial K} & = & \int_{\partial K} v\ w\ |\boldsymbol{\beta} \cdot \mathbf{n}_K| ds \quad ; \quad \langle\langle v \rangle\rangle_{\partial K} = \sqrt{\langle v,v \rangle_{\partial K}} \end{array}\right\} \tag{5}$$

The discontinuous Galerkin method applied to (2) is written in the following abstract form:

Find $\hat{u} \in V_p(\mathcal{P}_h)$ such that

$$\sum_{K \in \mathcal{P}_h} B_K(\hat{u}, \hat{v}) = \sum_{K \in \mathcal{P}_h} L_K(\hat{v}) \ , \text{ for every } \hat{v} \in V_p(\mathcal{P}_h) \tag{6}$$

where (see [4])

$$B_K(\hat{u},\hat{v}) \stackrel{\text{def}}{=} (\hat{u}_\beta + a\hat{u}, \hat{v} + \delta\frac{h_K}{p_K^2}\hat{v}_\beta)_K + (1+\delta\frac{h_K}{p_K^2})\langle\hat{u}^+ - \hat{u}^-, \hat{v}^+\rangle_{\partial K_-\setminus\Gamma_-}$$
$$+ (1+\delta\frac{h_K}{p_K^2})\langle\hat{u}^+, \hat{v}^+\rangle_{\partial K_-\cap\Gamma_-} \tag{7}$$
$$L_K(\hat{v}) \stackrel{\text{def}}{=} (f, \hat{v} + \delta\frac{h_K}{p_K^2}\hat{v}_\beta)_K + (1+\delta\frac{h_K}{p_K^2})\langle g, \hat{v}\rangle_{\partial K_-\cap\Gamma_-} \tag{8}$$

and δ is a parameter with a value of 0 or 1. The method with $\delta = 1$ in (7) and (8) is the so-called streamline-upwind discontinuous Galerkin method. The additional term $\frac{h_K}{p_K^2}\hat{v}_\beta$ in the element integrals adds diffusion in the streamline direction without compromising the accuracy of the approximation. The method with $\delta = 0$ is the standard discontinuous Galerkin method which can be viewed as a higher-order extension of a cell-centered finite volume method where the coefficients of the higher-order terms in the polynomial approximation of the solution in an element are obtained from the conservation law and not by reconstruction. Integrating the first-order terms by parts in (6) with $\delta = 0$ and manipulating the result yields the familiar numerical flux formulation of the finite volume method

$$B_K(\hat{u},\hat{v}) = (\hat{u}, a\hat{v} - \hat{v}_\beta)_K + \int_{\partial K} \hat{q}(\hat{u}^{\text{int } K}, \hat{u}^{\text{ext } K})\hat{v}\,ds \tag{9}$$

where

$$\hat{q}(\hat{u}^{\text{int } K}, \hat{u}^{\text{ext } K}) = \frac{1}{2}\left(\hat{u}^{\text{int } K} + \hat{u}^{\text{ext } K}\right) - \frac{1}{2}|\boldsymbol{\beta}\cdot\mathbf{n}_K|\left(\hat{u}^{\text{ext } K} - \hat{u}^{\text{int } K}\right) \tag{10}$$

The error in the discontinuous Galerkin solution satisfies the following *a priori* estimate [4]:

Theorem 1 *Let $u \in H^s(\Omega)$ be a solution to (2), let $\hat{u}$ be a solution to (6), and let (4) hold. Then there exists a positive constant C, independent of h_K, p_K, and u, such that the approximation error, $e = u - \hat{u}$, satisfies the following estimate*

$$|||e|||_{hp,\beta} \le C\left\{\sum_{K\in\mathcal{P}_h}\left[\frac{h_K^{2\mu_K}}{p_K^{2\nu_K}}\max\left(1, \frac{h_K}{p_K^2}\right)||u||_{s,K}^2\right]\right\}^{\frac{1}{2}} \tag{11}$$

where $\mu_K = \min(p_K+1, s) - \frac{1}{2}$, $\nu_K = s-1$, *and*

$$|||e|||_{hp,\beta} = \left\{\sum_{K\in\mathcal{P}_h}\left[\frac{h_K}{p_K^2}||e_\beta||_K^2 + ||e||_K^2 + \langle\langle e^+ - e^-\rangle\rangle_{\partial K_-\setminus\Gamma_-}^2 + \langle\langle e\rangle\rangle_{\partial K\cap\partial\Omega}^2\right]\right\}^{\frac{1}{2}} \tag{12}$$

The *a priori* estimate (11) establishes convergence of the method and is useful for predicting how the error in numerical solutions behaves with h-refinement or p-enrichment. Unfortunately, its usefulness in assessing the accuracy of a given numerical solution is limited since the estimate involves unknown constants and the exact solution.

A POSTERIORI ERROR ESTIMATION

A posteriori error estimates used here are based on extensions of an element residual method of Ainsworth and Oden [6]. Element error indicators are computed by solving a suitably-constructed local problem with the element residual as data. These local indicators are used in the adaptive strategy to assess the accuracy of the solution in an element. Moreover, they contribute to a global error estimate which is accurate enough to provide a reliable assessment of the quality of the approximate solution. Detailed derivation of the *a posteriori* estimate can be found in [4].

The local problem is constructed to result in an upper bound on the error. Let ψ_K be the solution to the following local problem,

$$A_K^U(\psi_K, v_K) = B_K(e_K, v_K) = L_K(v_K) - B_K(\hat{u}_K, v_K) \quad \forall\, v \in V(\mathcal{P}_h) \tag{13}$$

where

$$A_K^U(\psi_K, v_K) \stackrel{\text{def}}{=} \frac{h_K}{p_K^2}(\boldsymbol{\beta} \cdot \nabla\psi_K, \boldsymbol{\beta} \cdot \nabla v_K)_K + \bar{a}(\psi_K, v_K)_K \tag{14}$$

and $\bar{a} > 0$ is a constant. Note that the local problem differs from the conservation law, in particular, it is symmetric and induces a norm on the space $V(K)$. The solution to the local problem, measured in the norm,

$$||\psi_K||_{A^U(K)} = \sqrt{A_K^U(\psi_K, \psi_K)} \tag{15}$$

serves as an element error indicator in the adaptive strategy. The global error estimate is a sum of element contributions given by

$$||\psi||_{A^U} = \sqrt{\sum_{K\in\mathcal{P}_h} ||\psi_K||^2_{A^U(K)}} \tag{16}$$

The solution to the local problem (13) provides an upper bound on the global error in the following sense [4]:

Lemma 1 *Let $\psi \in V(\mathcal{P}_h)$ be the solution to the following problem:*

$$\sum_{K\in\mathcal{P}_h} A_K^U(\psi_K, v) = \sum_{K\in\mathcal{P}_h} B(e, v) \qquad \forall v \in V(\mathcal{P}_h) \tag{17}$$

Then there exists a positive constant k such that

$$||\psi||_{A^U} \geq k|||e|||_{hp,\beta} \tag{18}$$

An approximate solution to the local problem (13) in the corresponding norm serves as a local error indicator for the element. Since the discontinuous Galerkin solution satisfies the orthogonality condition,

$$B_K(e, v) = 0 \quad \forall v \in Q^{p_K}(K) \tag{19}$$

the error indicator must be approximated with a polynomial of degree $p_K + \sigma_K$ where $\sigma_K \geq 1$ in order for the discrete local problem to have a non-trivial solution. If a complete polynomial of degree $p_K + \sigma_K$ (on the master element) is used to approximate the solution to the local problem, then the discrete local problem requires the solution of a system of order $(p_K + \sigma_K + 1)^2$. This system can be fairly large compared to the system of $(p_K + 1)^2$ equations used to obtain the approximate solution for which we are estimating the error. Since $(p_K + 1)^2$ terms on the right hand side of the discrete local problem (corresponding to (19)) are zero, a simplification is made by approximating the solution to the local problem in the space $Q^{p_K + \sigma_K}(K) \setminus Q^{p_K}(K)$. In other words, the solution to the local problem is approximated with incomplete polynomials of degree $p_K + \sigma_K$ by neglecting the terms associated with polynomials of degree p_K. This simplification results in a system of $\sigma_K(\sigma_K + 2p_K + 2)$ equations for each element.

THE *hp*-ADAPTIVE STRATEGY

The *hp*-adaptive strategy used here is an extension of the 3-step strategy developed by Oden, Patra, and Feng [7] for a large class of elliptic problems and, in several applications, was shown to yield exponential rates of convergence with respect to both CPU time and the number of unknowns.

The goal of the adaptive strategy is to deliver a solution with the specified level of error in three adaptive steps: (1) estimate the error in the solution obtained on an initial mesh (2) construct a new mesh using h-refinement of the initial mesh, solve the problem on the new mesh, and estimate the error, and (3) enrich the approximation in regions where the solution is smooth by increasing the spectral order of the elements in the mesh from step (2), and if necessary, perform h-refinement in regions where the solution is of low regularity. If the level of error after step (3) exceeds the specified level, it is necessary to repeat steps (2) and (3) until the desired error is attained.

The *hp*-adaptive strategy is based on the assumption that the *a posteriori* estimate is a reasonable approximation to the actual error in a particular solution. The *a priori* estimate (11) and some additional assumptions (see [4]) lead to expressions for estimating the local regularity of the solution and for predicting the mesh required to reduce the error to the specified level. The entire procedure is outlined below. Detailed development of the *hp*-adaptive strategy can be found in [4].

(i) Specify a target normalized error, η_T. The target error is normalized by the solution in the same norm. Specify the parameter α to determine the intermediate target error,

$\eta_I = \alpha\eta_T$. Specify the parameters α_S and α_D which establish reduction factors for the error in smooth and non-smooth regions as described below. Specify the parameters μ_K and ν_K in the *a priori* estimate (11). Formally these parameters depend on the global regularity of the solution. While there is little theoretical justification, local values can be used by computing the rate of convergence of the local error for a uniform h-refinement and p-enrichment of a coarse mesh.

(ii) Construct an initial mesh $\mathcal{P}_0$ containing $N(\mathcal{P}_0)$ elements. The elements in $\mathcal{P}_0$ have uniform $p_K = p_0$ and essentially uniform $h_K \approx h_0$. Find the approximate solution $\hat{u}_0 \in V_{p_0}(\mathcal{P}_0)$. Estimate the error θ_0 where

$$\theta_0 = \sqrt{\sum_{K\in\mathcal{P}_0} \theta_{0,K}^2} = \sqrt{\sum_{K\in\mathcal{P}_0} ||\psi_K||_{A^U}^2} \tag{20}$$

and ψ_K is the solution to the local problem (13).

(iii) Construct a mesh $\mathcal{P}_1$ by subdividing each element in $\mathcal{P}_0$ into the number of elements, n_K, required to equally distribute the error and reduce it to $\theta_I = \eta_I(||\hat{u}_0||_{A^U} + \theta_0)$. The number of elements, n_K, is obtained by iteratively solving the following two equations:

$$n_K = \left(\frac{\theta_{0,K}^2}{\theta_I^2} N(\mathcal{P}_1)\right)^{\frac{1}{\mu_K+1}} \tag{21}$$

$$N(\mathcal{P}_1) = \sum_{K\in\mathcal{P}_0} n_K \tag{22}$$

Find the approximate solution $\hat{u}_1 \in V_{p_0}(\mathcal{P}_1)$ and estimate the error θ_1.

(iv) Estimate the local regularity of the solution by computing the rate of convergence of the local error

$$\mu_K = \frac{\log\theta_{0,K} - \log\sqrt{\sum_{L=1}^{n_K}\theta_{h,L}^2}}{\log h_K - \log\frac{h_K}{\sqrt{n_K}}}, \quad K = 1,\cdots,N(\mathcal{P}_0) \tag{23}$$

The value of μ_K given by (23) is associated with an element K in the initial mesh and is simply inherited by the new elements generated by subdividing the element K. The expected rate of convergence for smooth solutions is $p_K + \frac{1}{2}$, according to the *a priori* estimate (11). Divide the error into two contributions according to the value of μ_K:

$$\theta_D = \sqrt{\sum_{K\in\Omega_D}\theta_{1,K}^2}\,; \quad \Omega_D = \{K \in \mathcal{P}_1 : \mu_K < p_K + \frac{1}{2}\} \tag{24}$$

$$\theta_S = \sqrt{\sum_{K\in\Omega_S}\theta_{1,K}^2}\,; \quad \Omega_S = \mathcal{P}_1 \setminus \Omega_D \tag{25}$$

Subdivide the elements in Ω_D into the number of elements required to equally distribute the error and reduce it to $\alpha_D\theta_D$. Enrich the approximation in Ω_S to equally distribute the error and reduce it to $\alpha_S\theta_S$ by increasing p_K according to

$$p_K = p_0 \left(\frac{\theta_{1,K}\sqrt{N(\Omega_S)}}{\alpha_S\theta_S} \right)^{\frac{1}{\nu_K}} \tag{26}$$

Find the approximate solution on the new mesh and estimate the error.

(v) If the estimated error in (iv) is larger than the target error, repeat step (iii) and (iv) until the target error is reached.

In the current implementation, h-refinement is accomplished by successive bisection of an element and is limited to two levels for a particular adaptive step. The h-refinement in (iv) is necessary only when the error θ_D exceeds the target error.

NUMERICAL EXAMPLES

The discontinuous Galerkin method with $\delta = 1$ in (6) is used to solve the model problem (2) to assess the reliability of the error estimate and to investigate the performance of the hp-adaptive strategy for problems with discontinuous solutions.

Example 1

We solve the linear model problem (2) with the following data:

(i) $\Omega = (-1,1) \times (-1,1)$

(ii) $\boldsymbol{\beta} = (1.0, 0.0)^T$

(iii) $a(\mathbf{x}) = 1.0$

(iv) $g = \begin{cases} 3e^{-5(1+y^2)} & \text{if } y < 0 \\ -3e^{-5(1+y^2)} & \text{otherwise} \end{cases}$

The source term f in (2) is chosen so that the exact solution is the discontinuous function given by

$$u(x,y) = \begin{cases} 3e^{-5(x^2+y^2)} & \text{if } y < 0 \\ -3e^{-5(x^2+y^2)} & \text{otherwise} \end{cases} \tag{27}$$

and shown in Fig. 1. The discontinuity is aligned with element interfaces at $y = 0$ to illustrate the advantage of using a discontinuous method to capture discontinuities, particularly if the adaptive scheme includes some shock fitting which aligns the grid with the discontinuity.

The problem was solved using a variety of uniform meshes with h-refinements, p-enrichments, and the hp-adaptive strategy. The error history for an hp-adaptive solution starting from an initial 8×8 mesh of $p = 1$ elements is listed in Table 1. The target error was nearly achieved at each step in the adaptive process. Recall that the target error is a global quantity obtained as the square root of the sum of the squares of the element error indicators (16). Therefore the error in a single element cannot exceed the target error. The global effectivity index, the ratio of the estimated error to the actual error, is also listed in Table 1. An effectivity index close to unity indicates that the error estimate is reliable and provides a good approximation to the actual error. The effectivity indices in Table 1 are slightly less than unity, indicating that the actual error is larger than the estimated error. However, the estimated error is sufficiently close to the actual error to result in an effective adaptive strategy.

Adaptive step	Target error	Achieved error	Effectivity index
initial (8×8 mesh, $p = 1$)	—	15.4%	0.998
h-refinement	7.5%	3.3%	0.996
p-enrichment	5.0%	5.5%	0.901

Table 1: Example 1 - Error history for an hp-adaptive solution

The rate of convergence of the estimated and exact error is compared in Fig. 2. The exact error (denoted by a solid line in the figure) and the estimated error (denoted by a dashed line) are in close agreement, indicating the reliability of the estimate. Note that with the discontinuity aligned with element interfaces, the error behaves as if the solution is smooth; that is, algebraic rates of convergence are achieved with respect to mesh refinement, and exponential rates of convergence are achieved with respect to p-enrichment. When the discontinuity is aligned with the element interfaces, the most significant error reduction with fewest degrees of freedom results by specifying a target error for the h-step which is closer to the initial error than to the final target error. This is verified by the curves corresponding to two hp-adaptive solutions in Fig. 2. The error corresponding to the hp-adaptive solutions in Fig. 2 exhibits super-linear rates of convergence.

The element residual method gives a global error estimate which bounds the actual global error, however, nothing in the theory indicates the reliability of the local element indicators. Since the local error indicators are used in the adaptive strategy, it is important that the indicators give an accurate approximation of the actual element error. The hp-adapted mesh resulting from an initial 8×8 mesh of $p = 1$ elements and the local effectivity index for the element error indicators, η_K, are shown in Fig. 3. Although there are some elements with low effectivity indices (indicating that that actual error is much larger than the estimate), the local effectivity index for most of the elements falls between 0.8 and 1.2 indicating that the local error indicators are sufficiently accurate for use in the adaptive strategy.

Example 2

The following data is used in (2):

(i) $\Omega = (-1,1) \times (-1,1)$

(ii) $\boldsymbol{\beta} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$

(iii) $a(\mathbf{x}) = 1.0$

(iv) $g(x,y) = \begin{cases} 5e^{-[\frac{1}{4}+y^2]} + 3e^{-[1+(y-\frac{1}{2})^2]} & x = -1 \\ -1 - 8e^{-5[(x-\frac{1}{2})^2+\frac{1}{4}]} & y = -1 \end{cases}$

The source term f in (2) is chosen so that the exact solution is a function which is discontinuous along the domain diagonal given by

$$u(x,y) = \begin{cases} 5e^{-[(x+\frac{1}{2})^2+y^2]} + 3e^{-[x^2+(y-\frac{1}{2})^2]} & \text{if } y > x \\ -1 - 8e^{-5[(x-\frac{1}{2})^2+(y+\frac{1}{2})^2]} & \text{otherwise} \end{cases} \qquad (28)$$

and shown in Fig. 4.

The global estimated error for a sequence of uniform refinements and for several adaptive *hp*-meshes is shown in Fig. 5. The labels *hp*-adaptive in Fig. 5 refer to the adaptive strategy with only *p*-enrichment in the third adaptive step. The labels *hhp*-adaptive refer to the strategy with both *h*-refinement and *p*-enrichment in the third adaptive step. The *hp*-adaptive strategy delivers nearly linear rates of convergence with respect to the number of degrees of freedom. The rates of convergence (the slope of the lines in Fig. 5) for the adaptive strategy are higher than the rates of convergence for uniform refinement, indicating that a more accurate solution is obtained with far fewer degrees of freedom when using the *hp*-strategy. The rate of convergence obtained with the adaptive strategy determines the efficiency of the overall process, and as seen in Fig. 5, the rate of convergence depends significantly on the target and intermediate error specified.

The error history for the *hp*-adaptive solution denoted by the solid triangles in Fig. 5 is listed in Table 2. The target error was nearly achieved at each step in the adaptive process. The effectivity index (the ratio of the estimated error to the exact error) is on the order of 0.6, quite good for a discontinuous solution, but indicating that the actual error is larger than the estimated error.

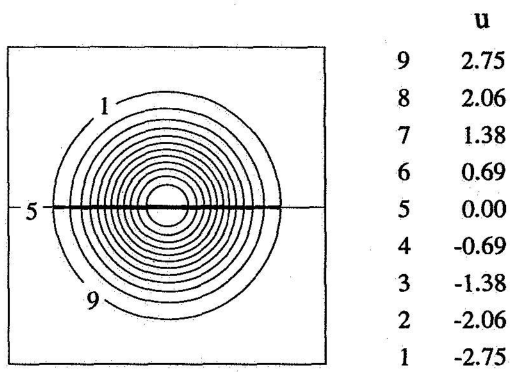

Figure 1: Example 1 - Exact solution.

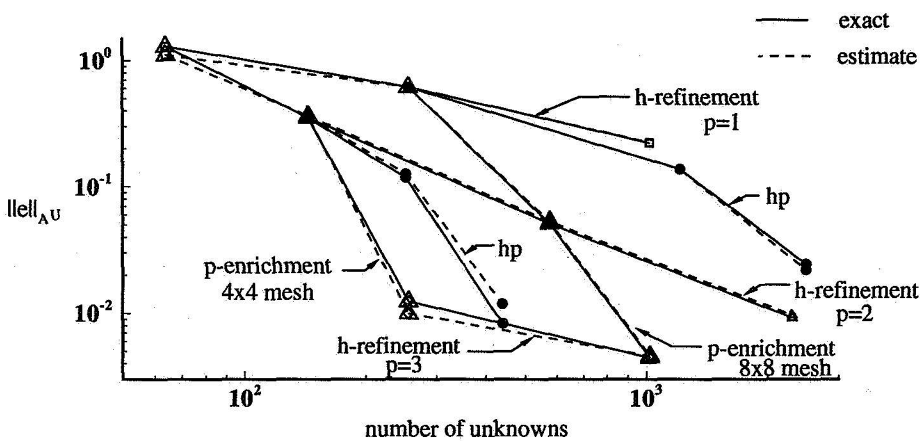

Figure 2: Example 1 - Rates of convergence of the global error with respect to the total number of unknowns.

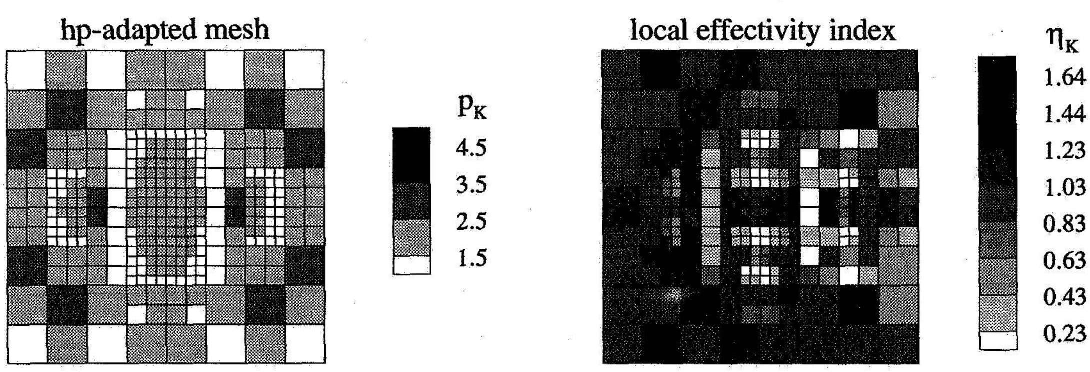

Figure 3: Example 1 - Reliability of the local error indicators for an *hp*-adapted mesh.

Adaptive step	Target error	Achieved error	Effectivity index
initial (16×16 mesh, $p = 1$)	—	7.22%	0.62
h-step	3.6%	4.1%	0.58
hp-step	2.2%	2.8%	0.57

Table 2: Example 2 - Error history for an hp-adaptive solution

Recall that the global error is a sum of element error indicators. The primary source of the under-estimation of the global error is the under-estimation of the element error indicators near the discontinuity as shown in Fig. 6. Although the local error estimate provides a qualitative measure of the error at the discontinuity, the low local effectivity index indicates some severe under-estimation of the error in that region. Note, however, that the local error estimate in smooth regions is very accurate with effectivity indices near unity.

CONCLUDING REMARKS

The development of an hp-adaptive discontinuous Galerkin method for hyperbolic conservation laws is presented in this work. The emphasis of the work is on a model class of linear hyperbolic conservation laws for which it is possible to develop *a priori* error estimates and reliable *a posteriori* estimates which provide bounds on the actual error. These estimates are obtained using a mesh-dependent norm which reflects the dependence of the error on the local element size and the local order of the approximation.

The hp-adaptive strategy is designed to deliver solutions to a specified error level in an efficient way. This is accomplished using a three-step procedure in which the *a posteriori* estimate is used to determine the error in the solution at a particular adaptive step and the *a priori* estimate is used to predict the mesh required to deliver a solution with the specified level of error. The hp-adaptive strategy makes further use of the *a priori* estimate to provide detection of discontinuities in the solution thereby identifying regions where h-refinement and p-enrichment are appropriate.

Numerical experiments demonstrate the effectiveness of the *a posteriori* estimates in providing reliable estimates of the actual error in the numerical solution. Although local error estimates near discontinuities under-estimate the actual error, the local error estimates are very accurate in smooth regions. The numerical examples also illustrate the ability of the hp-adaptive strategy to deliver a final solution with the specified error. While the hp-adaptive strategy provides super-linear convergence rates with respect to the number of unknowns in the problem, the rate of convergence depends on the level of error requested at each step in the adaptive process. More numerical experiments are needed to provide guidelines for selecting the optimum user-specified parameters.

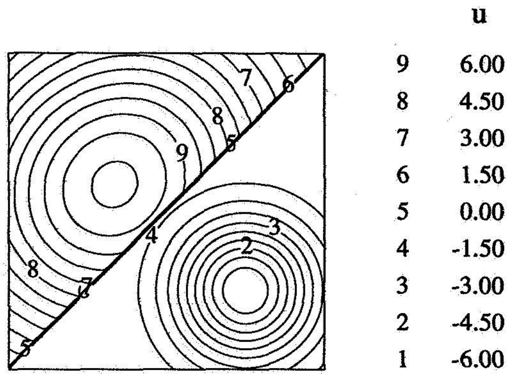

Figure 4: Example 2 - Exact solution.

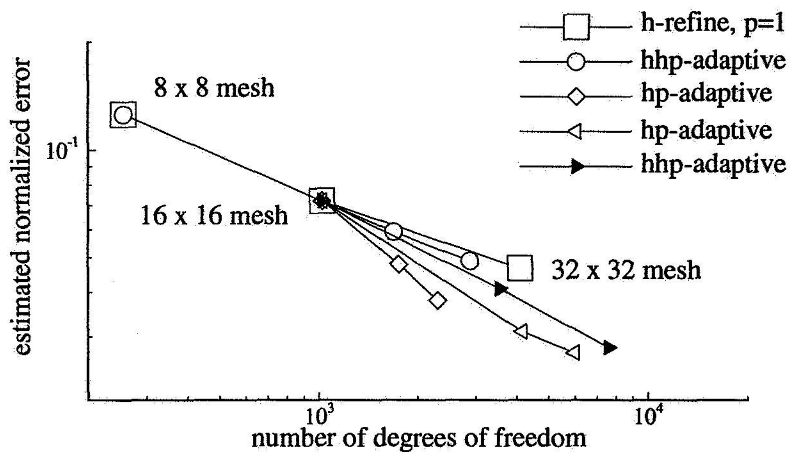

Figure 5: Example 2 - Rates of convergence of the estimated global error with the number of unknowns.

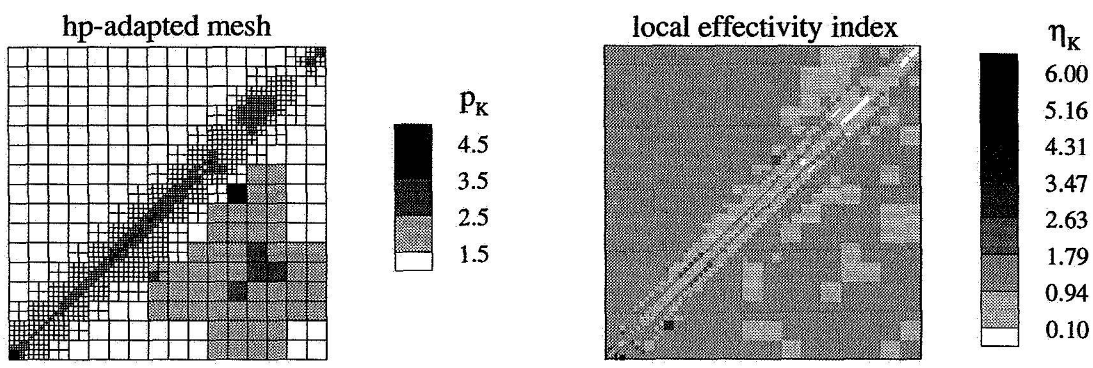

Figure 6: Example 2 - Reliability of the local error indicators for an *hp*-adapted mesh.

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