https://ntrs.nasa.gov/search.jsp?R=19960021104 2020-06-16T05:09:00+00:00Z

NASA-CR-200909

FINAL
IN-64-CR
OCIT-
43933

DEPARTMENT OF MATHEMATICS & STATISTICS
COLLEGE OF SCIENCES
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529

SINGULARITY PRESERVING NUMERICAL METHODS FOR BOUNDARY INTEGRAL EQUATIONS

By

Dr. Hideaki Kaneko, Principal Investigator

Final Report
For the period ended May 7, 1996

Prepared for
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-0001

Under
Research Grant NCC1-213
Peter Padilla, Technical Monitor
IETD-Systems Validation Methods Branch

May 1996

Old Dominion University Research Foundation

DEPARTMENT OF MATHEMATICS & STATISTICS
COLLEGE OF SCIENCES
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529

SINGULARITY PRESERVING NUMERICAL METHODS FOR BOUNDARY INTEGRAL EQUATIONS

By

Dr. Hideaki Kaneko, Principal Investigator

Final Report
For the period ended May 7, 1996

Prepared for
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23681-0001

Under
Research Grant NCC1-213
Peter Padilla, Technical Monitor
IETD-Systems Validation Methods Branch

Submitted by the
Old Dominion University Research Foundation
P.O. Box 6369
Norfolk, VA 23508-0369

May 1996

FINAL REPORT NCC1-213 (ODURF #152741)

prepared by **Hideaki Kaneko** and **Peter A. Padilla**

In the past twelve months (May 8, 1995 - May 8, 1996), under the cooperative agreement with Division of Multidisciplinary Optimization at NASA Langley, we have accomplished the following five projects:

1. A note on the finite element method with singular basis functions
2. Numerical quadrature for weakly singular integrals
3. Superconvergence of degenerate kernel method
4. Superconvergence of the iterated collocation method for Hammersteion equations
5. Singularity preserving Galerkin method for Hammerstein equations with logarithmic kernel.

This final report consists of five papers describing these projects. Each project is preceeded by a brief abstract. The first author, Hideaki Kaneko, would like to thank Dr. Thomas Zang, Director of Division of Multidisciplinary Optimization at NASA Langley, for the financial support that he provided.

PROJECT 1

HIDEAKI KANEKO and PETER A. PADILLA

A Note on the Finite Element Method with Singular Basis Functions

Recently there has been considerable interest in the finite element analysis that incorporates singular element functions. A need for introducing some singular elements as part of basis functions in certain finite element analysis arises out of the following considerations. The solution of certain problems, such as a field problem, exhibits highly singular behavior due to geometric features of the spatial domain. It is thought that an incorporation of singular elements that emulate the solution with the standard polynomial elements may perhaps be desirable. In order to make the computations of the finite element method with singular elements more efficient, Hughes and Akin (*The Finite Element Method, T.J.R. Hughes, Prentice Hall*) established an algorithm for constructing interpolation functions that have the same interpolation properties of the Lagrange polynomials.

We pointed out in this research that the aforementioned algorithm of Hughes and Akin is sensitive to the locations of the interpolation points that correspond to the singular basis functions. Specifically, we demonstrated numerically that the rate of convergence of a finite element solution varies according to the locations of these points. A general theoretical explanation is provided for this variance in the rates of convergence.

A further invesigation is required toward the establishment of mathematical theory that guarantees the optimal rate of convergence of a finite element method that uses singular functions as part of its basis.

A Note on the Finite Element Method with Singular Basis Functions

Hideaki Kaneko *

Department of Mathematics and Statistics

Old Dominion University

Norfolk, Virginia 23529-0077

Peter A. Padilla

Multidisciplinary Design and Optimization Branch

NASA-Langley Research Center

MS-159

Hampton, VA 23681

Abstract

In this note, we make a few comments concerning the paper of Hughes and Akin [3]. First it is noted that algorithm reported in [3] to produce a new set of functions is subject to the location of collocation points. Second we make a note concerning the rate of convergence of numerical solutions of the finite element method with singular basis functions.

*This author is partially supported by NASA under grant NCC1-213

1 Finite Element Method With Singular Basis Functions

In the paper [3], Hughes and Akin made an interesting observation concerning the finite element analysis that incorporates singular element functions. A need for introducing some singular elements as part of basis functions in certain finite element analysis arises out of the following considerations. The solution of certain problems, such as a field problem [1], exhibits highly singular behavior due to geometric features of the spatial domain. On the other hand, in other circumstances, the solution is overwhelmingly affected by the nature of loading and the problem of singularity can be ignored. To satisfy both situations just described, it is thought that an incorporation of singular elements that emulate the solution with the standard polynomial elements may perhaps be desirable. This is the point that was exploited in [3] by Hughes and Akin. In order to make the computations of the finite element method with singular elements more efficient, they consider the following algorithm for constructing interpolation functions. A construction of such algorithm was motivated by the idea that "it is of practical interest to develop techniques for systematically defining shape functions for singularity modeling (and for developing special elements in general), which circumvent the interpolation problem" ([4] p.176). The algorithm that they developed go as follows:

ALGORITHM Suppose that there are n shape functions N_a, $a = 1, 2, \ldots, n$ which satisfy the interpolation property on the first m nodes r_b, viz., $N_a(r_b) = \delta_{ab}$, $a, b = 1, 2, \ldots, m$ $(m < n)$. Their idea here is to reshape N_a's so that the interpolation property is satisfied on all n nodes. The algorithm is given by

$$\text{Step 1} \quad N_{m+1} \leftarrow \frac{N_{m+1}(r) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r)}{N_{m+1}(r_{m+1}) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r_{m+1})}$$

$$\text{Step 2} \quad N_a(r) \leftarrow N_a(r) - N_a(r_{m+1}) N_{m+1}(r), \qquad a = 1, 2, \ldots, m$$

Step 3 If $m + 1 < n$, replace m by $m + 1$ and repeat Steps 1 to 3.
If $m + 1 = n$, stop.

To demonstrate this algorithm, we borrow one of the examples from [4]. Let $r_1 = 0$, $r_2 = \frac{1}{2}$ and $r_3 = 1$. The shape functions that we reconstruct are $N_1(r) = 1 - 2r$, $N_2(r) = 2r$ and $N_3(r) = r^\alpha$ where α representing some real number. Note that $N_a(r_b) = \delta_{ab}$, $1 \leq a, b \leq 2$. An

application of the above algorithm gives

$$\begin{aligned} N_1(r) &\leftarrow 1 - 2r + \left[\frac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1-2(\frac{1}{2})^\alpha}\right] \\ N_2(r) &\leftarrow 2r - 2\left[\frac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1-2(\frac{1}{2})^\alpha}\right] \\ N_3(r) &\leftarrow \frac{r^\alpha - 2(\frac{1}{2})^\alpha r}{1-2(\frac{1}{2})^\alpha}. \end{aligned}$$

Of course, the newly defined shape functions satisfy

$$N_a(r_b) = \delta_{ab} \qquad 1 \leq a, b \leq 3. \tag{1}$$

What is not addressed in [3] [4] is that the algorithm is subject to the location of the interpolation points r_b, $m+1 \leq b \leq n$. Clearly, step 1 of algorithm does not work if these interpolation points are such that

$$N_{m+1}(r_{m+1}) - \sum_{a=1}^{m} N_{m+1}(r_a) N_a(r_{m+1}) = 0 \tag{2}$$

Out of this observation, there seems to arise a profound and difficult problem in the area of approximation theory. The problem is important in that the success of the finite element method using the collocation scheme hinges on a resolution of this problem. To describe it, let W_p^k denote the Sobolev space,

$$W_p^k = \{f | f^{(k)} \in L_p(\Omega)\}$$

where $f^{(k)}$ denote the kth generalized derivative and Ω is a bounded region in R. The theory extends easily to higher dimensions. Now let $U \equiv \text{span}[N_a]_{a=m+1}^n$. Also denote an approximation space by S_h^k. Here S_h^k is usually taken as the space of piecewise polynomials of degree $k-1$ with length of each subinterval h. Our goal is to approximate each element of $U \oplus W_p^k$ by an element from $U \oplus S_h^k$ by interpolation. That is, if $S_h^k = \text{span}[N_a]_{a=1}^m$, then for each $f \in U \oplus W_p^k$, we must find $v \in U \oplus S_h^k$ that satisfies

$$v(r_b) = f(r_b) \qquad b = 1, \ldots, n. \tag{3}$$

Denote the interpolation projector of $U \oplus W_p^k$ to $U \oplus S_h^k$ by P_h. Namely P_h is defined so that

$$P_h f(s) = v(s) \qquad s \in \Omega. \tag{4}$$

Notice that $P_h^2 = P_h$. In order to achieve a convergence by the collocation scheme in the finite element method, we must examine the following inequality. Here we assume that the number of singular basis functions, $n - m$, is fixed.

$$|\det[N_a(r_b)]|_{a,b=1}^n \geq \epsilon > 0 \qquad \text{for all } n. \tag{5}$$

This inequality is a necessary and sufficient condition for the algorithm of Hughes and Akin to work. It is important to remark at this point that the success of algorithm depends upon the existence of a solution to the interpolation problem (3) which in turn is equivalent to the condition (5). For each fixed index n (hence for fixed m) , it is not difficult to find $n-m$ interpolation points r_b, $m+1 \leq b \leq n$, that correspond to the singular basis elements N_a, $m+1 \leq a \leq n$ for which the inequality in (5) is satisfied. What is difficult here is the question of locating $n-m$ points for as many singular elements for which condition (5) is satisfied. The problem of finding $n-m$ interpolation points for singular basis functions that work for all n is currently under investigations. In the following section, we proceed our discussion of the finite element analysis assuming condition (5). The analysis will provide information concerning the rate of convergence of numerical solutions. Interpolation examples at the end of the section show that rates of convergence are quite sensitive to locations of interpolation points for singular elements.

2 Convergence Analysis

When condition (5) is satistied, one can deduce the rate of convergence of the projector P_h to the identity operator I. As is well known -e.g., [4], the convergence rate of such interpolation projectors determine the rate of convergence of the finite element method that uses collocation scheme. The following theorem of Cao and Xu [2] is useful. We sketch a proof for completeness.

Lemma 2.1 *Let X be a Banach space. Assume that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Moreover assume that $P_1: X \to U_1$ and $P_2: X \to U_2$ are linear operators. If P_2 is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \qquad \textit{for all } x \in X.$$

Proof: Let $x \in X$. We write

$$x - P_2 x = (x - P_1 x) + (P_1 x - P_2 x).$$

Since $P_1 x \in U_1$ and $U_1 \subseteq U_2$, we have $P_2 P_1 x = P_1 x$. Hence,

$$\begin{aligned} x - P_2 x &= x - P_1 x + P_2 P_1 x - P_2 x \\ &= (I - P_2)(x - P_1 x). \end{aligned}$$

It follows that

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \ \textit{for all } x \in X.$$

□ Using lemma and a well known fact about the approximation power of piecewise polynomials of degree $k-1$ to approximate elements in W_p^k [5], we see immediately that

Theorem 2.2 *Assume that $y = u + v$ with $u \in U$ and $v \in W_p^k$. Then*

$$\|P_h y - y\|_p \leq C h^n \|v^{(k)}\|_\infty$$

where $1 \leq p \leq \infty$ and C is a constant.

To demonstrate the sensitivity of the location of interpolation points for singular elements, consider the following interpolation problem.

EXAMPLE: Let $f(x) = \sqrt{x} + \sqrt{1-x} + x^2$. We wish to approximate f over $[0,1]$ by an element from $U \oplus S_h^2$, where $U = \text{span}[\sqrt{x}, \sqrt{1-x}]$. Let $\{x_i\}_{i=0}^n$ be the uniform partition of $[0,1]$ defined by $x_i = \frac{i}{n}$, $i = 0, 1, \ldots, n$ and $h = \frac{1}{n}$. The interpolation points used to define an element from S_h^2 are taken to be the zeros of the second degree Legendre polynomials transformed into $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. The following data shows that (a) when the interpolation points for the singular elements are taken to be $t_1 = \frac{1}{5}$ and $t_2 = \frac{4}{5}$ for each n, the convergence is $O(h^{1/2})$, whereas (b) when $t_1 = \frac{1}{2^n}$ and $t_2 = 1 - t_1$, then the convergence is of the order $O(h^2)$.

interpolation point t_1 =	$1/2^n$	1/5
n = 4	0.0513168	0.0513168
n = 6	0.0008437	0.0235540
convergence exponent =	1.99	0.56

Table 1. Error and convergence rate data for the example

References

[1] R. E. Barnhill and J. R. Whiteman, Error Analysis of Finite Element Methods with Triangles for Elliptic Boundary Value Problems, *Mathematics of Finite Element Methods and Applications,* ed. J. R. Whiteman, London: Academic Press, (1973), 83-112.

[2] Y. Cao and Y. Xu, Singularity Preserving Galerkin Methods for Weakly Singular Fredholm Integral Equations, *Journal of Integral Equations and Applications*, (in press)

[3] T. J. R. Hughes and J. E. Akin, Techniques for developing 'special' finite element shape functions with particular reference to singularities, *International Journal for Numerical Methods in Engineering*, 15(1980), 733-751.

[4] T. J. R. Hughes, *The Finite Element Method, Linear Static and Dynamical Finite Element Analysis* , Prentice-Hall Inc. Englewood Cliffs, N. J. (1987).

[5] L. schumaker, *Theory of Splines* , John Wiley.

PROJECT 2

HIDEAKI KANEKO and PETER A. PADILLA

Numerical Quadratures for Weakly Singular Integrals

In this report, a Fortran program for approximating weakly singular integrals is given. In particular, we are interested in approximating integrals of functions having algebraic or logarithmic end point singularities. Program is based upon the idea that was recently developed in the paper by Kaneko and Xu *Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of the second kind- Mathematics of Computations, Vol. 62, (1994), 739-753.* The idea of the quadrature scheme in the aforementioned paper is based upon the nonlinear splines approximation of weakly singular functions using a certain set of nonuniform knots that was originally developed by John Rice.

The program developed here is useful for approximating solutions of weakly singular integral equations. This is demonstrated in Reports # 3 and # 4 in which the quadrature is used to approximate the solution of weakly singular Hammerstein equation that arises as a reformulation of the Dirichlet problem with a certain class of nonlinear boundary conditions.

Numerical Quadratures for Weakly Singular Integrals

Hideaki Kaneko*
Department of Mathematics and Statistics
Old Dominion University
Norfolk, VA 23529

Peter A. Padilla
NASA - Langley Research Center
M/S 130
Hampton, VA 23681

Abstract

In this report, Fortran programs for approximating weakly singular integrals are given. In particular, we are interested in approximating integrals of functions having algebraic or logarithmic end point singularities. Programs are based upon the idea that was recently developed in the paper by Kaneko and Xu [1].

*This author is partially supported by NASA under grant NCC1-213

1 Theoretical Background

In this report, we develop Fortran codes for Gauss-Legendre-type quadratures for weakly singular integrals. The idea of the quadratures was recently developed by Kaneko and Xu [1]. The integrals which we intend to approximate can be described in the form

$$\int_a^b f(x)w(x)\,dx \tag{1}$$

where f is a smooth function and w takes one of the following forms;

1. $(x-a)^\alpha(b-x)^\beta$
2. $(x-a)^\alpha(b-x)^\beta log(x-a)$
3. $(x-a)^\alpha(b-x)^\beta log(b-x)$
4. $(x-a)^\alpha(b-x)^\beta log(x-a)log(b-x)$

where $\alpha, \beta > -1$. To familiarize the reader with the basic theory that supports the current program, we shall consider the following problem. The reader who is interested in a more rigorous and broader development of the theory can find necessary materials in [1].

Consider

$$I(f) = \int_0^1 f(x)w(x)dx \tag{2}$$

where $w(x) = x^\alpha$, $\alpha > -1$, -i.e., w belongs to type 1 above and f is a smooth function. Define the index of parameter that measures the degree of singularity by

$$q = \frac{2k+1}{\alpha+1}$$

where k is a positive integer that corresponds to the degree $k-1$ of polynomials used to approximate fw over $[0,1]$. In the case of the logarithmic singularity $w(x) = \log x$, we set $\alpha = 0$. A partition π_α of $[0,1]$ is defined according to the parameter α by

$$\pi_\alpha: t_0 = 0, t_1 = n^{-q}, t_j = j^q t_1, \qquad j = 2,3,\ldots,n. \tag{3}$$

The more severe the singularity of w is at 0, the closer t_1 is to 0. In the ordinary Gauss-Legendre quadrature scheme to approximate the integral $\int_0^1 f(x)dx$, f is approximated by the polynomial S_k of degree $k-1$ that interpolates f at the k points $\{x_i\}_{i=1}^k$ where $\{x_i\}_{i=1}^k$ are k zeros of the kth degree Legendre polynomial transformed into $[0,1]$. The kth degree Legendre polynomial is described by

$$\frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k, \qquad x \in [-1,1].$$

The quadrature then takes the form

$$\sum_{i=1}^{n} w_i f(x_i)$$

where w_i's are weights for the quadratue. It is well known [2] that the error of the Gauss-Legendre quadrature is given by

$$\frac{|f^{(2n)}(\xi)|}{(2n)!}\left|\int_0^1 \prod_{i=1}^{n}(x-x_i)^2 dx\right| \qquad \text{if } f \in C^{(2n)}[0,1] \tag{4}$$

As evidenced by this formula for the error of the Gauss-Legendre quadrature, the scheme does not provide an optimal degree of approximation when the integrand f is not sufficiently smooth. It is exactly this reason why we require some special treatment for approximating the integrals in (2) that produces an optimal convergence rate. To achieve this goal, in the current method, a piecewise polynomial S_k of degree $k-1$ is constructed by the following rule; $S_k(x) = 0$, $x \in [t_0, t_1)$ and $S_k(x)$ is the Lagrange polynomial of degree $k-1$ interpolating fw at $\{x_j^{(i)}\}_{j=1}^k$ for $x \in [t_i, t_{i+1})$, $i = 1, 2, \ldots, n-2$ and for $x \in [t_{n-1}, t_n]$. Here $\{x_j^{(i)}\}_{j=1}^k$ denote the zeros of the kth degree Legendre polynomial transformed into $[t_i, t_{i+1})$. Over the first interval, because of the endpoint singularity of fw, formula (4) is not at our disposal. Dispite this, we are able to recover the optimal rate of convergence due to sufficiently small size of the first interval that was determined by the index q of singularity corresponding to fw. To describe, in a more detailed way, the error of the current quadrature scheme, let

$$E_{k,i} \equiv \int_{t_i}^{t_{i+1}} f(x)w(x)dx - \int_{t_i}^{t_{i+1}} S_k(x)dx, \qquad i = 0, 1, \ldots, n-1. \tag{5}$$

Then

$$\begin{aligned} |E_{k,0}| &\le \textstyle\int_{t_0}^{t_1} |f(x)w(x) - S_k(x)|dx \\ &\le \textstyle\int_{t_0}^{t_1} |f(x)w(x)|dx \\ &\le C \textstyle\int_{t_0}^{t_1} x^\alpha dx \\ &\le C t_1^{\alpha+1} = Cn^{-2k-1}. \end{aligned}$$

C denotes a generic constant whose value may change as it appears. For $i \ge 1$, by using (4), we obtain

$$\begin{aligned} |E_{k,i}| &\le \tfrac{|(fw)^{(2k)}(\xi_i)|}{(2k)!}\left|\textstyle\int_{t_i}^{t_{i+1}} \prod_{j=1}^{k}(x - x_j^{(i)})^2 dx\right| \\ &\le |\xi_i|^{\alpha-2k}(t_{i+1} - t_i)^{2k+1} \\ &\le C t_i^{\alpha-2k}(t_{i+1} - t_i)^{2k+1} \\ &= C(i^q t_1)^{\alpha-2k}[(i+1)^q - i^q]^{2k+1} t_1^{2k+1} \\ &\le C q^{2k+1}(i+1)^{(q-1)(2k+1)} i^{q(\alpha-2k)} t_1^{\alpha+1} \\ &= Cn^{-2k-1}. \end{aligned}$$

The last equality was obtained using

$$q(\alpha - 2k) = q(\alpha + 1) - q(2k + 1) = (2k + 1) - q(2k + 1) = (1 - q)(2k + 1).$$

Combining local errors above, the total error of approximation E_k is given by

$$|E_k| = |\sum_{i=0}^{n-1} E_{k,i}| \leq Cn^{-2k} \tag{6}$$

obtaining the optimal convergence rate of approximation.

References

[1] H. Kaneko and Y. Xu, "Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of the second kind", Mathematics of Computations, Vol. 62, (1994), 739-753.

[2] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Second Ed. Academic Press, Inc.,(1984).

goog.larc.nasa.gov:pap

test_dri.c

Tue Apr 9 11:16:54 1996

lp01 / lp01

```
lp01 goog.larc.nasa.gov:pap  Job: test_dri.c  Date: Tue Apr  9 11:16:54 1996

lp01 goog.larc.nasa.gov:pap  Job: test_dri.c  Date: Tue Apr  9 11:16:54 1996

lp01 goog.larc.nasa.gov:pap  Job: test_dri.c  Date: Tue Apr  9 11:16:54 1996

lp01 goog.larc.nasa.gov:pap  Job: test_dri.c  Date: Tue Apr  9 11:16:54 1996
```

```
/* Author: Peter A. Padilla

   Summary: This program test the function sintegrate ability to integrate
            a function with weak singularities at both ends of the interval.
            The program calculates the integral in two ways, first we
            calculate the value by using sintegrate ability to automatically
            split the interval into two sections, in the second method we
            split the interval manually and feed sintegrate the appropriate
            parameters for each half of the interval.

*/



#include <stdio.h>
#include <stdlib.h>
#include <math.h>

#include "sintegrate.h"


/* ----- Function Definitions ----- */


double one(double x)
{
return(1.0);
}

double fun(double x)
{
 return(log(1-x)*sqrt(x));
}


/* ----- Main ----- */


main(int argc, char argv[])
{
  double v1, v2, v3, v, (*g1)(), (*g2)();
  int n;

/* Example 3.2 of (1) */

  n = 16;

  g2 = &one;

  g1 = &fun;

  v = sintegrate(g1,g2, 0.0, 1.0, n, 1, 1, 1, 0.5, 0.0);

  printf("Results:  %g for n = %i\n",v,n);

 n = 24;

  g2 = &one;

  g1 = &fun;

  v = sintegrate(g1,g2, 0.0, 1.0, n, 1, 1, 1, 0.5, 0.0);

  printf("Results:  %g for n = %i\n",v,n);

 n = 28;
```

goog.larc.nasa.gov:pap

sintegra.h

Tue Apr 9 11:16:35 1996

lp01 / lp01

```
lp01 goog.larc.nasa.gov:pap  Job: sintegra.h  Date: Tue Apr  9 11:16:35 1996
lp01 goog.larc.nasa.gov:pap  Job: sintegra.h  Date: Tue Apr  9 11:16:35 1996
lp01 goog.larc.nasa.gov:pap  Job: sintegra.h  Date: Tue Apr  9 11:16:35 1996
lp01 goog.larc.nasa.gov:pap  Job: sintegra.h  Date: Tue Apr  9 11:16:35 1996
```

```
    g2 = &one;

    g1 = &fun;

    v = sintegrate(g1,g2, 0.0, 1.0, n, 1, 1, 1, 0.5, 0.0);

    printf("Results:  %g for n = %i\n",v,n);

    exit(EXIT_SUCCESS);
}
```

```
/* Arguments:

        1) f is declared a pointer to a function that returns a double and takes
           a double as input. (double = double precision). This pointer must
           be set to point to the weakly singular function for the interval in
           question, see test_driver.c for an example.

        2) weight is declared a pointer to a function that returns a double
           and takes a double as input. This pointer must be set to point to
           the smooth part of the kernel, see test_driver.c and test_driver_1.c
           for examples.

        3) a and b are the lower and upper bounds of the integration interval.

        4) n integer. Need I say more.

        5) left_singularity is a flag, 0 if the singular point for the interval
           is not at the left (point a) of the interval, 1 if it is.

        6) right_singularity is another flag. Can you guess how is defined?

        7) with_log is another flag. Set it to 1 if the function f is of types
           2, 3, or 4, i.e., it involves log()'s.

        8) alpha is a double precision variable and must be set to the index of
           the singularity at x = a.

        9) beta is a double precision variable and must be set to the index of
           the singularity at x = b.

*/


double sintegrate(double (*f)(double x), double (*weight)(double x),
     double a, double b, int n, int left_singularity,
     int right_singularity, int with_log, double alpha, double beta);
```

goog.larc.nasa.gov:pap

sintegra.c

Tue Apr 9 11:13:19 1996

lp01 / lp01

```
lp01 goog.larc.nasa.gov:pap  Job: sintegra.c  Date: Tue Apr  9 11:13:19 1996

lp01 goog.larc.nasa.gov:pap  Job: sintegra.c  Date: Tue Apr  9 11:13:19 1996

lp01 goog.larc.nasa.gov:pap  Job: sintegra.c  Date: Tue Apr  9 11:13:19 1996

lp01 goog.larc.nasa.gov:pap  Job: sintegra.c  Date: Tue Apr  9 11:13:19 1996
```

```
/* Author: Peter A. Padilla
   Date: 9/21/95
   Summary: Implements the numerical quadrature algorithm for weakly singular
            integrals as described in the references.

   Ref: (1) H. Kaneko and Y. Xu, "Gauss-type quadrature for weakly singular
            integrals and their application to Fredholm integral equations of
            the second kind", Mathematics of Computations, Vol. 62, (1994),
            739-753.

   Calling symtax: See sintegrate.h

   Notes:

   The function sintegrate is the external interface to the code. No other
   function needs to be made accessible to the outside world. If other
   interpolation polynomials are desired, then, the function legendre should
   be modified to implement the appropriate interpolation functions and the
   preprocessor constant K (line: "#define K 2.0") should be set accordingly.

*/

#include <math.h>
#include <stdio.h>
#include <stdlib.h>


/* ----- Do not changed below this line unless you know what your doing ----- */

#define K 2.0                       /* DO NOT CHANGE, K is determined by the
                                       interpolation function as described in the
                                       paper */


/* ----- Function to generate a Partition ----- */

double * points(int n, double a, double b, double q, int flag)
 { int i;
   double t1;
   double *res;

   res = (double *) malloc(sizeof(double)*(n+1));

   if (q != 0)
    if (flag == 0)
       { t1 = (b-a)/pow(n,q);
         for (i=0; i<=n; i++) res[i] = a + t1*pow(i,q);
       }
    else
       { t1 = (b-a)/pow(n,q);
         for (i=0; i<=n; i++) res[n-i] = b - t1*pow(i,q);
       }
   else
       { printf("Error: q = 0 not allowed.\n");
         free(res);
         exit(0);
       }
    return(res);
 }

/* ----- Function to generate the interpolation points. ----- */
double * inter_points(double p[], int n)
{
 int i,j;
 double eps;
 double * tp;
```

```
 eps = 1/sqrt(3.0);

 tp = (double *) malloc(sizeof(double)*(2*n));

 for (i=0; i<n; i++)
  { j = 2*i;
    tp[j] = p[i] + (p[i+1] - p[i])/2.0*(1.0-eps);
    tp[j+1] = p[i] + (p[i+1] - p[i])/2.0*(1.0+eps);
  }
 return(tp);
}

/* ----- Interpolation Function, Corresponds to k = 2 in the paper ----- */

double legendre(double x, int j, int k, double * p)  /* j=1..n */
 {
  double point;
  double ret;

  point = (2.0*x - p[j] - p[j-1])/(p[j] - p[j-1]);
  if ((point < -1.0) || (point >= 1.0)) return(0.0);
  if (k == 1) ret = (1.0 - point*sqrt(3))/2.0;
  else if (k == 2) ret = (1.0 + point*sqrt(3))/2.0;
  else ret = 0.0;
  return(ret);
 }

/* -----   Gaussian Quadrature with three points. ----- */

double quad(double (*f)(), double a, double b, int j, int k, double * p)
{
 double c1,c2,c3,x1,x2,x3,alpha,beta,Konst,temp,temp1;
 alpha = (b-a)/2.0;
 beta = (b+a)/2.0;
 Konst = alpha;
 x1 = -sqrt(15.0)/5.0;
 x2 = 0.0;
 x3 = -x1;
 c1 = 1.0/(3.0*pow(x3,2));
 c2 = 2.0/3.0 * (3.0 - 1.0/pow(x3,2));
 c3 = c1;
 temp = Konst * (c1 * (*f)(alpha*x1+beta)*legendre(alpha*x1+beta,j,k,p) +
                 c2 * (*f)(alpha*x2+beta)*legendre(alpha*x2+beta,j,k,p) +
                 c3 * (*f)(alpha*x3+beta)*legendre(alpha*x3+beta,j,k,p));
 return(temp);
}

/* ----- sintegrate function ----- */

double sintegrate(double (*f)(), double (*weight)(),
     double a, double b, int n, int left_singularity,
     int right_singularity, int with_log, double alpha, double beta)
{
 double q_alpha, q_beta, * grid, * grid1;
 double * interpolation_grid, * interpolation_grid1, * coefs, * coefs1;
 double * vals, value, * vals1, value1;
 int i,j,k,index;

 if (with_log == 0)
   {
    q_alpha = (2.0*K+1.0)/(alpha+1.0);
    q_beta = (2.0*K+1.0)/(beta+1.0);
   }
 else
   {
```

```
    if (alpha >= 0)
        q_alpha = 2.0*K+1.0;
    else
        q_alpha = (2.0*K+1.0)/(alpha+1.0);

    if (beta >= 0)
        q_beta = 2.0*K+1.0;
    else
        q_beta = (2.0*K+1.0)/(beta+1.0);

   }

if (left_singularity == 0) q_alpha = 0.0;
if (right_singularity == 0) q_beta = 0.0;

if ((q_alpha != 0.0) && (q_beta == 0.0))
   {
    grid = points(n,a,b,q_alpha,right_singularity);
    interpolation_grid = inter_points(grid,n);
    coefs = (double *) malloc(sizeof(double)*(2*n));
    vals = (double *) malloc(sizeof(double)*(2*n));
    for (i=0; i<=2*n-1; i++) vals[i] = (*weight)(interpolation_grid[i]);
    for (i=1; i<=n; i++)
     for (k=1; k<=2; k++)
      {
       index = 2 * (i - 1) + k - 1;
       coefs[index] = 0.0;
       for (j=0; j<=n-1; j++)
        coefs[index] = coefs[index] + quad(f, grid[j], grid[j+1], i, k, grid);
      }
    value = 0.0;
    for (i=0; i<=2*n-1; i++) value = value + coefs[i]*vals[i];
    free((void *) coefs);
    free((void *) vals);
    free((void *) grid);
    free((void *) interpolation_grid);
   }
 else if ((q_alpha == 0.0) && (q_beta != 0.0))
   {
    grid = points(n,a,b,q_beta,right_singularity);
    interpolation_grid = inter_points(grid,n);
    coefs = (double *) malloc(sizeof(double)*(2*n));
    vals = (double *) malloc(sizeof(double)*(2*n));
    for (i=0; i<=2*n-1; i++) vals[i] = (*weight)(interpolation_grid[i]);
    for (i=1; i<=n; i++)
     for (k=1; k<=2; k++)
      {
       index = 2 * (i - 1) + k - 1;
       coefs[index] = 0.0;
       for (j=0; j<=n-1; j++)
        coefs[index] = coefs[index] + quad(f, grid[j], grid[j+1], i, k, grid);
      }
    value = 0.0;
    for (i=0; i<=2*n-1; i++) value = value + coefs[i]*vals[i];
    free((void *) coefs);
    free((void *) vals);
    free((void *) grid);
    free((void *) interpolation_grid);
   }
  else if ((q_alpha != 0.0) && (q_beta != 0.0))
   {
    double mid;

    mid = (a+b)/2.0;
    grid = points(n,a,mid,q_alpha,0);
    grid1 = points(n,mid,b,q_beta,1);
```

```
    interpolation_grid = inter_points(grid,n);
    interpolation_grid1 = inter_points(grid1,n);
    coefs = (double *) malloc(sizeof(double)*(2*n));
    coefs1 = (double *) malloc(sizeof(double)*(2*n));
    vals = (double *) malloc(sizeof(double)*(2*n));
    vals1 = (double *) malloc(sizeof(double)*(2*n));
    for (i=0; i<=2*n-1; i++)
       { vals[i] = (*weight)(interpolation_grid[i]);
         vals1[i] = (*weight)(interpolation_grid1[i]);
       }
    for (i=1; i<=n; i++)
     for (k=1; k<=2; k++)
      {
       index = 2 * (i - 1) + k - 1;
       coefs[index] = 0.0;
       coefs1[index] = 0.0;
       for (j=0; j<=n-1; j++)
        {
         coefs[index] = coefs[index]+quad(f,grid[j],grid[j+1],i,k,grid);
         coefs1[index] = coefs1[index]+quad(f,grid1[j],grid1[j+1],i,k,grid1);
        }
      }
    value = 0.0;
    value1 = 0.0;
    for (i=0; i<=2*n-1; i++)
        { value = value + coefs[i]*vals[i];
          value1 = value1 + coefs1[i]*vals1[i];
        }
    value += value1;
    free((void *) coefs);
    free((void *) vals);
    free((void *) grid);
    free((void *) interpolation_grid);
    free((void *) coefs1);
    free((void *) vals1);
    free((void *) grid1);
    free((void *) interpolation_grid1);
   }
  else if ((q_alpha == 0.0) && (q_beta == 0.0))
   {
    printf("Internal Error: q == 0.\n");
    exit(0);
   }


 return(value);
}
```

PROJECT 3

HIDEAKI KANEKO and PETER A. PADILLA

Superconvergence of Degenerate Kernel Method

The degenerate kernel method is a classical method for finding approximate solutions of the second kind Fredholm integral equations ($x - Kx = f$ in operator form). The basic principle of the method is to approximate a kernel, a bivariate function, as a finite sum of univariate functions. An advantage of the method lies in its simplicity, whereas its disadvantage lies in the high cost of computations. If x_n denotes a degenerate kernel approximation, then by the *iterate* of x_n, we mean $x_n' = f + Kx_n$.

We discovered that the rate of convergence of the iterates of degenerate kernel approximations is determined by the method under which the kernel is decomposed. We proved and demonstrated numerically that, when the decomposition is done as a least squares approximation or as an interpolation approximation using a certain set of interpolation points, then the iterates converge twice as fast as the original degenerate kernel solutions provided that the kernel is sufficiently smooth.

To reduce the high computational cost, we propose to introduce a class of wavelets for a decomposition process. We expect to obtain a system of linear equations whose corresponding matrix is sparse as opposed to the normal full matrix that we encounter in the degenerate kernel method.

Superconvergence of Degenerate Kernel Method

Hideaki Kaneko*

Department of Mathematics and Statistics

Old Dominion University

Norfolk, Virginia 23529-0077

Peter Padilla

NASA Langley Research Center

MS-159

Hampton, Virginia 23681-0001

*This author is partially supported by NASA, project title NCCI-213

SUPERCONVERGENCE OF DEGENEREATE KERNEL MWETHOD

H. Kaneko and P. Padilla

Abstract

In this paper, a general theory for the iterated operator approximation is developed. Some of the known results on superconvergence of various iterated schemes can be formulated as special cases of our theory. The method is then subsequently used to obtain some results on the superconvergence of the iterated degenerate kernel method for the Fredholm equations of the second kind and the Hammerstein equations.

1 Introduction

In this paper, we consider the Fredholm integral equation of the second kind

$$x(t) - \int_a^b k(t,s)x(s)ds = f(t) \qquad a \le t \le b \tag{1.1}$$

and the Hammerstein equation

$$x(t) - \int_a^b k(t,s)\psi(s,x(s))ds = f(t) \qquad a \le t \le b \tag{1.2}$$

where k, f in (1.1) and (1.2) and ψ in (1.2) are known functions and in both cases x is the function to be determined. A number of numerical methods are available for obtaining approximate solutions of (1.1) and (1.2). It is well known that popularly used methods such as the Galerkin method and the collocation method can be viewed as special cases of the projection method. The projection method for the Fredholm integral equation of the second kind (1.1) can be described as follows; Let X denote the Banach space of functions and $\{X_n\}$ a sequence of finite dimensional subspaces of X that is essentially dense in X,-i.e. $cl \cup X_n = X$ where clS denotes the closure of a set S. We denote a family of projections of X onto X_n by $\{P_n\}$. Now we define

$$Kx(t) \equiv \int_a^b k(t,s)x(s)ds \qquad \text{for each } t \in (a,b). \tag{1.3}$$

Equation (1.1) can be written in operator form as

$$x = f + Kx. \tag{1.4}$$

Here x is assumed to be an element of X. The underlying principle that supports the projection method is to seek an element $x_n \in X_n$ for which the residual $r_n \equiv f - (x_n - Kx_n)$ disappears

under the projection P_n, -i.e. $P_n r_n = 0$ for each n. In the Galerkin method, the projection is orthogonal. On the other hand, in the collocation method, the projection is interpolatory. A similar discussion for the projection method for the nonlinear Hammerstein equation (1.2) will be left to the reader (see, e.g. [2] [3] [4] [8] [9]).

Let x_n denote a numerical solution of equation (1.4). The (Sloan) iterated solution x_n' corresponding to x_n is defined by

$$x_n'(t) = f(t) + \int_a^b k(t,s)x_n(s)ds. \tag{1.5}$$

In the Galerkin and the collocation solutions x_n, it is known [6] that, under suitable conditions on the kernel k and the forcing term f, their corresponding iterated solutions converge to x more rapidly than x_n does to x, a phenomenon commonly known as superconvergence. One of the purposes of this paper is to develop a general iterated approximation scheme for equation (1.1). This will be done in Section 2. The generality of the theory is demonstrated by showing that the results on the superconvergence of the Galerkin and the collocation iterates examined by Sloan et al are special cases of our iterated operator approximation scheme. We note that the results obtained in [6] concerning the superconvergence of the iterated Galerkin and the iterated collocation method have been recently generalized to hold for Hammerstein equation (1.2) [10] [7]. Moreover, in these papers superconvergence of the iterated solutions for weakly singular Hammerstein equations are also observed. It appears that these results on superconvergence of the weakly singular equations are new even for the Fredholm equations.

Results obtained in Section 2 are subsequently used to establish the superconvergence of the iterated degenerate kernel method for equations (1.1) and (1.2). The degenerate kernel method for solving equation (1.1) is discussed in [1] and that for equation (1.2) is discussed in [11]. It turns out that the superconvergence of the iterated degenerate kernel method depends upon the ways in which the kernel k in equations (1.1) and (1.2) are decomposed as a finite sum of products of univariate functions. This will be illustrated in detail in Section 3. In Section 4, the iterated degenerate kernel method for Hammerstein equation (1.2) is discussed. Even though the material in Section 4 does not have direct relation with the general theory presented in Section 1, we feel that it is appropriate to include it here since it extends the results of Section 3 as well as those in [11]. Also included in Section 4 is a discussion on a computational consideration for solving the system of nonlinear equations that must be solved in order to obtain the degenerate kernel solution of (1.2). We make use of the devise introduced by Kumar and Sloan [12] to

'linearlize' the system of nonlinear equations. That is, the integrals that must be evaluated repeatedly at each stage of the iterations, when a conventional iterated scheme is employed, are now evaluated only once at the beginning of computation.

Examples are provided in Sections 3 and 4 to demonstrate our theory.

2 The Iterated Approximation for the Fredholm Integral Equations of the Second Kind

In this section, we consider the Fredholm integral equation of the second kind given by (1.1). We denote by $C[a,b]$ the Banach space of all continuous functions defined on $[a,b]$ equipped with the uniform norm $\|\cdot\|_\infty$. Also we denote by $L_p[a,b]$, $1 \le p \le \infty$, the Banach space of measurable functions whose pth power are integrable ($1 \le p < \infty$) or the space of essentially bounded functions ($p = \infty$) on $[a,b]$. In equation (1.1), we assume that

$$f \in C[a,b] \tag{2.1}$$

and that, with $k_t(s) \equiv k(t,s)$,

$$\lim_{t\to\tau} \int_a^b |k_t(s) - k_\tau(s)|ds = 0 \quad \text{for each } \tau \in [a,b]. \tag{2.2}$$

Under the condition (2.2), the linear operator K defined by (1.3) is a compact linear operator of $C[a,b]$ into $C[a,b]$. In order to establish a general iterated approximation scheme, we assume that $\{K_n\}$ is a sequence of operators converging to K in operator norm. That is,

$$\|K_n - K\|_p \to 0 \text{ as } n \to \infty \text{ for some } 1 \le p \le \infty. \tag{2.3}$$

For each $n \ge 1$, we assume that we have an equation whose solution approximates the solution of (1.4). We denote this approximating equation by

$$x_n = f_n + K_n x_n. \tag{2.4}$$

For example, in the case of the projection method, equation (2.4) is identified by letting $K_n = P_n K$ and $f_n = P_n f$ where P_n is a projection of $C[a,b]$ onto a finite dimensional subspace of $C[a,b]$. In the case of the degenerate kernel method, K_n denotes the finite rank separable operator, -i.e. $K_n x(t) = \int_a^b \sum_{i=1}^n a_i(t) b_i(s) x(s) ds$ where $\{a_i\}_{i=1}^n$ is a linearly independent family

of functions and $f_n = f$ for each $n \geq 1$. We define the iterated approximation corresponding to (2.4) by

$$x'_n = f + Kx_n. \tag{2.5}$$

As was indicated in Introduction, the iterated approximations for the Galerkin and for the collocation methods exhibit, under suitable smoothness conditions on the kernel k and on the forcing term f, the global superconvergence. It is not known, however, that the similar superconvergence result can be obtained for the iterated approximations for the degenerate kernel method. We are now in a position to prove the main theorem of this section. Advantages of the theory presented are twofold. First, it can be used to describe the superconvergence results of Sloan et al mentioned as special cases of this theorem. Second, it can be used to establish the superconvergence of the iterated degenerate kernel method. The latter will be done in Section 3.

Theorem 2.1 *Consider equation (1.1) in a Banach space* $(X, \|\cdot\|)$. *We assume that* 1 *is not an eigenvalue of* K. *Let* x_n *and* x'_n *satisfy equations (2.4) and (2.5) respectively. Then, for sufficiently large* n, *there exists a constant* $c > 0$, *independent of* n, *such that*

$$\|x - x'_n\| \leq c\{\|K - K_n\|^2 + \|K(K - K_n)x\| + \|K - K_n\|\,\|f - f_n\| + \|K(f - f_n)\|\}. \tag{2.6}$$

Proof: From (1.4) and (2.5),

$$x - x'_n = K(x - x_n). \tag{2.7}$$

Applying K on both sides of (1.4) and (2.4), we obtain

$$Kx = Kf + K^2x \tag{2.8}$$

and

$$Kx_n = KK_nx_n + Kf_n. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned} K(x - x_n) &= K^2x - KK_nx_n + K(f - f_n) \\ &= K(Kx - K_nx_n) + K_n(Kx - Kx_n) - K_n(Kx - Kx_n) + K(f - f_n) \\ &= K_n(Kx - Kx_n) + (K - K_n)(Kx - Kx_n) + K(K - K_n)x_n + K(f - f_n). \end{aligned} \tag{2.10}$$

Since $\|K_n - K\| \to 0$ as $n \to \infty$ and $(I-K)^{-1}$ exists by assumption, we conclude [1] that $(I-K_n)^{-1}$ exists and uniformly bounded for sufficiently large n. Therefore,

$$K(x - x_n) = (I - K_n)^{-1}\{(K - K_n)(Kx - Kx_n) + K(K - K_n)x_n + K(f - f_n)\}.$$

Taking the norm on both sides,

$$\begin{aligned} \|K(x-x_n)\| &\le \|(I-K_n)^{-1}\|\{\|K-K_n\|\|K\|\|x-x_n\| + \|K(K-K_n)x_n\| \\ &\quad + \|K(K-K_n)(x-x_n)\| + \|K(f-f_n)\|\} \\ &\le \|(I-K_n)^{-1}\|\{2\|K\|\|K-K_n\|\|x-x_n\| + \|K(K-K_n)x_n\| \\ &\quad + \|K(f-f_n)\|\}. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned} x - x_n &= Kx - K_n x_n + f - f_n \\ &= Kx - K_n x + K_n x - K_n x_n + f - f_n \end{aligned}$$

we obtain

$$(I - K_n)(x - x_n) = Kx - K_n x + f - f_n,$$

or

$$x - x_n = (I - K_n)^{-1}\{(K - K_n)x + f - f_n\}. \tag{2.12}$$

From (2.11) and (2.12),

$$\begin{aligned} \|x - x_n'\| &= \|K(x - x_n)\| \\ &\le c\{\|K - K_n\|\|x - x_n\| + \|K(K - K_n)x_n\| + \|K(f - f_n)\|\} \\ &\le c\{\|K - K_n\|^2 + \|K - K_n\|\|f - f_n\| + \|K(K - K_n)x_n\| \\ &\quad + \|K(f - f_n)\|\}. \end{aligned}$$

This completes our proof. □

Corollary 2.2 *For the iterated approximation scheme (2.5), if $f_n = f$ for all n in (2.4), then*

$$\|x - x_n'\| \le c\{\|K - K_n\|^2 + \|K(K - K_n)x_n\|\}.$$

In the next section, this corollary will be used to establish a superconvergence result for the iterated degenerate kernel scheme. In order to see that Theorem 2.1 includes the results of superconvegence of the iterated Galerkin and the iterated collocation schemes of Sloan et al [6], we need some definitions. First we let W_p^m, $1 \le p \le \infty$, m nonnegative integer, denote the

Sobolev space of functions defined over $[a,b]$. Namely $g \in W_p^m$ if and only if $g^{(k)} \in L_p[a,b]$ for $k = 0, 1, \ldots, m$ where $g^{(k)}$ denotes the kth distributional derivative of g. Let

$$\Pi_n : a = t_0 < t_1 < \cdots < t_n = b$$

be a partition of $[a,b]$. Let $h = \max_{1 \le i \le n}(t_i - t_{i-1})$ and assume $h \to 0$ as $n \to \infty$. Let ν be an integer and r a positive integer such that $0 \le \nu < r$. Let $S_{r,n}^{\nu}$ denote the space of splines of order r and continuity ν, namely $\varphi_n \in S_{r,n}^{\nu}$ if and only if φ_n is a piecewise polynomial of degree $\le r-1$ on each $[x_{i-1}, x_i]$ and has $\nu - 1$ continuous derivatives on (a,b). Let P_n^G denote an orthogonal projection of $C[a,b]$ onto $S_{r,n}^{\nu}$. In the Galerkin method, equation (2.4) becomes

$$x_n^G - P_n^G K x_n^G = P_n^G f \tag{2.13}$$

-i.e. $K_n = P_n^G K$ and $f_n = P_n^G f$. The corresponding (Sloan) iteration approximation to (2.5) is given by

$$x_n^{G'} = f + K x_n^G. \tag{2.14}$$

If $f \in W_p^m, (m \ge 0)$, then there exists $\psi_n \in S_{r,n}^{\nu}$ $(0 \le \nu < r)$ such that

$$\|f - \psi_n\|_p \le c h^{\min\{m,r\}} \|f\|_{m,p} \tag{2.15}$$

where c is a constant independent of n and $\|f\|_{m,p} = \sum_{k=0}^{m} \|f^{(k)}\|_p$ (see e.g. [13]). Under the assumption of the quasiuniform mesh, -i.e.

$$\frac{h}{\min_{1 \le i \le n}(t_i - t_{i-1})} < c \qquad \text{for each } n,$$

it can be shown that

$$\sup_n \|P_n^G\|_{L_p \to L_p} \le c_2. \tag{2.16}$$

for some constant c_2. Since

$$\begin{aligned} \|f - P_n^G f\|_p &= \|f - \psi_n + P_n^G \psi_n - P_n^G f\|_p \\ &\le (1 + \|P_n^G\|) \|f - \psi_n\|_p, \end{aligned} \tag{2.17}$$

from (2.15), (2.16) and (2.17), we obtain

$$\|f - P_n^G f\|_p \le c h^{\min\{m,r\}} \|f\|_{m,p}. \tag{2.18}$$

Now let $\xi(t) = \int_a^b k(t,s)x_n^G(s)ds$. Then

$$\begin{aligned} |K(K-K_n)x_n^G(t)| &= |\int_a^b k(t,u)\{\xi(u) - P_n^G\xi(u)\}du| \\ &= |(k_t, \xi - P_n^G\xi)| \\ &= |(k_t - \varphi_n, \xi - P_n^G\xi)| \text{ for every } \varphi_n \in S_{r,n}^\nu \\ &\le \|k_t - \varphi_n\|_q \|\xi - P_n^G\xi\|_p, \end{aligned} \tag{2.19}$$

where $\frac{1}{q} + \frac{1}{p} = 1$ with convention that if $p = 1$, then $q = \infty$. In (2.19), we have used the orthogonality in the third equality and the Hölder inequality in the last step. If $k_t \in W_p^m$, then from (2.15) there exists $\varphi_n \in S_{r,n}^\nu$ such that $\|k_t - \varphi_n\|_q \le ch^{\min\{m,r\}}\|k_t^{(m)}\|_{m,q}$. Finally from (2.19) we obtain

$$\|K(K-K_n)x_n^G\|_\infty \le ch^{2\min\{m,r\}}.$$

Similarly, we can show that whenever $f \in W_p^m$,

$$\|K(f - P_n^G f)\|_\infty \le ch^{2\min\{m,r\}}$$

and that

$$\|K - K_n\|_\infty \le ch^m.$$

Using the estimate (2.6), the above discussion leads us to the following corollary.

Corollary 2.3 *(Graham, Joe and Sloan [[6] :Theorem 4.1]) Let x_n^G and $x_n^{G'}$ denote the solutions for (2.13) and (2.14) respectively. Suppose that $k_t \in W_p^m$ $(0 \le m \le r)$ with $\|k_t\|_{m,q}$ bounded independently of t and that $f, \xi \in W_p^m$ where $\xi(t) \equiv \int_a^b k(t,s)x_n^G(s)ds$ with x_n^G. Then*

$$\|x - x_n^{G'}\|_\infty \le ch^{2\min\{m,r\}}$$

where c is independent of n.

Now in the partition Π_n, for each i, we select $\{t_{ij}\}_{j=1}^r$ such that

$$x_{i-1} \le t_{i1} < t_{i2} < \cdots < t_{ir} \le x_i.$$

Let P_n^C denote the interpolatory projector of $C[a,b]$ onto $S_{r,n}^\nu$ defined by $P_n^C x(t_{ij}) = x(t_{ij})$ for each $i = 1, \ldots, n$ and $j = 1, 2, \ldots, r$. In the collocation method, equation (2.4) becomes

$$x_n^C - P_n^C K x_n^C = P_n^C f \tag{2.20}$$

-i.e. $K_n = P_n^C K$ and $f_n = P_n^C f$. The corresponding iterated collocation solution is defined by

$$x_n^{C'} = f + K x_n^C. \tag{2.21}$$

As in Corollary 2.3 of the iterated Galerkin method, to see that the iterated collocation method of (2.21) is a special case of Theorem 2.1, we must examine the terms in the right side of (2.6). The second term of (2.6) in this case is analyzed as follows: Let $y(t) = \int_a^b k(t,s)x_n^C(s)ds$. Then

$$\begin{aligned} K(K - K_n)x_n^C(t) &= (k_t, y - P_n^C y) \\ &\quad (k_t - \varphi_{n,t}, y - P_n^C y) + (\varphi_{n,t}, (I - P_n^C)(y - \psi_n)) \\ &\quad +(\varphi n, t, (I - P_n^C)\psi_n). \end{aligned} \tag{2.22}$$

where $\varphi_{n,t} \in S^0_{m,n}$ and $\psi_n \in S^0_{l,n}$. Now arguing exactly as in the proof of theorem 4.2 [6], we obtain

$$\|k(K - K_n)x_n^C\|_\infty \leq ch^{2\min\{l, m+r\}}$$

where c is a constant independent of n. The other terms in (2.6) can be bounded similarly.

Corollary 2.4 *(Graham, Joe and Sloan [[6] :Theorem 4.2]) Let x_n^C and $x_n^{C'}$ be the solutions of (2.20) and (2.21) respectively. Suppose $f \in C[a,b]$, $x \in W_1^l$ $(0 < l \leq 2r)$ and $k_t \in W_1^m$ $(0 < m \leq r)$, with $\|k_t\|_{m,1}$ bounded independently of t. Then*

$$\|x - x_n^{C'}\|_\infty \leq ch^{2\min\{l, r+m\}}$$

where c is independent of n.

3 The Iterated Degenerate Kernel Methods

The purpose of this section is to use Theorem 2.1, Corollary 2.2 in particular, to establish superconvergence of the iterated degenerate kernel method. A generalization to the iterated degenerate kernel method for the Hammerstein equations will be done in Section 4.

Consider equation (1.1). The degenerate kernel method for approximating the solution of (1.1) requires us to approximate the kernel k in (1.1) by some degenerate kernel whose general form can be described as

$$k_n(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varphi_i(s)\varphi_j(t) \tag{3.1}$$

where $\{\varphi_i\}_{i=1}^n$ is a set of linearly independent functions in $C[a,b]$. The operator K in (1.3) is then approximated by a sequence of operators

$$K_n y(t) = \int_a^b k_n(t,s)y(s)ds. \tag{3.2}$$

Subsequently an approximate solution x_n is found by solving

$$x_n(t) - \int_a^b k_n(t,s)x_n(s)ds = f(t) \qquad a \le t \le b. \tag{3.3}$$

Equation (3.3) can be written as

$$x_n(t) - \sum_{i=1}^n \varphi_i(t)\{\sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)x_n(s)ds\} = f(t) \qquad a \le t \le b.$$

If we put

$$c_i \equiv \sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)x_n(s)ds, \tag{3.4}$$

then x_n can be written as

$$x_n(t) = f(t) + \sum_{i=1}^n c_i\varphi_i(t). \tag{3.5}$$

Upon substituting (3.5) into (3.4), we obtain the following $n \times n$ system of linear equations for c_i.

$$c_i - \sum_{l=1}^n c_l \sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)\varphi_k(s)ds = \sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)f(s)ds \qquad 1 \le i \le n. \tag{3.6}$$

Finally, once these c_i's are found by solving (3.6), equation (3.5) gives the required approximate solution for the degenerate kernel method. Equation (3.3) is written in operator form as

$$x_n - K_n x_n = f \tag{3.7}$$

which is a special form of (2.4) with $f_n = f$ for all n. When the degenerate kernel solution x_n is iterated as in (2.5), an interesting question arises. The question is of course under what conditions superconvergence of the iterates in this case is guaranteed. It is the purpose of this section that we provide some answers to this question. It turns out that the superconvergence of the degenerate kernel method hinges critically upon the ways that one decomposes the kernel k in (3.1). Here we demonstrate two different methods that guarantee the superconvergence of the iterates of the degenerate kernel method.

In the first method, we examine the least-squares approximation. For each positive integer k, assume that a partition Π_k satisfies the quasiuniform condition (ref. Section 2)

$$\frac{h}{\min_{1\le i\le n}(t_i - t_{i-1})} < c \qquad \text{for each } k$$

Let n denote the dimension of the spline space $S^{\nu}_{r,k}$ and $B_1, B_2, \ldots, B_n$ be the B-spline basis for $S^{\nu}_{r,k}$. As in Section 2, r and ν are integers such that $0 \leq \nu < r$. Assume that $k_n(t,s)$ is the least-squares approximation of $k(t,s)$,-i.e. assume that a_{ij} in (3.1) are such that

$$\int_a^b \int_a^b |k(t,s) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}B_i(s)B_j(t)|^2 dsdt = \min_{b_{ij}\in R} \int_a^b \int_a^b |k(t,s) - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}B_i(s)B_j(t)|^2 dsdt.$$

We are now in a position to present the first theorem of this section.

Theorem 3.1 *Let x be the solution of (1.1) and x_n the solution of (3.7) where (3.1) is defined by the least-squares approximation. Assume that $k \in W_2^m([a,b]\times[a,b])$, $0 \leq m \leq r$, and $k_s(\cdot)x(\cdot) \in W_2^l([a,b]\times[a,b]))$ for each $s \in [a,b]$, where $0 \leq l \leq r$ and $k_s(t) = k(s,t)$. Then*

$$\|x - x_n'\|_\infty = O(h^\nu)$$

with $\nu = \min\{m+l, 2m\}$.

Proof: Using corollary 2.2, we obtain

$$\|x - x_n'\|_\infty = O(h^{2m}) + O(\|K(K-K_n)x_n\|_\infty). \tag{3.8}$$

Hence we only need to estimate the order of convergence of $\|K(K-K_n)x_n\|_\infty$. Note that

$$\begin{aligned} |K(K-K_n)x_n(t)| &= |\int_a^b k(t,u)\int_a^b [k(u,s)-k_n(u,s)]x_n(s)dsdu| \\ &= \int_a^b \int_a^b k(t,u)[k(u,s)-k_n(u,s)]x_n(s)dsdu|. \end{aligned}$$

Let $\psi_t(u,s) = k(t,u)x_n(s)$ and let $\varphi_n(u,s) = \sum_{i=1}^n \sum_{j=1}^n b_{ij}B_i(u)B_j(s)$ be any element that is a tensor product of B-splines. Then since k_n is the least-squares approximation of k,

$$\int_a^b \int_a^b \varphi_n(u,s)[k(u,s)-k_n(u,s)]dsdu = 0,$$

therefore

$$|K(K-K_n)x_n(t)| = |\int_a^b \int_a^b [\psi_t(u,s)-\varphi_n(u,s)][k(u,s)-k_n(u,s)]dsdu.$$

Applying the Cauchy-Schwartz inequality,

$$|K(K-K_n)x(t)| \leq \|\psi - \varphi_n\|_2 \|k-k_n\|_2.$$

Here of course $\|\cdot\|_2$ denotes the L_2 norm defined on the space of bivariate functions $W_2^l([a,b]\times[a,b])$. Noting that $\|k-k_n\|_2 = O(h^m)$ and $\|\psi-\varphi_n\|_2 = O(h^l)$, (3.8) proves the desired result. □

The second method that produces superconvergence of the iterates of the degenerate kernel solutions is based upon approximating k by interpolation. Let

$$\Pi_N : a = t_0 < t_1 < \cdots < t_N = b$$

be a partition of $[a,b]$. Let $h = \max_{1\le i\le N}(t_i - t_{i-1})$ and assume as in Section 2 that $h \to 0$ as $N \to \infty$. Let $\xi_1, \xi_2, \ldots, \xi_r$ be the zeros of the rth degree Legendre polynomial in $[-1,1]$. We shift these points to each subinterval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$ to obtain $\{\tau_{ij}\}_{j=1}^{r}$. Denote the interpolation polynomials by φ_{ij}, -i.e.

$$\varphi_{ij}(\tau_{\alpha,\beta}) = \begin{cases} 1 & \text{if } (i,j) = (\alpha,\beta) \\ 0 & \text{if } (i,j) \neq (\alpha,\beta) \end{cases} \tag{3.9}$$

An approximating degenerate kernel k_n ($n = Nr$) is now defined by

$$k_n(s,t) = \sum_{i=1}^{N}\sum_{j=1}^{r}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{r} k(\tau_{ij}, \tau_{\alpha,\beta})\varphi_{ij}(s)\varphi_{\alpha,\beta}(t). \tag{3.10}$$

Let the interpolation projector of $C([a,b]\times[a,b])$ into the tensor product space $S^0_{r,n} \otimes S^0_{r,n}$ be denoted by P_n. That is,

$$P_n k(s,t) = k_n(s,t)$$

where k_n is defined in (3.10). Also let x_n^I denote the solution of equation (2.4) when K_n possesses the kernel k_n defined by (3.10) and $x_n^{I'}$ denote the corresponding iterate defined by (2.5). The following theorem demonstrates the superconvergence of the iterated degenerate kernel method when the kernel is decomposed by interpolation.

Theorem 3.2 *Assume that in equation (1.1) $k \in W_1^m([a,b]\times[a,b])$, $0 < m \le r$, and $k_s(\cdot)x_n(\cdot) \in W_1^l([a,b]\times[a,b])$, $0 < l \le 2r$, where $k_t(u) = k(t,u)$. Then*

$$\|x - x_n^{I'}\|_\infty = O(h^\nu), \qquad \nu = \min\{2m, r+l\}.$$

Proof: As in the proof of Theorem 3.1, we need to estimate the error of $\|K(K-K_n)x_n\|_\infty$. By taking $\varphi_n \in S^0_{l,n} \otimes S^0_{l,n}$ and $\psi_n \in S^0_{m,n} \otimes S^0_{m,n}$, for each $t \in [a,b]$,

$$\begin{aligned} K(K-K_n)x(t) &= \int_a^b k(t,u)\int_a^b [k(u,s) - k_n(u,s)]x_n(s)dsdu \\ &= \int_a^b\int_a^b k(t,u)x_n(s)[k(u,s) - k_n(u,s)]dsdu \\ &= (k_t(u)x_n(s), k(u,s) - k_n(u,s)) \\ &= (k_t(u)x_n(s) - \varphi_n(u,s), k(u,s) - k_n(u,s)) \\ &\quad +(\varphi_n(u,s), (I-P_n)(k(u,s) - \psi_n(u,s)) + (\varphi_n(u,s), (I-P_n)\psi_n(u,s)) \end{aligned}$$

The rest of proof follows by an argument similar to the one give in the proof of theorem 4.2 of Graham, Joe and Sloan [6]. A modification needed here however is to accomodate the bivariate functions. An approximation of a bivariate function by an element from a space of tensor product of finite dimensional univariate functions and its approximation power is well ducumented, -e.g., see [5]. □

4 The Iterated Degenerate Kernel Method for Hammerstein Equations

In this section, we extend the results of the previous section to obtain superconvergence of the iterated degenerate kernel method for Hammerstein equation described in (1.2). We assume throughout this section, unless stated otherwise, the following conditions on k, f and ψ:

1. $\lim_{t\to\tau} \|k_t - k_\tau\|_\infty = 0, \qquad \tau \in [a,b]$;
2. $M \equiv \sup_{a\le s\le b} \int_a^b |k(t,s)|dt < \infty$;
3. $f \in C[a,b]$;
4. $\psi(t,x)$ is continuous in $t \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which
$$|\psi(t,x_1) - \psi(t,x_2)| \le C_1|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty);$$
5. the partial derivative $\psi^{(a,b)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that
$$|\psi^{(a,b)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \le C_2|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty); \qquad (1.2)$$
6. for $x \in C[0,1]$, $\psi(.,x(.)), \psi^{(0,1)}(.,x(.)) \in C[a,b]$.

Additional assumptions will be given later as needed. A comprehensive study of the degenerate kernel method for Hammerstein equations was made by Kaneko and Xu [11]. The degenerate kernel method for Hammerstein equations consists of replacing K in (1.2) by k_n of (3.1) and approximating the solution x of equation (1.2) by x_n which satisfies

$$x_n(t) - \int_a^b k_n(t,s)\psi(s,x_n(s))\,ds = f(t) \qquad a \le t \le b. \qquad (4.1)$$

Following analogously the development made in (3.5) and (3.6), for

$$c_i \equiv \sum_{j=1}^{n} \int_a^b a_{ij}\varphi_j(s)\psi(s, x_n(s))ds, \tag{4.2}$$

x_n can be written as

$$x_n(t) = f(t) + \sum_{i=1}^{n} c_i\varphi_i(t). \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain the following n nonlinear equations in n unknowns $c_1, \cdots, c_n$,

$$c_i = \sum_{j=1}^{n} \int_a^b a_{ij}\varphi_j(s)\psi(s, f(s) + \sum_{l=1}^{n} c_l\varphi_l(s))ds, \qquad 1 \le i \le n. \tag{4.4}$$

Define

$$K\Psi x(t) \equiv \int_a^b k(t,s)\psi(s, x(s))ds$$

so that (1.2) can be written as

$$x - K\Psi x = f. \tag{4.5}$$

Similarly we write equation (4.1) as

$$x_n - K_n\Psi x_n = f \tag{4.6}$$

The iterated solution x_n' is now obtained by

$$x_n' = f + K\Psi x_n. \tag{4.7}$$

The Fréchet derivative of $K\Psi$ at φ_0 is denoted and defined by

$$(K\Psi)'(\varphi_0)(\varphi)(t) = \int_a^b k(t,s)\psi_2(s, \varphi_0(s))\varphi(s)ds$$

with ψ_2 denoting the first partial derivative of ψ with respect to the second variable. The following theorem describes superconvergence phenomenon of x_n' to x. Here we assume that the decomposition (3.1) is done via interpolation (the second method described in the previous section). The case of the least-squares approximation is similar and will be left to the reader to supply the detail.

Theorem 4.1 *Assume that in equation (1.2)* $k \in W_1^m([a,b]\times[a,b])$, $0 < m \le r$, *and* $k_s(\cdot)\psi(\cdot, x_n(\cdot)) \in W_1^l([a,b]\times[a,b])$, $0 < l \le 2r$, *where* x_n *is the solution of (4.6). Assume also that* 1 *is not an eigenvalue* $(K\Psi)'(\xi)$ *for each* ξ *between* $f + K_n\Psi x_n$ *and* $f + K\Psi x$. *Then*

$$\|x - x_n^{I'}\|_\infty = O(h^\nu), \qquad \nu = \min\{2m, r+l\}.$$

Proof: From (4.5) and (4.6),

$$x - x_n' = K\Psi x - K\Psi x_n. \tag{4.8}$$

Now

$$\begin{aligned} K\Psi x - K\Psi x_n &= K\Psi(f + K\Psi x) - K\Psi(f + K_n\Psi x_n) \\ &= (K\Psi)'(\theta(f + K_n\Psi x_n) + (1-\theta)(f + K\Psi x))(K\Psi x - K_n\Psi x_n) \\ &\qquad \text{for some} 0 \le \theta \le 1 \\ &= K_\theta(K\Psi x - K_n\Psi x_n + (K\Psi x - K\Psi x_n) - (K\Psi x - K\Psi x_n)), \end{aligned}$$

where $K_\theta \equiv (K\Psi)'(\theta(f + K_n\Psi x_n) + (1-\theta)(f + K\Psi x))$. Since 1 is not an eigenvalue of K_θ for each $0 \le \theta \le 1$, we obtain

$$K\Psi x - K\Psi x_n = (I - K_\theta)^{-1}K_\theta(K - K_n)\Psi x_n. \tag{4.9}$$

Combining (4.8) and (4.9), and taking the norm on both sides, we obtain

$$\|x - x_n'\|_\infty \le c\|K_\theta(K - K_n)\Psi x_n\|_\infty.$$

Arguing as in the proof of Theorem 3.2, we obtain the desired result. □

Finally we consider a computational problem associated with (4.4). It is customary that the system of nonlinear equations (4.4) is solved by an iterative scheme. For example, the fixed point iteration scheme for (4.4) is to generate $\{c_i^{(k)}\}_{i=1}^n$ for $k \ge 1$ by

$$c_i^{(k+1)} = \sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)\psi(s, f(s) + \sum_{l=1}^n c_l^{(k)}\varphi_l(s))ds, \qquad 1 \le i \le n. \tag{4.10}$$

At each step of iteration, the integrals in (4.10) must be computed since the integrands contain the different values of $c_i^{(k)}$. To circumvent this difficulty. we propose the following device that is originated from [12]. We let

$$z_n(t) = \psi(t, x_n(t)) \tag{4.11}$$

where x_n is defined in (4.3). We have, assuming that k_n takes the form of (3.1),

$$z_n(t) = \psi(t, f(t) + \sum_{i=1}^n a_{ij}\varphi_i(t) \int_a^b \sum_{j=1}^n \varphi_j(s)z_n(s)ds). \tag{4.12}$$

The equation (4.12) can be solved by the collocation-type scheme that was developed by Kumar and Sloan [12]. Namely let $\{\eta_i\}_{i=1}^n$ be n functions defined on $[a, b]$ and let $\{t_j\}_{j=1}^n$ be n distinct points for which

$$\det(\eta_i(t_j)) \ne 0. \tag{4.13}$$

z_n in (4.11) is now approximated in the form $\sum_{j=1}^{n} \alpha_j \eta_j$. The α_j's can be found by solving the following nonlinear equations. Note that the constants α_j's are moved out of the integrals. This makes the repeated computations of the integrals unnecessary when the system of nonlinear equations is to be solved by an iterated scheme.

$$\sum_{j=1}^{n} \alpha_j \eta_j(t_k) = \psi(t_k, f(t_k) + \sum_{i=1}^{n} a_{ij} \varphi_i(t_k) \sum_{l=1}^{n} \alpha_l \int_a^b \sum_{j=1}^{n} \varphi_j(s) \eta_l(s) ds). \tag{4.14}$$

for $1 \leq k \leq n$. If we denote $A \equiv [\eta_j(t_i)]$ and the right side of (4.14) by $\psi_i(\bar{\alpha})$, then with $\bar{\psi}(\bar{\alpha}) \equiv (\psi_i(\bar{\alpha}))$ and $\bar{\alpha}^{(k)} \equiv (\alpha_i^{(k)})$, (4.14) may be solved by the fixed point iteration scheme that can be described as

$$\bar{\alpha}^{(k)} = A^{-1} \bar{\psi}(\bar{\alpha}^{(k-1)}). \tag{4.14}$$

5 Numerical Examples

In this section we present a numerical example using least-squares and interpolation to approximate $k(s,t)$. Let $k(s,t) = e^{st}$ and $f(t) = 1 - \frac{1-e^x}{x}$. Then, the computed errors for the least squares method are shown in the following table.

	Errors	
n	non-iterated	iterated
3	.00622970884	1.57623196E-05
4	.00356820367	4.97616686E-06
5	.00230562003	2.01950876E-06
convergence rate $\approx$	2	4

References

[1] K. E. Atkinson, *A Survey of Numerical Methods for the Solution of fredholm Integral Equations of the Second Kind*,SIAM Philadephia, PA. (1976)

[2] K. E. Atkinson, A survey of numerical methods for solving nonlinear integral equations, *J. Int. Equ. Appl.* 4 (1992), 15-46.

[3] K. E. Atkinson and J. Flores, The discrete collocation method for nonlinear integral equations, Report on Computational Mathematics, No. 10, the University of Iowa, 1991.

[4] K. E. Atkinson and F. Potra, Projection and iterated projection methods for nonlinear integral equations, *SIAM J. Numer. Anal.* 24 (1987), 1352-1373.

[5] E. W. Cheney, *Multivariate Approximation Theory: Selected Topics,* CBMS-NSF 51, Regional deries in Applied Math. SIAM (1986).

[6] I. Graham, S. Joe and I. Sloan, Iterated Galerkin versus iterated collocation for integral equations of the second kind, *IMA J. Numer. Anal.* 5 (1985), 355-369.

[7] H. Kaneko, R. Noren and P. Padilla, Superconvergence of the iterated Collocation Method for Hammerstein equations, (in preparation)

[8] H. Kaneko, R. Noren and Y. Xu, Numerical solutions for weakly singular Hammerstein equations and their superconvergence, *J. Int. Eqs. Appl.* 4 (1992), 391-407.

[9] H. Kaneko, R. Noren and Y. Xu, Regularity of the solution of Hammerstein equations with weakly singular kernels, *Int. Eqs. Op. Thy.* 13 (1990), 660-670.

[10] H. Kaneko and Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, *SIAM Jl. Num. Anal.* (in press).

[11] H. Kaneko and Y. Xu, Degenerate kernel method for Hammerstein equations, *Math. Comp.* 56 (1991), 141-148.

[12] S. Kumar and I. H. Sloan, A New Collocation-type Method for Hammerstein Integral Equations, *Math. Comp.* 48 (1987), 585-593.

[13] L. Schumaker, *Theory of Splines, A Basis Theory,* Wiley

PROJECT 4

HIDEAKI KANEKO, RICHARD D. NOREN and PETER A. PADILLA

Superconvergence of Collocation Method for Hammerstein Equations

The Hammerstein equation arises as a reformulation of a class of boundary value problems with nonlinear boundary conditions. The collocation method is one of the widely used numerical methods to approximate the solution of such equations due to its reasonable computation cost.

We proved that the iterates of the collocation solutions for Hammerstein equation converge faster than the original collocation solution, a phenomenon commonly known as a superconvergence. This result extends the results obtained by Kaneko and Xu (*Superconvergence of the Iterated Galerkin Mathods for Hammerstein Equations* -SIAM Jl. Num. Anal. June 1996 (to appear)) concerning a similar outcome for the Galerkin method. The degree of improvement in the rate of convergence depends upon the smoothness of the kernel involved.

Superconvergence of the Iterated Collocation Methods for Hammerstein Equations

Hideaki Kaneko* **Richard D. Noren**
Department of Mathematics and Statistics
Old Dominion University
Norfolk, Virginia 23529-0077, USA
and
Peter A. Padilla
NASA Langley Research Center
MS-130
Hampton, Virginia 23681-0001, USA

*This author is partially supported by NASA under grant NCC1-213

SUPERCONVERGENCE OF THE ITERATED COLLOCATION METHODS FOR HAMMERSTEIN EQUATIONS

H. Kaneko, R. D. Noren and P. A. Padilla

Abstract

In this paper, we analyse the iterated collocation method for Hammerstein equations with smooth and weakly singular kernels. The paper expands the study which began in [14] concerning the superconvergence of the iterated Galerkin method for Hammerstein equations. We obtain in this paper a similar superconvergence result for the iterated collocation method for Hammerstein equations. We also discuss the discrete collocation method for weakly singular Hammerstein equations. Some discrete collocation methods for Hammerstein equations with smooth kernels were given previously in [3] and [18].

Key words: the iterated collocation method, the discrete collocation method, Hammerstein equations with weakly singular kernels, superconvergence.
Mathematics Subject Classification (1990): 65B05, 45L10.

1 Introduction

In this paper, we investigate the superconvergence property of the iterated collocation method for Hammerstein equations. In the recent paper [14], the superconvergence of the iterated Galerkin method for Hammerstein equations with smooth as well as weakly singular kernels was established. The paper generalizes the previously reported results on the superconvergence of the iterated Galerkin method for the Fredholm integral equations of the second kind [8], [9] [20]. A more important contribution made in [14] lies in the fact that the superconvergence result was established under weaker assumptions (Theorem 3.3 [14]). The approach used in [14] to establish the superconvergence of the iterated Galerkin method can easily be adopted to prove the results of Graham, Joe and Sloan [8], Joe [9] and Sloan [20] under weaker conditions imposed upon the Fredholm equations. This will be demonstrated in Section 3. In Section 2, we review the collocation method for Hammerstein equations as well as some necessary known results that will be pertinent to the matreials in the ensuing sections. We recall that the collocation method for weakly singular Hammerstein equations was discussed and some superconvergence results of the numerical solutions at the collocation points were discovered by Kaneko, Noren and Xu in [11]. In Section 3, the supererconvergence of the iterated collocation method for Hammerstein equations is established. The results obtained there encompass Hammerstein equations with smooth as well

as weakly singular kernels. Finally, in Section 4, we discuss the discrete collocation method for Hammerstein equations with weakly singular kernels. The result obtained in this section extends the results of [3] and [18] which deals with the discrete collocation methods for Hammeratein equations with smooth kernels. Some examples are also included in this section.

We note that there have been several other research papers published in recent years that describe various numerical methods for Hammerstein equations. A variant of Nystöm method was proposed by Lardy [19]. The degenerate kernel method was studied by Kaneko and Xu [16]. We point out that a superconvergence of the iterates of the degenerate kernel method was recently observed when a decomposition of the kernel is done properly. This will be reported in a future paper [17]. The reader who is interested in more information on numerical methods for a wider class of nonlinear integral equations may find necessary materials in [2] and [5].

2 The Collocation Method

In this section, the collocation method for Hammerstein equations is presented. Some materials from the approximation theory are also reviewed in this section to make the present paper self-contained. We consider the following Hammerstein equation

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))ds = f(t), \quad 0 \le t \le 1, \tag{2.1}$$

where k, f and ψ are known functions and x is the function to be determined. Define $k_t(s) \equiv k(t,s)$ for $t,s \in [0,1]$ to be the t section of k. We assume throughout this paper unless stated otherwise, the following conditions on k, f and ψ:

1. $\lim_{t\to\tau} \|k_t - k_\tau\|_\infty = 0, \qquad \tau \in [0,1]$;
2. $M \equiv \sup_t \int_0^1 |k(t,s)|ds < \infty$;
3. $f \in C[0,1]$;
4. $\psi(s,x)$ is continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which

$$|\psi(s,x_1) - \psi(s,x_2)| \le C_1|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty);$$

5. the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

$$|\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \le C_2|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty); \tag{2.2}$$

6. for $x \in C[0,1]$, $\psi(.,x(.)), \psi^{(0,1)}(.,x(.)) \in C[0,1]$.

We let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t,s)\psi(s,x(s))ds.$$

With this notation, equation (2.1) takes the following operator form

$$x - K\Psi x = f. \tag{2.3}$$

For any positive integer n, we let

$$\Pi_n : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$$

be a partition of $[0,1]$. Let r and ν be nonnegative integers satisfying $0 \leq \nu < r$. Let $S_r^\nu(\Pi_n)$ denote the space of splines of order r, continuity ν, with knots at Π_n, that is

$$S_r^\nu(\Pi_n) = \{x \in C^\nu[0,1] : x|_{[t_i,t_{i+1}]} \in \mathcal{P}_{r-1}, \text{ for each } i = 0,1,\ldots,n-1\}$$

where $\mathcal{P}_{r-1}$ denotes the space of polynomials of degree $\leq r-1$. For the collocation method, we are interested in the cases $\nu = 0$ or 1. That is, it is posible to work with the space of piecewise polynomials with no continuity at the knots or with the space of continuous piecewise polynomials with no continuity requirement on the derivatives at the knots. We assume that the sequence of partitions Π_n of $[0,1]$ satisfies the condition that there exists a constant $C > 0$, independent of n, with the property:

$$\frac{\max_{1\leq i\leq n}(t_i - t_{i-1})}{\min_{1\leq i\leq n}(t_i - t_{i-1})} \leq C, \text{ for all } n. \tag{2.4}$$

In many cases, equation (2.1) possesses multiple solutions (see e.g. [16]). Hence, it is assumed for the remainder of this paper that we treat an isolated solution x_0 of (2.1). Let $I_i = (t_{i-1}, t_i)$ for each $i = 1,\ldots,n$. Then for $\nu = 0$, we let $\tau_{i1}, \tau_{i2}, \ldots, \tau_{ir}$ be the Gaussian points (the zeros of the rth degree Legendre polynomial on $[-1,1]$) shifted to the interval I_i. We define

$$G_0 = \{\tau_{ij} : 1 \leq i \leq n, 1 \leq j \leq r\}. \tag{2.5}$$

The points in G_0 give rise to the piecewise collocation method where no continuity between polynomials is assumed. This is the approach taken by Graham, Joe and Sloan [8]. Joe [9], on the other hand, considered the continuous piecewise polynomial collocation method. His method corresponds with taking $\nu = 1$. Here we define the set G_1 of the collocation points to be the set consisting of the knots along with the Labatto points (the zeros of the first derivative of the $r-1$th degree Legendre polynomial) shifted to the interval I_i. Namely, let $\xi_{r-1} = 1$ and for

$1 \le l \le r-2$ $(r \ge 3)$, let ξ_l denotes the lth Labatto point. If $|I_i|$ denotes the length of I_i, then G_1 contains

$$\tau_{(i-1)(r-1)+l+1} = \frac{1}{2}(t_{i-1} + t_i + |I_i|\xi_l), \qquad 1 \le i \le n, 1 \le l \le r-1, \text{ and } \tau_1 = t_0 = 0 \tag{2.6}$$

The analyses of [8] and [9] are very similar. We therefore confine ourselves to developing the collocation method for Hammerstein equations that is analogous to the method of [8]. An obvious extension to the continuous piecewise collocation method will be left to the reader. Define the interpolatory projection P_n from $C[0,1] + S_r^\nu(\Pi_n)$ to $S_r^\nu(\Pi_n)$ by requiring that, for $x \in C[0,1] + S_r^\nu(\Pi_n)$,

$$P_n x(\tau_{ij}) = x(\tau_{ij}), \qquad \text{for all } \tau_{ij} \in G_0. \tag{2.7}$$

Then we have, for $x \in C[0,1] + S_r^\nu(\Pi_n)$

$$P_n x \to x, \qquad \text{as } n \to \infty \tag{2.8a}$$

and consequently

$$\sup_n \|P_n\| < c. \tag{2.8b}$$

The collocation equation corresponding to (2.3) can be written as

$$x_n - P_n \Psi x_n = P_n f \tag{2.9}$$

where $x_n \in S_r^\nu(\Pi_n)$. Now we let

$$\hat{T}x \equiv f + K\Psi x$$

and

$$T_n x_n \equiv P_n f + P_n K \Psi x_n$$

so that equations (2.3) and (2.9) can be written respectively as $x = \hat{T}x$ and $x_n = T_n x_n$. We obtain;

Theorem 2.1 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. Then the collocation approximation equation (2.9) has a unique solution $x_n \in B(x_0, \delta)$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\alpha_n}{1-q}, \tag{2.10}$$

where $\alpha_n \equiv \|(I - T_n'(x_0))^{-1}(T_n(x_0) - \hat{T}(x_0))\|_\infty$. Finally,

$$E_n(x_0) \le \|x_n - x_0\|_\infty \le C E_n(x_0), \tag{2.11}$$

where C is a constant independent of n and $E_n(x_0) = \inf_{u \in X_n} \|x_0 - u\|_\infty$.

A proof is a straight application of Theorem 2 of Vainikko [23] and is demonstrated in the proof of Theorem 2.1 [11]. We denote by $W_p^m[0,1]$, $1 \le p \le \infty$, the Sobolev space of functions g whose m-th generalized derivative $g^{(m)}$ belongs to $L_p[0,1]$. The space $W_p^m[0,1]$ is equipped with the norm

$$\|g\|_{W_p^m} \equiv \sum_{k=0}^{m} \|g^{(k)}\|_p.$$

It is known from Demko [6] and De Vore [7] that if $0 \le \nu < r$, $1 \le p \le \infty$, $m \ge 0$ and $x \in W_p^m$, then for each $n \ge 1$, there exists $u_n \in S_r^\nu(\Pi_n)$ such that

$$\|x - u_n\|_p \le Ch^\mu \|x\|_{W_p^\mu}, \tag{2.12}$$

where $\mu = \min\{m, r\}$ and $h = \max_{1 \le i \le n}(t_i - t_{i-1})$. The inequality (2.12) when combined with Theorem 2.1 yields the following theorem;

Theorem 2.2 *Let x_0 be an isolated solution of equation (2.3) and let x_n be the solution of equation (2.9) in a neighborhood of x_0. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. If $x_0 \in W_\infty^l$, then*

$$\|x_0 - x_n\|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W_p^l$ $(1 \le p < \infty)$, then

$$\|x_0 - x_n\|_\infty = O(h^\nu),$$

where $\nu = \min\{l-1, r\}$.

When the kernel k is of weakly singular type, namely if

$$k(t,s) = m(t,s)g_\alpha(|t-s|), \tag{2.13}$$

where $m \in C^{\mu+1}([0,1] \times [0,1])$ and

$$g_\alpha(s) = \begin{cases} s^{\alpha-1}, & 0 < \alpha < 1, \\ \log s, & \alpha = 1. \end{cases} \tag{2.14}$$

then the solution x_0 of equation (2.3) does not, in general, belong to W_p^m. To better characterize the regularity of the soution of (2.3) with weakly singular kernel, consider a finite set S in $[0,1]$ and define the function $\omega_S(t) = \inf\{|t-s| : s \in S\}$. A function x is said to be of $Type(\alpha, k, S)$, for $-1 < \alpha < 0$, if

$$|x^{(k)}(t)| \le C[\omega_S(t)]^{\alpha-k} \quad t \notin S,$$

and for $\alpha > 0$, if the above condition holds and $x \in Lip(\alpha)$. Here $Lip(\alpha) = \{x: |x(t) - x(s)| \le C|t-s|^\alpha\}$. It was proved by Kaneko, Noren and Xu [12] that if f is of $Type(\beta, \mu, \{0,1\})$, then

a solution of equation (1.1) is of $Type(\gamma, \mu, \{0,1\})$, where $\gamma = \min\{\alpha, \beta\}$. The optimal rate of convergence of the collocation solution x_n to x_0 can be recovered by selecting the knots that are defined by

$$\begin{aligned} t_i &= (1/2)(2i/n)^q, \quad 0 \le i \le n/2, \\ t_i &= 1 - t_{n-i}, \qquad n/2 < i \le n, \end{aligned} \tag{2.15}$$

where $q = r/\gamma$ denotes the index of singularity. Details can be found in [11].

3 The Iterated Collocation Method

The faster convergence of the iterated Galerkin method for the Fredholm integral equations of the second kind compared to the Galerkin method was first observed by Sloan in [21] and [22]. On the other hand, the superconvergence of the iterated collocation method was studied in [8] and [9]. Given the equation of the second kind

$$x - Kx = f, \tag{3.1}$$

where K is a compact operator on $X \equiv C[0,1]$ and $x, f \in X$, the collocation approximation x_n is the solution of the following projection equation

$$x_n - P_n K x_n = P_n f. \tag{3.2}$$

Here P_n is the interpolatory projection of (2.7). The iterated collocation method obtains a solution x_n' by

$$x_n' = f + Kx_n. \tag{3.3}$$

Under the assumption of

$$\|KP_n - K\| \to 0 \qquad \text{as } n \to \infty \tag{3.4}$$

it can be shown that

$$\|x - x_n'\| \le \|(I - KP_n)^{-1}\| \|K(x - P_n x)\|. \tag{3.5}$$

The assumption (3.4) is satisfied if $X = L_2$ and P_n is the orthogonal projection satisfying $\|P_n g - g\| \to 0$ for all g in the closure of the range of the adjoint K^* of K since in this case $\|KP_n - K\| = \|P_n K^* - K^*\|$. The results of Sloan were recently generalized to the iterated Galerkin method for Hammerstein equations by Kaneko and Xu [14]. The main theorem of [14], Theorem 3.3, that guarantees the superconvergence of the iterates was proved by making use of the collectively compact operator theory.

The purpose of this section is to study the superconvergence of the iterated collocation method. For the collocation solution x_n of (2.9), we define

$$x_n' = f + K\Psi x_n. \tag{3.6}$$

A standard argument shows that x_n' satisfies

$$x_n' = f + K\Psi P_n x_n'. \tag{3.7a}$$

We denote the right side of (3.7a) by $S_n x_n'$, namely

$$S_n x_n' \equiv f + K\Psi P_n x_n'. \tag{3.7b}$$

We recall the following two lemmas from [14].

Lemma 3.1 *Let $x_0 \in C[0,1]$ be an isolated solution of (2.3). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently large n, the operators $I - S_n'(x_0)$ are invertible and there exists a constant $L > 0$ such that*

$$\|(I - S_n'(x_0))^{-1}\|_\infty \leq L, \text{ for sufficeintly large } n.$$

Lemma 3.2 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3) and x_n be the unique solution of (2.9) in the sphere $B(x_0, \delta_1)$. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then for sufficiently large n, x_n' defined by the iterated scheme (3.6) is the unique solution of (3.7) in the sphere $B(x_0, \delta)$. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\beta_n}{1+q} \leq \|x_n' - x_0\|_\infty \leq \frac{\beta_n}{1-q},$$

where $\beta_n = \|(I - S_n'(x_0))^{-1}[S_n(x_0) - \hat{T}(x_0)]\|_\infty$. Finally,

$$\|x_n' - x_0\|_\infty \leq C E_n(x_0).$$

The definitions of δ and δ_1 are described in [14]. Following the development made in [14], we let

$$\psi(s,y) = \psi(s,y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))(y - y_0), \tag{3.8}$$

where $\theta := \theta(s, y_0, y)$ with $0 < \theta < 1$. Also let

$$g(t,s,y_0,y,\theta) = k(t,s)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)),$$

$$(G_n x)(t) = \int_0^1 g(t,s,P_n x_0(s), P_n x_n'(s), \theta) x(s) ds,$$

and $(Gx)(t) = \int_0^1 g_t(s)x(s)ds$, where $g_t(s) = k(t,s)\psi^{(0,1)}(s, x_0(s))$. Now we are ready to state and prove our main theorem of this paper. The proof is a combination of the idea used in [14] (Theorem 3.3) and the one used in [8] (Theorem 4.2).

Theorem 3.3 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3) and x_n be the unique solution of (2.9) in the sphere $B(x_0, \delta_1)$. Let x_n' be defined by the iterated scheme (3.7). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Assume that $x_0 \in W_1^l$ $(0 < l \leq 2r)$ and $g_t \in W_1^m$ $(0 < m \leq r)$ with $\|g_t\|_{W_1^m}$ bounded independently of t. Then*

$$\|x_0 - x_n'\|_\infty = O(h^\gamma), \qquad \text{where } \gamma = min\{l, r+m\}.$$

Proof: From equations (2.3) and (3.7), we obtain

$$x_0 - x_n' = K(\Psi x_0 - \Psi P_n x_n') = K(\Psi x_0 - \Psi P_n x_0) + K(\Psi P_n x_0 - \Psi P_n x_n'). \tag{3.9}$$

Using (3.8), the last term of (3.9) can be written as

$$K(\Psi P_n x_0 - \Psi P_n x_n')(t) = (G_n P_n(x_0 - x_n'))(t).$$

Equation (3.9) then becomes

$$x_0 - x_n' = K(\Psi x_0 - \Psi P_n x_0) + G_n P_n(x_0 - x_n'). \tag{3.10}$$

Using the Lipschitz condition (2.2) imposed on $\psi^{(0,1)}$, for $x \in C[0,1]$,

$$\|(G_n x) - (Gx)\|_\infty \leq C_2 \sup_{0\leq t\leq 1} \int_0^1 |k(t,s)|ds \|x\|_\infty (\|P_n x_0 - x_0\|_\infty + \|P_n\|_\infty \|x_n' - x_0\|_\infty).$$

This shows that

$$\|G_n - G\|_\infty \leq MC_2(\|P_n x_0 - x_0\|_\infty + c\|x_n' - x_0\|_\infty) \to 0 \quad \text{as} \quad n \to \infty.$$

Also, for each $x \in C[0,1]$,

$$\sup_{0\leq t\leq 1} |(GP_n x)(t) - (Gx)(t)| = \sup_{0\leq t\leq 1} |\int_0^1 g_t(s)[P_n x(s) - x(s)]ds| \leq MM_1 \|P_n x - x\|_\infty,$$

where

$$M_1 = \sup_{0\leq t\leq 1} |\psi^{(0,1)}(t, x_0(t))| < +\infty.$$

It follows that $GP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. Again since P_n is uniformly bounded, we have for each $x \in C[0,1]$,

$$\|G_n P_n x - Gx\|_\infty \leq \|G_n - G\|_\infty \|P_n\|_\infty \|x\|_\infty + \|GP_n x - Gx\|_\infty.$$

Thus, $G_n P_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. By Assumptions 2, 5, and 6, we see that there exists a constant $C > 0$ such that for all n

$$|\psi^{(0,1)}(s, P_n x_0(s) + \theta(P_n x_n'(s) - P_n x_0(s)))| \leq C_2 \|P_n x_0 - x_0\|_\infty + \theta C_2 P \|x_n' - x_0\|_\infty + M_1 \leq C.$$

This implies that $\{G_nP_n\}$ is a family of collectively compact operators. Since $G = (K\Psi)'(x_0)$ is compact and $(I-G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I-G_nP_n)^{-1}$ exists and is uniformly bounded for sufficiently large n. Now using (3.10), we see that

$$\|x_0 - x_n'\|_\infty \leq C\|K(\Psi x_0 - \Psi P_n x_0)\|.$$

Hence we need to estimate $\|K(\Psi x_0 - \Psi P_n x_0)\|$. The following four inequalities are known (Theorem 4.2 [8]). Let $\psi_n \in S_l^0(\Pi_n)$ be such that

$$\sum_{i=1}^{n} \|(x_0 - \psi_n)^{(j)}\|_{W_1^m(I_i)} \leq ch^{l-j}\|x_0\|_{W_1^l}, \qquad 0 \leq j \leq l, \tag{3.11}$$

$$\max_{1\leq i\leq n} \|\psi_n^{(j)}\|_{W_\infty^m(I_i)} \leq c\|x_0\|_{W_1^l}, \qquad j \geq 0. \tag{3.12}$$

Also for each $t \in [0,1]$, there exists $\varphi_{n,t} \in S_m^0(\Pi_n)$ such that

$$\sum_{i=1}^{n} \|(g_t - \varphi_{n,t})^{(j)}\|_{W_1^m(I_i)} \leq ch^{m-j}K_m, \qquad 0 \leq j \leq m, \tag{3.13}$$

$$\max_{1\leq i\leq n} \|\varphi_{n,t}^{(j)}\|_{W_\infty^m(I_i)} \leq cK_m, \qquad j \geq 0, \tag{3.14}$$

where $K_m = \sup_{0\leq t\leq 1}\|k_t\|_{W_1^m} < \infty$. Now for $t \in [0,1]$ we have

$$\begin{aligned} K(\Psi x_0 - \Psi P_n x_0)(t) \;&= (g_t - \varphi_{n,t}, x_0 - P_n x_0) + (\varphi_{n,t}, (I-P_n)(x_0 - \psi_n)) \\ &\quad + (\varphi_{n,t}, (I-P_n)\psi_n). \end{aligned} \tag{3.15}$$

Using equations (3.11)-(3.14) along with the arguments from [8] (p.362) we can show that each of the three terms is bounded by ch^γ uniformly in t. This completes our proof. □

One way to establish the superconvergence of the iterated collocation method for the Fredholm equation is to assume (3.4). In the context of the present discussion, (3.4) is equivalent to assuming

$$\|(K\Psi)'(x_0)(I-P_n)|_{C[a,b]}\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{3.16}$$

Theorem 3.3 was thus proved under weaker assumptions. The idea used to prove Theorem 3.3 originates from [4] (section 6) in which the superconvergence of the iterated collocation method for the Fredholm equations was established by showing that $\{KP_n\}$ is a family of collectively compact operators.

Finally in this section, we investigate the superconvergence of the iterated collocation method for weakly singular Hammerstein equation. Specifically, we consider equation (2.3) with kernel given by (2.13) and (2.14). An enhancement in the rate of convergence is given in the following theorem.

Theorem 3.4 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3) and x_n be the unique solution of (2.9) in the sphere $B(x_0, \delta_1)$ with kernel defined by (2.13) and (2.14) and knots defined by (2.15). Let x_n' be defined by the iterated scheme (3.7). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$ and that $\psi^{(0,1)}(\cdot, x_0(\cdot))$ is of Type$(\alpha, r, \{0,1\})$ for $\alpha > 0$ whenever x_0 is of the same type. Then*

$$\|x_0 - x_n'\|_\infty = O(h^{r+\alpha}).$$

Proof: We follow the proof of Theorem 3.3 exactly the same way to (3.15), which is

$$\begin{aligned} K(\Psi x_0 - \Psi P_n x_0)(t) &= (g_t - \varphi_{n,t}, x_0 - P_n x_0) + (\varphi_{n,t}, (I - P_n)(x_0 - \psi_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n)\psi_n). \end{aligned}$$

The difference in superconvergence arises from the degree to which we may bound the first term. As in Kaneko and Xu [14] (Theorem 3.6), using an argument similar to [15], it can be proved that there exists $u \in S_r^\nu(\Pi_n)$ with knots Π_n given by (2.15) such that $\|g_t - u\|_1 = O(h^\alpha)$. Here $h = \max_{1 \le i \le n}\{x_i - x_{i-1}\}$. Then

$$\begin{aligned} |(g_t - \varphi_{n,t}, x_0 - P_n x_0)| &\le \|g_t - \varphi_{n,t}\|_1 \|x_0 - P_n x_0\|_\infty \\ &= O(h^{\alpha+r}). \end{aligned}$$

The rest of proof follows in the same way as described in [8] (p.362).□

4 The Discrete Collocation Method for Weakly Singular Hammerstein Equations

Several papers have been written on the subject of the discrete collocation method. Joe [10] gave an analysis of discrete collocation method for second kind Fredholm integral equations. A discrete collocation-type method for Hammerstein equations was described by Kumar in [18]. Most recently Atkinson and Flores [3] put together the general analysis of the discrete collocation methods for nonlinear integral equations. In this section, we describe a discrete collocation method for weakly singular Hammerstein equations. In the aforementioned papers [10, 18, 3], their discussions are primarily concerned with integral equations with smooth kernels. Even though, in principle, an analysis for the discrete collocation method for weakly singular Hammerstein equations is similar to the one given in [3], we feel that a detailed discussion on some specific points pertinent to weakly singular equations, -e.g.,a selection of a particular quadrature scheme and a convergence analysis etc, will be of great interest to practioners. Our convergence analysis of the discrete collocation method presented in this section is different from the one given

in [3] in that it is based upon theorem 2 of Vainikko [23]. The idea of the quadrature used here was recently developed by Kaneko and Xu [15] and a complete Fortran program based on the idea is being developed by Kaneko and Padilla [13]. A particular case of the quadrature schemes developed in [14] is concerned with an approximation of the integral

$$I(f) = \int_0^1 f(x)dx, \tag{4.1}$$

where $f \in Type(\alpha, 2r, S)$ with $\alpha > -1$. For simplicity of demonstration, we assume $S = \{0\}$. We define $q = \frac{2r+1}{\alpha+1}$ and a partition

$$\pi_\alpha: x_0 = 0, x_1 = n^{-q}, x_j = j^q t_1, \qquad j = 2, 3, \ldots, n. \tag{4.2}$$

Now we construct a piecewise polynomial S_r of degree $r-1$ by the following rule; $S_r(x) = 0$, $x \in [x_0, x_1)$ and $S_r(x)$ is the Lagrange polynomial of degree $r-1$ interpolating f at $\{u_j^{(i)}\}_{j=1}^r$ for $x \in [x_i, x_{i+1})$, $i = 1, 2, \ldots, n-2$ and for $x \in [x_{n-1}, x_n]$. Here $\{u_j^{(i)}\}_{j=1}^r$ denote the zeros of the rth degree Legendre polynomial transformed into $[x_i, x_{i+1})$. Our approximation process consists of two stages. First, $I(f)$ is approximated by

$$\hat{I}(f) = \int_{x_1}^1 f(x)dx = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} f(x)dx. \tag{4.3}$$

Second, $\hat{I}(f)$ is approximated by $\hat{I}(S_r) = \int_{x_1}^1 S_r(x)dx$. A computation of $\hat{I}(S_r)$ can be accomplished as follows; let $s: [x_i, x_{i+1}] \to [-1, 1]$ be defined by $s = \frac{2x-(x_{i+1}+x_i)}{x_{i+1}-x_i}$ so that

$$\hat{I}(f) = \int_{-1}^1 F_f(s)ds \tag{4.4}$$

where

$$F_f(s) = \sum_{i=1}^{n-1} \frac{1}{2}(x_{i+1} - x_i) f(\frac{1}{2}(x_{i+1} - x_i)s + \frac{1}{2}(x_{i+1} + x_i)).$$

If $\{u_i: i = 1, 2 \ldots, r\}$ denotes the zeros of the Legendre polynomial of degree r, then

$$S_r(x) = \sum_{i=1}^r F_f(u_i) l_i(x)$$

with $l_i(x)$ the fundamental Lagrange polynomial of degree $r-1$ so that

$$\hat{I}(S_r) = \sum_{i=1}^r w_i F_f(u_i), \qquad \text{where } w_i = \int_{-1}^1 l_i(x)dx. \tag{4.5}$$

It was proved in [15] that

$$|I(f) - \hat{I}(S_r)| = O(n^{-2r}). \tag{4.6}$$

In this section, we examine equation (2.1) with the kernel k defined by (2.13) and (2.14). When the knots are selected according to (2.15), as stated earlier, it was shown in [11] that the solution x_n of the collocation equation (2.9) converges to the solution x of (2.1) in the rate that is optimal to the degree of polynomials used. Specifically, x_n must be found by solving

$$x_n(u_j^{(i)}) - \int_0^1 g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)}, s)\psi(s, x_n(s))ds = f(u_j^{(i)}) \tag{4.7}$$

where $i = 0, 1, \ldots n-1$ and $j = 1, 2, \ldots r$.

The discrete collocation method for equation (2.1) is obtained when the integral in (4.7) is replaced by a numerical quadrature given in (4.5). Let $k_{ij}(s) \equiv g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)}, s)$. Then

$$\begin{aligned} \int_0^1 g_\alpha(|u_j^{(i)} - s|)m(u_j^{(i)}, s)\psi(s, x_n(s))ds &= \int_0^1 k_{ij}(s)\psi(s, x_n(s))ds \\ &= \int_0^{u_j^{(i)}} + \int_{u_j^{(i)}}^1 k_{ij}(s)\psi(s, x_n(s))ds. \end{aligned} \tag{4.8}$$

The integrals in the last expression of (4.8) represent two weakly singular integrals which can be approximated to within $O(n^{-2r})$ order of accuracy by (4.5) by transforming them to $[-1, 1]$ and selecting the points in (4.2) appropriately.

Writing (4.7) as

$$P_n x_n - P_n K \Psi x_n = P_n f, \tag{4.9}$$

we consider the approximation $\tilde{x}_n$ to x_n defined as the solution of

$$\tilde{x}_n = Q_n \tilde{x}_n \equiv P_n K_n \Psi \tilde{x}_n + P_n f, \tag{4.10}$$

where K_n is the discrete collocation approximation to the integrals in (4.8) described above.

We will use Theorem 2 of [23] to find a unique solution to (4.10) in some δ neighborhood of x_n, where n is sufficiently large. Clearly, $Q_n'(x) = P_n K_n \Psi'(x)$, where $\Psi'(x)[y](s) = \psi^{0,1)}(s, x(s))y(s)$. For sufficiently large n, (4.9) has a unique solution in some δ neighborhood of x. To see that $I - Q_n'(x_n)$ is continuously invertible with $\{(I - Q_n'(x_n))^{-1}\}_{n=N}^\infty$ uniformly bounded, it is enough to observe that $\{Q_n'(x_n)\}_{n=1}^\infty$ is collectively compact, and to do this we will show that

$$| Q_n'(x_n)[x](t) - Q_n'(x_n)[x](t') | = | P_n K_n \Psi'(x_n)x(t) - P_n K_n \Psi'(x_n)x(t') | \to 0 \tag{4.11}$$

as $t \to t'$, for each $x \in C[0, 1]$, [1]. Here N is some sufficiently large number.

If we show (4.11), then part (a) of Theorem 2 [23] is also verified. In order to verify part (b) of Theorem 2 [23], we only need to establish (because of the uniform boundedness of $\{(I - Q_n(x_n))^{-1}\}_{n=N}^\infty$) that

$$\| Q_n'(x) - Q_n'(x_n) \|_\infty \leq L \| x - x_n \|_\infty \leq L\delta, \tag{4.12}$$

for some constant L, and

$$\| Q_n(x_n) - T_n(x_n) \| \to 0 \text{ as } n \to \infty. \tag{4.13}$$

Once this is done, Theorem 2 [23] applies yielding a unique solution $\tilde{x}_n$ in some neighborhood of x_n (for sufficiently large n) and

$$\| x_n - \tilde{x}_n \| \leq L\tilde{\alpha}_n \leq L \| Q_n(x_n) - T_n(x_n) \|_\infty . \tag{4.14}$$

(Here and throughout the remainder of the section, L denotes a generic constant, the exact value of which may differ at each occurance.) This inequality will be used to obtain the order of convergence.

Considering (4.11), the right hand side is bounded by $T_1 + T_2 + T_3$, where

$$\begin{aligned} T_1 &= | P_n K_n \Psi'(x_n)x(t) - P_n K \Psi'(x_n)x(t) |, \\ T_2 &= | P_n K \Psi'(x_n)x(t) - P_n K \Psi'(x_n)x(t') |, \\ T_3 &= | P_n K_n \Psi'(x_n)x(t') - P_n K_n \Psi'(x_n)x(t') | . \end{aligned}$$

Let $\epsilon > 0$. Since $\{P_n\}_{n=1}^{\infty}$ is uniformly bounded, $T_1 + T_3 < \frac{2\epsilon}{3}$ by applying (4.6) with $f(s) = \psi^{(0,1)}(s, x_n(s))x(s)$ and letting n be sufficiently large. For T_2 we have
$T_2 \leq M \int_0^1 | k(t,s) - k(t',s) | ds \leq M(S_1 + S_2)$,
where

$$S_1 = \int_0^1 g_\alpha(| s - t |) | m(t,s) - m(t',s) | ds$$

and

$$S_2 = \int_0^1 | g_\alpha(| t - s |) - g_\alpha(| t' - s |) || m(t',s) | ds.$$

but

$$\begin{aligned} S_1 &\leq \sup_{0 \leq s \leq 1} | m(t,s) - m(t',s) | \int_0^1 g_\alpha(| t - s |) ds \\ &\leq L \sup_{0 \leq s \leq 1} | m(t,s) - m(t',s) | \to 0 \text{ as } t \to t', \end{aligned}$$

and

$$\begin{aligned} S_2 &\leq L \int_0^1 | g_\alpha(| t - s |) - g_\alpha(| t' - s |) | ds \\ &= \tfrac{L}{\alpha} \{ | t^\alpha - (t')^\alpha | + | (1-t)^\alpha - (1-t')^\alpha | + \tfrac{4}{2^\alpha} | t - t' |^\alpha \} \\ &\to 0 \text{ as } t \to t'. \end{aligned}$$

Hence (4.11) holds. For (4.12),

$$\| Q_n'(x) - Q_n'(x_n) \|_\infty = \| P_n K_n (\Psi'(x) - \Psi'(x_n)) \| \leq MC \| x - x_n \| \leq M\delta = q < 1$$

for δ sufficiently small. Note that we have used the uniform boundedness of $\{P_n\}, \{K_n\}$ and because $\Psi^{(0,1)}(s, y(s))$ is locally Lipschitz, so is the operator
$\Psi' : C[0,1] \to B(C[0,1], C[0,1])$ (the space of bounded linear operators from $C[0,1]$ into $C[0,1]$).

For (4.13), we have

$$\| Q_n(x_n) - T_n(x_n) \|_\infty = \| P_n(K_n \Psi x_n - K \Psi x_n) \| \leq L \| (K_n - K)\Psi(x_n) \leq L(R_1 + R_2 + R_3) \quad (4.15)$$

where

$$R_1 = \| K_n \Psi(x_n) - K_n \Psi(x_0) \|, \; R_2 = \| K_n \Psi(x_0) - K \Psi(x_0) \|, \; R_3 = \| K \Psi(x_0) - K \Psi(x_n) \| . \quad (4.16)$$

But

$$R_1 \leq L \| \Psi(x_n) - \Psi(x_0) \| \leq C_1 L \| x_n - x_0 \| \quad (4.17)$$

because Ψ is a Lipschitz operator and $\{K_n\}$ is uniformly bounded, and also

$$R_3 \leq M \| \Psi(x_0) - \Psi(x_n) \| \leq C_1 M \| x_n - x_0 \| . \quad (4.18)$$

Finally,

$$R_2 = O(n^{-2r}) \quad (4.19)$$

by (4.6) using $f(s) = \Psi(x, x_0(s))$.

Thus Vainniikko's Theorem yields a unique solution $\tilde{x}_n$ for n sufficiently large and (4.14) holds. Now (4.14) and (4.15) - (4.19) show that

$$\| x_n - \tilde{x}_n \| = O(n^{-\beta}) \quad (4.20)$$

where β is the minimum of $2r$ and the order of convergence of $\| x_0 - x_n \|$. We summerize the results obtained above in the following theorem:

Theorem 4.1 *Let x_0 be an isolated solution of equation (2.3) and let x_n be the solution of equation (2.9) in a neighborhood of x_0. Moreover, let $\tilde{x}_n$ be the solution of (4.10). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. If $x_0 \in W_\infty^l$, then*

$$\|x_0 - \tilde{x}_n\|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W_p^l$ $(1 \leq p < \infty)$, then

$$\|x_0 - \tilde{x}_n\|_\infty = O(h^\nu),$$

where $\nu = \min\{l - 1, r\}$.

References

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[2] K. E. Atkinson, A survey of numerical methods for solving nonlinear integral equations, *J. Int. Equ. Appl.* 4 (1992), 15-46.

[3] K. E. Atkinson and J. Flores, The discrete collocation method for nonlinear integral equations, Report on Computational Mathematics, No. 10, the University of Iowa, 1991.

[4] K. E. Atkinson, I. Graham and I. Sloan, Piecewise continuous collocation for integral equations, *SIAM J. Numer. Anal.* 20 (1983), 172-186.

[5] K. E. Atkinson and F. Potra, Projection and iterated projection methods for nonlinear integral equations, *SIAM J. Numer. Anal.* 24 (1987), 1352-1373.

[6] S. Demko, Splines approximation in Banach function spaces, in *Theory of Approximation with Applications* (Law, A. G. & Sahney, B. N., Eds), pp. 146-154, New York, Academic Press, 1976.

[7] R. A. De Vore, Degree of approximation, in *Approximation Theory II* (Lorentz, G. G., Chui, C. K. & Schumaker, L. L., Eds) pp. 117-161, New York, Academic Press, 1976.

[8] I. Graham, S. Joe and I. Sloan, Iterated Galerkin versus iterated collocation for integral equations of the second kind, *IMA J. Numer. Anal.* 5 (1985), 355-369.

[9] S. Joe, Collocation methods using piecewise polynomials for second kind integral equations, *Jl. of Comp. and Appl. Math.* 12&13 (1985), 391-400.

[10] S. Joe, Discrete collocation methods for second kind Fredholm integral equations, *SIAM Jl. Numer. Anal.*22 (1985), 1167-1177.

[11] H. Kaneko, R. Noren and Y. Xu, Numerical solutions for weakly singular Hammerstein equations and their superconvergence, *J. Int. Eqs. Appl.* 4 (1992), 391-407.

[12] H. Kaneko, R. Noren and Y. Xu, Regularity of the solution of Hammerstein equations with weakly singular kernels, *Int. Eqs. Op. Thy.* 13 (1990), 660-670.

[13] H. Kaneko and P. Padilla, Numerical Quadratures for Weakly Singular Integrals, *Technical Report NASA Langley Research Center*

[14] H. Kaneko and Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, *SIAM Jl. Num. Anal.* (in press).

[15] H. Kaneko and Y. Xu, Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of the second kind, *Math. Comp.* , 62 (1994), 739-753.

[16] H. Kaneko and Y. Xu, Degenerate kernel method for Hammerstein equations, *Math. Comp.* 56 (1991), 141-148.

[17] H. Kaneko, P. A. Padilla and Y. Xu, The iterated operator approximation method and its application to superconvergence of degenerate kernel method, (in preparation).

[18] S. Kumar, A discrete collocation-type method for Hammerstein equation, *SIAM J. Numer. Anal.* 25 (1988), 328-341.

[19] L. J. Lardy, A variation of Nystrom's method for Hammerstein equations, *J. Int. Eqs.* 3 (1981), 43-60.

[20] I. H. Sloan, Four variants of the Galerkin methods for integral equations of the second kind, *IMA J. Numer. Anal.* 4 (1984), 9-17.

[21] I. H. Sloan, Error analysis for a class of degenerate-kernel methods, *Num. Math.* 25 (1976), 231-238.

[22] I. H. Sloan, Improvement by iteration for compact operator equations, *Math. Comp.*30 (1976), 758-764.

[23] G. Vainikko, Perturbed Galerkin method and general theory of approximate methods for nonlinear equations, *Zh. Vychisl. Mat. Fiz.* 7 (1967), 723-751. Engl. Translation, *U.S.S.R. Comp. Math. and Math. Phys.* 7, No. 4 (1967), 1-41.

PROJECT 5

HIDEAKI KANEKO, RICHARD D. NOREN and PETER A. PADILLA

Singularity Preserving Galerkin Method for Hammerstein Equations With Logarithmic Kernel

In the recent paper *Singularity preserving Galerkin methods for weakly singular Fredholm integral equations, Jl. Int. Eqs. and Appl. 6 (1994) 303-334*, Y. Cao and Y. Xu established the Galerkin method for weakly singular Fredholm integral equations that preserves the singularity of the solution. Their Galerkin method provides a numerical solution that is a linear combination of a certain class of basis functions which includes elements that reflect the singularity of the solution. The purpose of this paper is to extend the result of Cao and Xu and to establish singularity preserving Galerkin method for Hammerstein equations with logarithmic kernel. First, a singularity expansion for the soultion of Hammerstein equation with logarithmic kernel is given. Secondly, this singularity expansion is used to obtain the numerical Galerkin scheme that preserves the singularity of the solution.

An application is given to a Dirichlet problem with a certain class of nonlinear boundary conditions. Numerical experiments are being performed.

Singularity Preserving Galerkin Method For Hammerstein Equations With Logarithmic Kernel

Hideaki Kaneko* **Richard D. Noren**
Department of Mathematics and Statistics
Old Dominion University
Norfolk, Virginia 23529-0077, USA
and
Peter A. Padilla
NASA Langley Research Center
MS-130
Hampton, Virginia 23681-0001, USA

*This author is partially supported by NASA under grant NCC1-213

SINGULARITY PRESERVING GALERKIN METHOD FOR HAMMERSTEIN EQUATIONS WITH LOGARITHMIC KERNEL

H. Kaneko, R. D. Noren and P. A. Padilla

Abstract

In a recent paper [3], Y. Cao and Y. Xu established the Galerkin method for weakly singular Fredholm integral equations that preserves the singularity of the solution. Their Galerkin method provides a numerical solution that is a linear combination of a certain class of basis functions which includes elements that reflect the singularity of the solution. The purpose of this paper is to extend the result of Cao and Xu and to establish singularity preserving Galerkin method for Hammerstein equations with logarithmic kernel. An application is given to a Dirichlet problem with nonlinear boundary condition.

Key words: Singularity preserving Galerkin method, Hammertein equations with logarithmic kernel.

Mathematics Subject Classification (1990): 65B05, 45L10.

1 Introduction

In this paper, we are concerned with the problem of obtaining a numerical solution of weakly singular Hammerstein equations with logarithmic kernel by the Galerkin method that preserves the singularity of the exact solution. Namely we establish a method that generates an approximate solution in terms of a collection of basis functions some of which are comprised of singular elements that reflect the characteristics of the singularity of the exact solution. The idea of the method originates in the recent paper by Cao and Xu [3]. Cao and Xu studied the characteristics of the singularities that are pertinent to solutions of the weakly singular Fredholm equations of the second kind. Let $C[0,1]$ denote the space of all continuous functions defined on $[0,1]$. The weakly singular Fredholm integral equations of the second kind can be described as

$$y(s) - \int_0^1 g_\alpha(|s-t|)m(s,t)y(t)dt = f(s), \qquad 0 \le s \le 1 \tag{1.1}$$

where $f \in C[0,1]$, m is sufficiently smooth and

$$g_\alpha(|s-t|) = \begin{cases} |s-t|^\alpha, & -1 < \alpha < 0, \\ \log|s-t|, & \alpha = 0, \end{cases} \tag{1.2}$$

where y is of course the function to be determined. It is well documented (see, e.g. [16],[13],[4],[20]) that the solutions of the equations described in (1.1) exhibit, in general, mild singularities even

in the case of a smooth forcing term f. Here by "mild" singularities, we mean the singularities in derivatives. The papers of Richter [13] and Graham [4] contain singularity expansions of the solutions of equation (1.1) in the case of $m(s,t) \equiv 1$. The results of Graham were recently generalized by Cao and Xu for equation (1.1). Information concerning the type of singularities that solutions have is useful when solving equation (1.1) numerically. In order to approximate functions with mild singularities, many investigators utilized the important theorem of Rice [14] that gives an optimal order of approximation to such functions. Rice's theorem is that of nonlinear approximation by splines using variable knots. Based upon this idea of approximating the solutions of equation (1.1) by splines defined on nonuniform knots, the collocation method, the Galerkin method and the product-integration method were established for equation (1.1) by Vainikko and Uba [21], by Graham [4] and by Schneider [17] respectively. A modified collocation method was introduced in [12] which also uses the idea of Rice. Recently there has been some considerable interest in the study of the following weakly singular Hammerstein equation:

$$y(s) - \int_0^1 g_\alpha(|s-t|)m(s,t)\psi(t,y(t))dt = f(s), \qquad 0 \le s \le 1 \tag{1.3}$$

where f, m and g_α are defined as in (1.1) and (1.2) and ψ is a known function. We will see in Section 4 that equation similar to (1.3) arises naturally in connection with Dirichlet problem with certain nonlinear boundary conditions. A study on the regularities of the solution y of equation (1.3) is reported in [9], extending the results of [16]. Subsequently, Kaneko, Noren and Xu used the regularity results to establish the collocation method for weakly singular Hammerstein equations in [10]. The approximate solutions provided by these methods for equations (1.1) and (1.3) are in the form of piecewise polynomials that are not always satisfactory as a tool for approximating functions with singularities. This observation is quite evident in the areas of finite element analysis. Hughes and Akin [6] list several problems (e.g. 'upwind' finite elements for treating convection operators [5],[8],[7]; boundary-layer elements [1] etc.) in which the finite element shape functions are constructed to include polynomials as well as singular functions. Singular shape functions are introduced to the set of basis functions through asymptotic analysis on the solution of the problem that is being considered. It should be pointed out that the analysis involved in the aforementioned papers on the finite element method is centered around the collocation method. The problems such as the choice for the extra collocation points for singular basis elements or the rate of convergence are not addressed in these papers. It should be pointed out that the location of additional collocation points for singular basis elements is critical in detemining the rate of convergence of numerical solutions. A detailed discussion on this subject can be found in [11]. A singularity preserving collocation method, because of the

reasons mentioned above, seems to be more difficult to establish.

In this paper, a singularity expansion for the solution of equation (1.3) with logarithmic kernel is given. This extends the results in [9] and [3]. Only the logarithmic kernel is considered here because of our primary interest in obtaining numerical solution of a Dirichlet problem with nonlinear boundary condition as described in Section 4. It is a routine matter, however, to establish, following the argument of Section 2, a singularity expansion for the solution of equation (1.3) with algebraic singularity. A detail is left to the reader. The paper is organized as follows: in Section 2, we study the regularity property of the soultion of (1.3) and establish its singularity expansion. The results obtained there generalize the results of [3] and [9]. The singularity expansion is then utilized in Section 3 to achieve the singularity preserving Galerkin method for equation (1.3). Finally, in Section 4, the singularity preserving Galerkin method is applied to a class of Dirichlet problems with nonlinear boundary condition. Examples are also included in this section.

2 Singularity Expansion for Weakly Singular Hammerstein Equations

In this section, we consider the following Hammerstein equation with logarithmic singularity,

$$y(s) - \int_0^1 \log|s-t| m(s,t)\psi(t,y(t))dt = f(s), \qquad 0 \le s \le 1 \tag{2.1}$$

(see (1.3) also). We let

$$K\Psi y(s) \equiv \int_0^1 \log|s-t| m(s,t)\psi(t,y(t))dt. \tag{2.2}$$

Then equation (2.1) can be written in operator form as

$$y - K\Psi y = f. \tag{2.3}$$

Let H^n denote the Sobolev space $H^n[0,1] = \{w : w^{(n)} \in L_2[0,1]\}$ equipped with the norm $\|u\|_{H^n} = \left(\sum_{i=0}^n \|u^{(i)}\|_2^2\right)^{1/2}$ where $w^{(i)}$ describes the ith generalized derivative of w. We also let $W = W_n$ be the linear space spanned by the functions $s^i \log^j s, (1-s)^i \log^j(1-s); i,j = 1,2,...,n-1$. Throughout the paper, we assume the following conditions:

$$m \in C^{2n}([0,1]\times[0,1]), n \ge 1,\ m \in C^1([0,1]\times[0,1]), n = 0. \tag{2.4}$$

$$\psi \in C^{2n+1}(R \times R) \tag{2.5}$$

$$f \in W \oplus H^n. \tag{2.6}$$

We define

$$Ky(s) \equiv \int_0^1 \log|s-t| m(s,t) y(t) dt. \tag{2.7}$$

Also let $u_1(s) = s^p \log^q s$, and $u_2(s) = (1-s)^p \log^q(1-s)$, where $p, q \geq 1$ are integers. First we quote the following result (lemma 4.4(2)) from [3].

Lemma 2.1 *Let $f \in H^{n-1}$ and assume $m \in C^{n+1}([0,1]\times[0,1])$. Then,*

$$(Kf)(s) = \sum_{j=1}^{n-1} \left[c_j s^j \log s + d_j (1-s)^j \log(1-s) \right] + v_n(s),$$

$$(Ku_1)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} s^j (\log s)^i + \sum_{j=q+1}^{n-1} d_j (1-s)^j \log(1-s) + v_n(s),$$

and

$$(Ku_2)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} (1-s)^j (\log(1-s))^i + \sum_{j=q+1}^{n-1} d_j s^j \log s + v_n(s).$$

Lemma 2.2 *If $u_1(s) = s^p \log^q s, u_2(s) = (1-s)^r \log^u(1-s)$, where $p, q, r, u \geq 1$ are integers, then $u_1 u_2 \in W \oplus H^n$.*

Proof: Expand u_1 in series about $t = 1$ and u_2 about $t = 0$:

$$\begin{aligned} u_1(t) &= \textstyle\sum_{i=0}^{n-1} b_i (1-t)^i + f_1(t), & u_2(t) &= \textstyle\sum_{i=0}^{n-1} a_i t^i + f_2(t), \\ &\equiv P_1(t) + f_1(t) & &\equiv P_2(t) + f_2(t) \end{aligned}$$

where $f_1^{(k)}(t) = O((1-t)^{n-k})$ near $t = 1$, f_1 is analytic at $t = 1$, and $f_1^{(k)} \sim u_1^{(k)}(t) - P_1^{(k)}(0)$ as $t \to 0+$; $f_2^{(k)}(t) = O(t^{n-k})$ near $t = 0$, f_2 is analytic at $t = 0$, and $f_2^{(k)}(t) \sim u_2^{(k)}(t) - P_2^{(k)}(1)$ as $t \to 1^-$.

Now $u_1 u_2 = P_1 P_2 + P_1 f_2 + P_2 f_1 + f_1 f_2$. Clearly $P_1 P_2$ is in H^n. For $f_1 f_2$, we have

$$\frac{d^n}{dt^n}(f_1(t) f_2(t)) = \sum_{i=0}^{n} \binom{n}{i} f_1^{(i)}(t) f_2^{(n-i)}(t).$$

Each term $f_1^{(i)}(t) f_2^{(n-i)}(t), i = 0, 1, ..., n$ satisfies

$$f_1^{(i)}(t) f_2^{(n-i)}(t) = O(f_1^{(i)}(t) t^i) = O([u_1^{(i)}(t) - P_1^{(i)}(0)] t^i) \to 0$$

as $t \to 0+$.

Similarly

$f_1^{(i)}(t)f_2^{(n-i)}(t) \to 0$ as $t \to 1^-$. Thus $f_1 f_2 \in C^n \subseteq H^n$. For $f_1 P_2$ we have $f_1(t)P_2(t) = (u_1(t) - P_1(t))P_2(t) = u_1(t)P_2(t) - P_1(t)P_2(t)$. Since P_2 is a polynomial, $u_1 \in W$, it is easy to see that $u_1 P_2 \in W \oplus H^n$ (see [[3], (4.7)]). So $f_1 P_2 \in H^n$. Similarly $f_2 P_1 \in W \oplus H^n$, and Lemma 2 has been verified. □

Lemma 2.3 *A product of an H^n function with a function in W is in $H^n \oplus W$.*

Proof: Let $g \in H^n$ and let u_1 and u_2 be defined as before prior to Lemma 1. For gu_1 we write

$$\begin{aligned} u_1(s)g(s) &= \textstyle\sum_{i=0}^{n-1} \frac{g^{(i)}(0)}{i!} s^{i+p} \log^q s + \frac{s^p \log^q s}{(n-1)!} \int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma \\ &\equiv T_1 + T_2. \end{aligned}$$

Since $T_1 \in W \oplus H^n$, we turn to T_2 and write

$$\begin{aligned} \frac{d^n T_2}{ds^n} &= \frac{1}{(n-1)!} \textstyle\sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s] \frac{d^{n-k}}{ds^{n-k}} [\int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma] \\ &= \frac{1}{(n-1)!} \textstyle\sum_{k=1}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s][(n-1))...k] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma \\ &\quad + s^p \log^q s g^{(n)}(s). \end{aligned}$$

But $s^p \log^q s \in L^\infty, g^{(n)} \in L_2[0,1]$ so $(s^p \log^q s) g^{(n)}(s) \in L^2$.

For the terms

$$b_n(s) \equiv \frac{d^k}{ds^k}[s^p \log^q s] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma$$

we have, for some constant M and nonegative integer α

$$\begin{aligned} |\, b_n(s) \,| &\le M \frac{(-\log s)^\alpha}{s^{k-1}} \int_0^s |\, g^{(n)}(\sigma) \,|\, s^{k-1} d\sigma \\ &= M s (-\log s)^\alpha \tfrac{1}{s} \int_0^s |\, g^{(n)}(\sigma) \,|\, d\sigma. \end{aligned}$$

But $g^{(n)} \in L_2[0,1]$, so by Hardy's inequality [15] (p. 72) $\frac{1}{s}\int_0^s |\, g^{(n)}(\sigma) \,|\, d\sigma \in L_2[0,1]$. Since $s(-\log s)^\alpha \in L^\infty$ it follows that $b_n \in L_2[0,1]$. Hence $\frac{d^n T_2}{ds^n} \in L_2[0,1]$, or $T_2 \in H^n$, This proves $gu_1 \in W \oplus H^n$.

The case for $gu_2 \in W \oplus H^n$ is similar. □

Finally we need the following:

Lemma 2.4 *The operator $K\Psi$ maps $W \oplus H^n$ into $W \oplus H^{n+1}$.*

Proof: Let $y = w + h$, $w \in W$, $h \in H^n$. We use Taylor's theorem in the form

$$\psi(x) = \sum_{k=0}^{n} \frac{1}{k!} \psi^{(k)}(a)(x-a)^k + \frac{1}{n!} \int_a^x (x-\sigma)^n \psi^{(n+1)}(\sigma) d\sigma. \tag{2.8}$$

Letting $x = y(s)$ and $a = h(s)$ allows us to write

$$\begin{aligned}(K\Psi)(y)(t) &= \sum_{k=0}^{n} \frac{1}{k!} \int_0^1 \log \mid t-s \mid m(t,s)\psi^{(k)}(h(s))w(s)^k ds \\ &+ \frac{1}{n!} \int_0^1 \log \mid t-s \mid m(t,s) \int_{h(s)}^{y(s)} \psi^{(n+1)}(\sigma)(y(s)-\sigma)^n d\sigma ds \\ &\equiv \sum_{k=0}^{n} \frac{1}{k!} A_k(t) + \frac{1}{n!} B(t).\end{aligned} \tag{2.9}$$

By (3), $\psi^{(k)}(h(s)) \in H^n, k = 0, 1, ..., n$, and by expanding $w(s)^k$ with the multinomial expansion, it is clear that $w(s)^k$ is a sum of terms in W as well as terms of the form $as^p \log^q s(1-s)^r \log^u(1-s)$, $p, q, r, u \geq 1$ are integers. The constant, a, depends on p, q, r and u. Since $\psi^{(k)}(h(s)) \in H^n$ and $w(s)^k \in W \oplus H^n, k = 0, 1, ..., n$, it follows from Lemma 3 that

$$\psi^{(k)}(h(s))w(s)^k \in W \oplus H^n. \tag{2.10}$$

By Lemma 1 and (2.10), we have

$$A_k \in W \oplus H^{n+1}. \tag{2.11}$$

For $B(t)$, if we prove that

$$F(s) \equiv \int_{h(s)}^{y(s)} \psi^{(n+1)}(\sigma)(y(s)-\sigma)^n d\sigma \in W \oplus H^n,$$

then, also by Lemma 1, $B(t) = K[F](t)$ will be in $W \oplus H^{n+1}$. This will complete the proof of this lemma. First of all, suppose $n \geq 1$. We write

$$F'(s) = -\psi^{(n+1)}(h(s))w(s)^n h'(s).$$

Since $h \in H^n, \psi \in C^{2n+1}, \psi^{(n+1)}(h(s)) \in H^n$. By Lemmas 2 and 3, $-\psi^{(n+1)}(h(s))w(s)^n \in H^n \oplus W$. Since $h' \in H^{n-1}$, it follows that $-\psi^{(n+1)}(h(s))w(s)^n h'(s) \in H^{n-1} \oplus W$ (Lemma 2). Since $F' \in H^{n-1} \oplus W$ it is clear that $F \in H^n \oplus W$. Second, let $n = 0$. Then $F(s) = \int_{h(s)}^{y(s)} \psi'(\sigma) d\sigma = \psi(y(s)) - \psi(h(s)) \in L_2[0,1] \subseteq W \oplus H^0$.
Thus

$$B(t) \in W \oplus H^n. \tag{2.12}$$

By (2.9), (2.11) and (2.12), it follows that $K\Psi$ maps $W \oplus H^n$ into $W \oplus H^{n+1}$. □

Using the lemmas which we proved above, we obtain the following main result of this section.

Theorem 2.5 *Suppose the conditions (2.4)-(2.6) hold and y is an isolated solution of (2.1). Then there are constants a_{ij} and b_{ij}, for $i, j = 1, 2, ..., n-1$, and there is a function v_n in H^n such that*

$$y(t) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} [a_{ij} t^i \log^j t + b_{ij}(1-t)^i \log^j(1-t)] + v_n(t). \tag{2.13}$$

Proof: For $n = 0$, this follows from Lemma 4 with $n = 0$. Assume that the result holds for $n = k$, that is, if $f \in H^k \oplus W_k$, then (2.13) holds with $n = k$. Say $y = w_k + v_k$, where $v_k \in H^h, w_k = \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} [a_{ij} t^i \log^j t + b_{ij}(1-t)^i \log^j(1-t)]$.

Now consider the case $n = k+1$ and suppose $f \in H^{k+1} \oplus W_{k+1}$.

Since $y = w_k + v_k$ we write $y = K\Psi y + f = K\Psi(w_k + v_k) + f$. From Lemma 1, $K\Psi(w_k + v_k) \in W_{k+1} \oplus H^{k+1}$. The proof is complete. □

3 Singularity Preserving Galerkin Method

In this section, we establish the singularity preserving Galerkin method for equation (2.1). First we recall the definition of the space of spline functions of order n. Define the partition of $[0,1]$ as

$$\Delta : 0 = t_0 < t_1 < ... < t_k < t_{k+1} = 1.$$

Let

$$h = \max_{1 \le i \le k+1} (t_i - t_{i-1}),$$

and assume $h \to 0$ as $k \to \infty$. Denote by Π_n the set of polynomials of degree $n-1$. Then the space of splines of order n with knots t_i's of multiplicity $n - 1 - \nu$ is defined as

$$S_h^n = S_h^{n,\nu}(\Delta) = \{s \in C^\nu[0,1] : \ s|_{I_i} \in \Pi_n, \},$$

where $0 \le \nu \le n-1$ and $I_i = (t_{i-1}, t_i)$ for $i = 1, 2, ..., k+1$. It is well known that the dimension of S_h^n is $d = n(k+1) - k(1+\nu)$. S_h^n is spanned by a basis consisting of B-splines $\{B_i\}_{i=1}^d$. We let

$$V_h^n \equiv W \oplus S_h^n \tag{3.1}$$

and denote the orthogonal projection of $L_2[0,1]$ into V_h^n by P_h^G. The singularity preserving Galerkin method for approximating the solution of equation (2.3) requires the solution $y_h \in V_h^n$ that satisfies the following equation:

$$y_h - P_h^G K\Psi y_h = P_h^G f. \tag{3.2}$$

More specifically, we need to find y_h in the form

$$y_h(s) = \sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \sum_{i=1}^{d} \gamma_i B_i(s) \tag{3.3}$$

where $\{\alpha_{ij}, \beta_{ij}\}_{i,j=1}^{n-1}$ and $\{\gamma_i\}_{i=1}^{d}$ are found by solving the following system of nonlinear equations:

$$
\begin{aligned}
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, s^p \log^q s) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), s^p \log^q s) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s)), s^p \log^q s) \\
& \qquad = (f, s^p \log^q s) \qquad p,q = 1,2,\ldots,n-1 \\
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, (1-s)^p \log^q(1-s)) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), (1-s)^p \log^q(1-s)) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} \beta_{i,j}(1-s)^i \log^j(1-s)), (1-s)^p \log^q(1-s)) \\
& \qquad = (f, (1-s)^p \log^q(1-s)) \qquad p,q = 1,2,\ldots,n-1 \\
\textstyle\sum_{i,j=1}^{n-1} \quad & \alpha_{ij}(s^i \log^j s, B_p) + \textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), B_p) \\
& -(K\Psi(\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \textstyle\sum_{i,j=1}^{n-1} b_{i,j}(1-s)^i \log^j(1-s)), B_p) \\
& \qquad = (f, B_p) \qquad p = 1,2,\ldots,d
\end{aligned}
$$

where $(\cdot,\cdot)$ denotes the usual inner product defined on $L_2[0,1]$. Now let P_h be the orthogonal projection of $L_2[0,1]$ into S_h^n. Then we have

$$P_h v \to v \qquad \text{for all } v \in L_2[0,1]. \tag{3.4}$$

It is well known (e.g. [18]) that if $g \in H^n$, $n \geq 0$, then for each $h > 0$, there exists $\phi_h \in S_h^n$ such that

$$\|g - \phi_h\|_2 \leq Ch^n \|g\|_{H^n}, \tag{3.5}$$

where $C > 0$ is a constant independent of h. By virtue of the fact that $P_h u$ is the best L_2 approximation of u from S_h^n, we see immediately that

$$\|P_h u - u\|_2 \leq \|u - \phi_h\|_2 \leq Ch^n \|u\|_{H^n}, \text{ for all } u \in H^n. \tag{3.6}$$

The following lemma from [3] is useful in the sequel.

Lemma 3.1 *Let X be a Banach space. Suppose that U_1 and U_2 are two subspaces of X with $U_1 \subseteq U_2$. Assume that $P_1 : X \to U_1$ and $P_2 : X \to U_2$ are linear operators. If P_2 is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \text{ for all } x \in X.$$

For convenience, we introduce operators $\hat{T}$ and T_n by letting

$$\hat{T} y \equiv f + K\Psi y \tag{3.7}$$

and

$$T_h y_h \equiv P_h^G f + P_h^G K \Psi y_h \tag{3.8}$$

so that equations (2.1) and (3.2) can be written respectively as $y = \hat{T}y$ and $y_h = T_h y_h$. The following theorem guarantees the existence of a solution of the singularity preserving Galerkin method (3.2) and describes the accuracy of its approximation.

Theorem 3.2 *Let $y \in L_2[0,1]$ be an isolated solution of equation (2.1). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(y)$, where $(K\Psi)'(y)$ denotes the Fréchet derivative of $K\Psi$ at y. Then the singularity preserving Galerkin approximation equation (3.2) has a unique solution y_h such that $\|y - y_h\|_2 < \delta$ for some $\delta > 0$ and for all $0 < h < h_0$ for some $h_0 > 0$. Moreover, there exists a constant $0 < q < 1$, independent of h, such that*

$$\frac{\alpha_h}{1+q} \le \|y - y_h\|_2 \le \frac{\alpha_h}{1-q}, \tag{3.9}$$

where $\alpha_h \equiv \|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2$. Finally, if $y = w + v$ with $w \in W$ and $v \in H^n$, then

$$\|y - y_h\|_2 \le Ch^n \|v\|_{H^n}, \qquad \text{whenever } 0 < h < h_0, \tag{3.10}$$

where $C > 0$ is a constant independent of h.

Proof: The existence of a unique solution y_h of equation (3.2) and the inequalities in (3.7) can be proved using Theorem 2 of Vainikko [19]. A detailed argument can be found in [10]. To get (3.10), first we note from Lemma 3.1, for $v \in L_2[0,1]$,

$$\|P_h^G v - v\|_2 \le (1 + \|P_h^G\|_2)\|P_h v - v\|_2. \tag{3.11}$$

Now, from (3.9),

$$\begin{aligned} \|y - y_h\|_2 &\le \tfrac{\alpha_h}{1-q} \\ &= \tfrac{1}{1-q}\|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_2 \\ &\le C\|P_h^G K\Psi y - K\Psi y + P_h^G f - f\|_2 \\ &= C\|P_h^G y - y\|_2. \end{aligned} \tag{3.12}$$

where C is independent of h. Using the uniform boundedness of $\{P_h^G\}$, (3.11) and (3.12), we obtain

$$\|y - y_h\|_2 \le Ch^n \|v\|_{H^n}.$$

□

4 Nonlinear Boundary Value Problem

In this section, we consider the following Laplace's equation with nonlinear boundary condition in R^2:

$$\begin{aligned} \triangle u(P) &= 0, && P \in D \\ \tfrac{\partial u}{\partial n_P}(P) &= -\psi(P, u(P)) + f(P), && P \in \Gamma = \partial D \end{aligned} \tag{4.1}$$

where D is a simply connected open region in R^2 with an open contour Γ and n_P denotes the exterior unit normal to Γ at P. The functions f and ψ are given and we assume that $f \in C(\Gamma)$ and $\psi \in C(\Gamma \times R)$. The solution u is to be found in $C^2(D) \cap C^1(\overline{D})$. The function ψ is assumed to be continuous on $\Gamma \times R$. The problem (4.1) in the case of a closed smooth boundary Γ was considered by Atkinson and Chandler in [2]. They employed the method of piecewise polynomial product integration and that of trigonometric product integration to approximate the solution of (4.1). In the current problem in which Γ describes a boundary that is open, one expects logarithmic singularities in u at the two ends of the boundary. Now it is well known that, using Green's representation formula for harmonic functions, the function u satisfies

$$u(P) = \frac{1}{2\pi}\int_\Gamma u(Q)\frac{\partial}{\partial n_Q}\log|P-Q|d\sigma(Q) - \frac{1}{2\pi}\int_\Gamma \frac{\partial u}{\partial n_Q}(Q)\log|P-Q|d\sigma(Q) \tag{4.2}$$

for all $P \in D$. Using the boundary condition in (4.1) and letting P approach to a point of Γ, we obtain

$$\begin{aligned} u(P) - \tfrac{1}{\pi}\textstyle\int_\Gamma \;& u(Q)\tfrac{\partial}{\partial n_Q}\log|P-Q|d\sigma(Q) - \tfrac{1}{\pi}\textstyle\int_\Gamma \psi(Q,u(Q))\log|P-Q|d\sigma(Q) \\ &= -\tfrac{1}{\pi}\textstyle\int_\Gamma f(Q)\log|P-Q|d\sigma(Q), \qquad P \in \Gamma. \end{aligned} \tag{4.3}$$

We denote the double layer operator by T,

$$Tv(P) = \frac{1}{\pi}\int_\Gamma v(Q)\frac{\partial}{\partial n_Q}\log|P-Q|d\sigma(Q), \qquad P \in \Gamma, \tag{4.4}$$

and the single layer operator by S,

$$Sv(P) = -\frac{1}{\pi}\int_\Gamma v(Q)\log|P-Q|d\sigma(Q), \qquad P \in \Gamma. \tag{4.5}$$

If we put $\Psi v(P) = \psi(P, v(P))$, $P \in \Gamma$, then equation (4.3) can be written in operator form as

$$u - Tu + S\Psi(u) = Sf. \tag{4.6}$$

Define the parametrization $r(t) = (\xi(t), \eta(t))$ for $t \in [0,1]$ and assume that $r \in C^\infty[0,1]$ and $|r'(t)| \neq 0$ for $t \in [0,1]$. The double layer operator and the single layer operator now become respectively,

$$Tv(t) = \frac{1}{\pi}\int_0^1 \frac{\eta'(s)[\xi(s)-\xi(t)] - \xi'(s)[\eta(s)-\eta(t)]}{[\xi(s)-\xi(t)]^2 + [\eta(s)-\eta(t)]^2}v(s)ds \tag{4.7}$$

for $v \in C[0,1]$ with the kernel having the value

$$\frac{\xi'(t)\eta''(t) - \eta'(t)\xi''(t)}{2[\xi'(t)^2 + \eta'(t)^2]}$$

when $s = t$ and

$$Sv(t) = -\frac{1}{\pi}\int_0^1 \log|s-t|v(s)|r'(s)|ds. \tag{4.8}$$

The kernel of T is well behaved whereas the kernel of S needs our scrutiny. As in [2], we write Sv as

$$\begin{aligned} Sv(t) = & -\frac{1}{\pi}\int_0^1 v(s)|r'(s)|\{\log|t-s| + \log|1-s+t| + \log|1-t+s|\}ds \\ & -\frac{1}{\pi}\int_0^1 v(s)|r'(s)|\log\left[\frac{|r(s)-r(t)|}{|s-t|(1-s+t)(1-t+s)}\right]ds. \end{aligned} \tag{4.9}$$

Due to the fact that the double layer operator T contains infinitely differential kernel, using the arguments to prove Lemma 2.4, we see that $T + S\Psi$ maps $W \oplus H^n$ into $W \oplus H^{n+1}$. Hence we obtain the following result that parallels the results obtained in Theorem 2.5.

Theorem 4.1 *Suppose the conditions (2.5) and (2.6) hold and u is an isolated solution of (4.6). Then there are constants a_{ij} and b_{ij}, for $i,j = 1,2,...,n-1$, and there is a function v_n in H^n such that*

$$y(t) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[a_{ij}t^i\log^j t + b_{ij}(1-t)^i\log^j(1-t)] + v_n(t). \tag{4.10}$$

Define

$$\tilde{T}u \equiv Tu - S\Psi(u) + Sf \tag{4.11}$$

so that equation (4.6) can be written as

$$u = \tilde{T}u. \tag{4.12}$$

The singularity preserving Galerkin method is now described by

$$u_h - P_h^G T u_h + P_h^G S\Psi(u_h) = P_h S f. \tag{4.13}$$

where $u_h \in V_h^n$ and P_h^G is the orthogonal projection of $L_2[0,1]$ into V_h^n as defined in the previous section. By letting

$$\tilde{T}_h u \equiv P_h^G T u - P_h^G S\Psi(u) + P_h^G S f, \tag{4.14}$$

equation (4.13) can be written as

$$u_h = \tilde{T}_h u_h. \tag{4.15}$$

The following theorem guarantees the existence of the solution of equation (4.15) and describes its accuracy as an approximation to u that is the soution of equation (4.12). A proof is an easy exercise of modifying the argument given in the proof of theorem 2.1 [10], hence we omit it.

Theorem 4.2 *Let $u \in L_2[0,1]$ be an isolated solution of equation (4.6). Assume that 1 is not an eigenvalue of the linear operator $T + (S\Psi)'(u)$. Then the singularity preserving Galerkin approximation equation (4.13) has a unique solution u_h such that $\|u - u_h\|_2 < \delta$ for some $\delta > 0$*

and for all $0 < h < h_0$ *for some* $h_0 > 0$. *Moreover, there exists a constant* $0 < q < 1$, *independent of* h, *such that*

$$\frac{\alpha_h}{1+q} \leq \|u - u_h\|_2 \leq \frac{\alpha_h}{1-q}, \tag{4.16}$$

where $\alpha_h \equiv \|(I - \tilde{T}_h'(u))^{-1}(\tilde{T}_h(y) - \tilde{T}(y))\|_2$. *Finally, if* $y = w + v$ *with* $w \in W$ *and* $v \in H^n$, *then*

$$\|u - u_h\|_2 \leq Ch^n \|v\|_{H^n}, \qquad \text{whenever } 0 < h < h_0, \tag{4.17}$$

where $C > 0$ *is a constant independent of* h.

References

[1] J. E. Akin, Physical bases for the design of special finite element interpolation functions, *Recent Advances in Engineering Science*, (G. C. Sih, ed.) Lehigh Univ. Press, (1977), 879-884.

[2] K. E. Atkinson and G. Chandler, Boundary integral equations methods for solving Laplace's equation with nonlinear boundary conditions: The smooth boundary case, *Tech Report Univ Iowa*

[3] Y. Cao and Y. Xu, Singularity preserving Galerkin methods for weakly singular Fredholm integral equations, *Jl. Int. Eqs. and Appl.*, 6 (1994) 303-334.

[4] I. Graham, Singularity expansions for the solution of the second kind Fredholm integral equations with weakly singular convolution kernels, *Jl. Int. Eqs.*, 4 (1982), 1-30.

[5] T. J. R. Hughes, A simple scheme for developing "upwind" finite elements, *Int. Jl. Num. Meth. Engng.*12 (1978), 1359-1365.

[6] T. J. R. Hughes and J. E. Akin, Techniques for developing 'special' finite element shape functions with particular reference to singularities, *Int. Jl. Num. Meth. Eng.*15 (1980), 733-751.

[7] J. C. Henrich, P. S. Huyakorn, A. R. Mitchell and O. C. Zienkiewicz, An upwind finite element scheme for two-dimensional convective transport equations, *Int. Jl. Num. Meth. Engng.*, 11 (1977), 131-143.

[8] T. J. R. Hughes, W. K. Liu and A. Brooks, Finite element analysis of incompressible viscous flows by the penalty function formulation, *Jl. Comp. Phys.*, 30 (1979), 1-60.

[9] H. Kaneko, R. Noren and Y. Xu, Regularity of the solution of Hammerstein equations with weakly singular kernels, *Int. Eqs. Op. Thy.* 13 (1990), 660-670.

[10] H. Kaneko, R. Noren and Y. Xu, Numerical solutions for weakly singular Hammerstein equations and their superconvergence, *J. Int. Eqs. Appl.* 4 (1992), 391-407.

[11] H. Kaneko and P. Padilla, A note on the finite element method with singular basis functions, (submitted for publication).

[12] H. Kaneko and Y. Xu, Numerical solutions for weakly singular Fredholm integral equations of the second kind, *Appl. Num. Math.*, 7 (1991), 167-177.

[13] G. R. Richter, On weakly singular Fredholm integral equations with displacement kernels, *Jl. Math. Anal. Appl.*, 55 (1976), 32-42.

[14] J. Rice, On the degree of convergence of nonlinear spline approximation, in *Approximation with Special Emphasis on Spline Functions,* ed. I. J. Schoenberg, Academic Press, N.Y. (1969), 349-365.

[15] W. Rudin, *Real and Complex Analysis*, 3rd. ed., McGraw-Hill Book Co. New York, (1987).

[16] C. Schneider, Regularity of the solution to a class of weakly singular Fredholm integral equations of the second kind, *Int. Eqs. Op. Thy.*2 (1979), 62-68.

[17] C. Schneider, Product integration for weakly singular integral equations, *Math. Comp.*36 (1981), 207-213.

[18] I.H. Sloan, Superconvergence, in *Numerical Solution of integral equations,* ed. M. A. Golberg, Plenum Press, N.Y., 35-70 (1990).

[19] G. Vainikko, Perturbed Galerkin method and general theory of approximate methods for nonlinear equations, *Zh. Vychisl. Mat. Fiz.* 7 (1967), 723-751. Engl. Translation, *U.S.S.R. Comp. Math. and Math. Phys.* 7, No. 4 (1967), 1-41.

[20] G. Vainikko and A. Pedas, The properties of solutions of weakly singular integral equations, *Jl. Austral. Math. Soc. Ser. B*22 (1981), 419-430.

[21] G. Vainikko and P. Ubas, A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel, *Jl. Austral. Math. Soc. Ser. B*22 (1981), 431-438.