S27-64
44845

RECENT ADVANCES IN WAVELET TECHNOLOGY

R. O. Wells, Jr.
Computational Mathematics Laboratory
Department of Mathematics
Rice University
Houston, Texas 77251
wells@rice.edu

March 24, 1994

1 INTRODUCTION AND OVERVIEW

In this paper I want to report on some recent developments in wavelet technology and, in particular, how it relates to some of the research activities at NASA. First, I want to indicate the nature of our research effort at Rice University in this direction. We have developed over the last four years a **Computational Mathematics Laboratory** (CML) housed in the **Computer and Information Technology Institute** (CITI) at Rice. This laboratory has as its primary focus research in the theory and applications of wavelets and more generally multiscale phenomena in mathematics, science and engineering. The researchers in the CML are:

- R. O. Wells, Jr., Professor of Mathematics (Rice), Director of CML
- C. S. Burrus, Professor of Electrical and Computer Engineering (Rice) and Director of CITI
- W. W. Symes, Professor and Chairman of Computational and Applied Mathematics (Rice)
- Roland Glowinski, Cullen Professor of Mathematics, University of Houston
- 4 Post Doctoral Fellows and 5 Graduate Students
- *Principal Support:* ARPA, NASA, Aware, Inc., Texas Instruments, Texas Higher Education Coordinating Board

Wavelet research has been developing rapidly over the past five years, and in particular in the academic world there has been significant activity at Rice, Yale, MIT, Delaware, Brown, S. Carolina, Washington Univ., Minnesota, Dartmouth, and numerous other universities. In the industrial world, there has been developments at Aware, Inc., Lockheed, Martin-Marietta, TRW, Kodak, Exxon, and many others. The government agencies supporting wavelet research and development include ARPA, ONR, AFOSR, NASA, and many other agencies. The recent literature in the past five years includes a recent book [6] which is an index of citations in the past decade on this subject, and it contains over 1,000 references and abstracts.

2 WAVELET MATHEMATICS

Fundamentally, wavelets are a new type of function which provide an excellent orthonormal basis for functions of one or more variables. They provide a localized basis, and can represent square-integrable functions, but also constant and, more generally, polynomial functions in a locally finite manner.

In 1988 Daubechies' fundamental paper on wavelets [1] appeared. In this paper we find for the first time a parametrized family of orthonormal systems of functions in $L^2(\mathbf{R})$ with certain important complementary properties. Each system of functions has the following properties:

- each system is generated from a scaling function $\varphi(x)$ and a wavelet function $\psi(x)$ by rescalings (by powers of an integer) and translations (e.g., $\varphi_{j,k}(x) := 2^{j/2}\varphi(2^j x - k)$ and $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$. The wavelet system
$$\{\psi_{0,k}(x), \psi_{j,k}(x), \quad j,k \in \mathbf{Z}, \quad j \geq 0\} \tag{1}$$
is an orthonormal basis for $L^2(\mathbf{R})$ and more general functions as well (including constants and higher order polynomials, depending on the wavelet system chosen).

- each element in a given system has compact support and is continuous or can be chosen to be smooth up to a given finite order and by the rescaling above, the supports of the basis functions becomes very small for large scaling index j.

- There are fast algorithms for computing the coefficients of the expansion of a given digitized (sampled function). This is the *discrete wavelet transform* (from the sampled function to the wavelet expansion coefficients), and it is an $O(N)$ algorithm.

- The classical discrete Fourier and cosine transforms appear as a special case of the general discrete wavelet transform (DWT)

- The discrete wavelet transform is parallelizable and can ben implemented on massively parallel machines as well as can be designed into specializedd VLSI chips (e.g., for digital video editing).

In general a scaling function and corresponding wavelet function satisfy the *scaling equation*

$$\varphi(x) = \sum_{k=0}^{2g-1} a_k \varphi(2x - k) \tag{2}$$

and the corresponding *wavelet defining equation*

$$\psi(x) := \sum_{k=0}^{2g-1} b_k \varphi(2x - k), \tag{3}$$

where the coefficients of the scaling equation a_k must satisfy linear and quadratic constraints of the form:

$$\sum a_k = 2, \tag{4}$$

$$\sum a_k a_{k+2l} = 2\delta_{l,0}, \tag{5}$$

and where $b_k := (-1)^{k+1} b_{2g-1-k}$.

One of the powers of wavelet technology is the ability to choose the defining coefficients for a given wavelet system to be best adapted to the given problem. Daubechies developed in her original paper [1] specific families of wavelet systems which had maximal vanishing moments of the ψ function and which were very good for representing polynomial behavior. In Figure 1 we see the corresponding Daubechies scaling and wavelet function for the case of 4 coefficient ($g = 2$) where

$$\{a_0, a_1, a_2, a_3\} = \{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\} \tag{6}$$

and

In Figure 2 we see the contrast between the Fourier representation and wavelet representation for a given example of a transitory signal, and that the wavelet representation does provide a superior representation for this particular example.

3 WAVELET MULTISCALE REPRESENTATION OF DATA

If we consider such a wavelet system, and assume that there is a certain amount of smoothness (C^2, for instance), then we can try to use these functions as basis elements for representing discrete data at different scales.

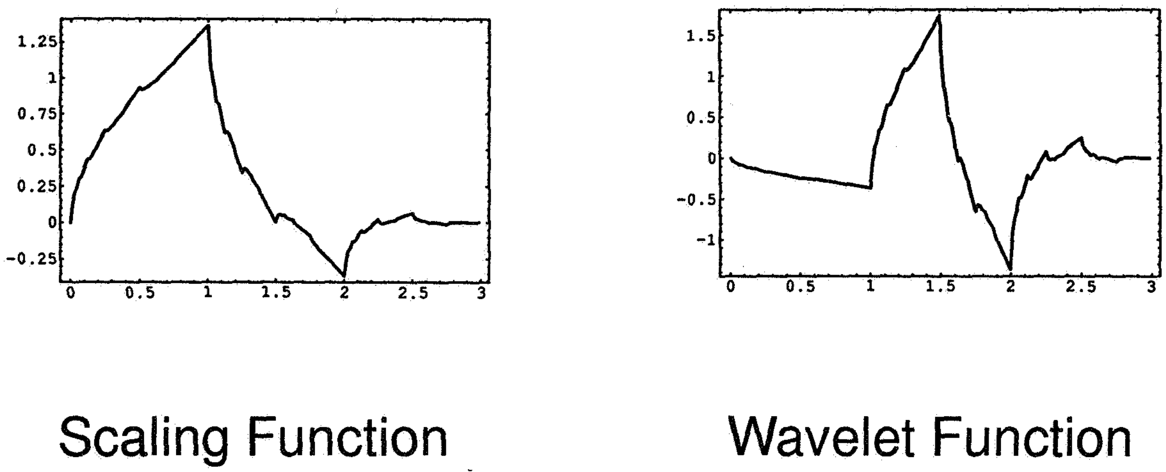

Figure 1: On the left is the 4-coefficient Daubechies scaling function and on the right is the corresponding wavelet function

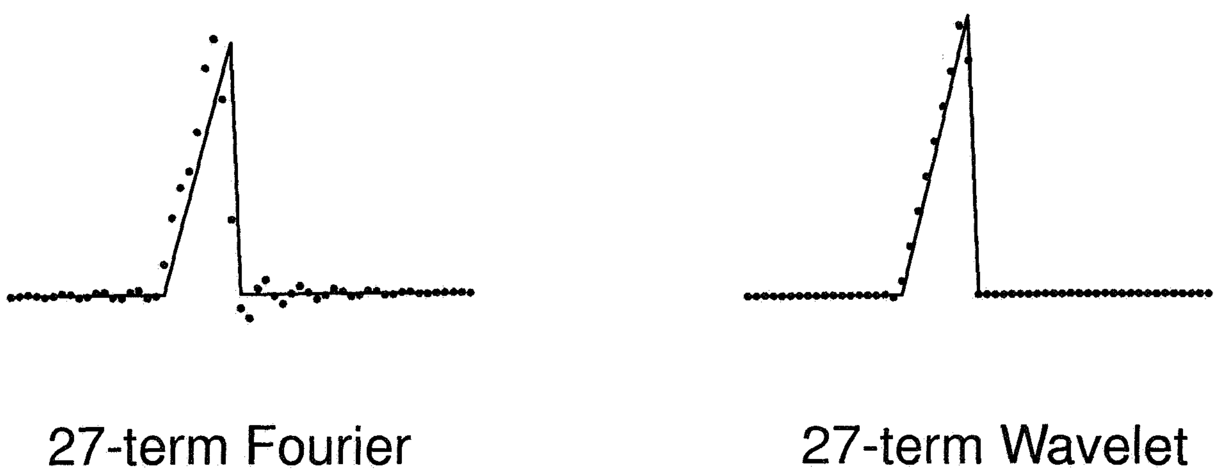

Figure 2: Comparison of a wavelet and Fourier representation of a transient signal

Namely, if we let for fixed $J \in \mathbf{N}$,

$$\tilde{f}(x) = \sum_k c_{Jk} \varphi_{Jk}(x), \tag{7}$$

where c_{Jk} represents a sampling of a given function $f(x)$ at the points $x = k/2^J$, then $\tilde{f}$ is a smooth wavelet interpolation of our original sampled $f(x)$ *at the scale* J (or, what is the same thing, on a mesh with mesh size $h = 1/2^J$). Mallat [5] showed that from the scaling equations defining φ and ψ one can reexpress $\tilde{f}$ in terms of scaling and wavelet functions at coarser scales, namely:

$$\tilde{f}(x) = \sum_k c_{Jk} \varphi_{Jk}(x) = \sum_k c_{0k} \varphi_{0k}(x) + \sum_k \sum_{j=0}^{J-1} d_{Jk} \psi_{jk}(x). \tag{8}$$

In (8) we see that the left hand side (LHS) represents the data at a single "fine" scale J, while the right hand side (RHS) gives a multiscale representation of the data at the coarser scales $\{0, 1, \ldots, J-1\}$. The Mallat transform consists of mapping the coefficients at the single scale on the LHS of (8) to the multiscale coefficients on the RHS of (8), and conversely (inverse Mallat transform). This transform consists of convolution with the filters which define the scaling and wavelet functions along with downsampling (and upsampling for the inverse transform).

4 IMAGE COMPRESSION AND TELECOMMUNICATIONS TECHNOLOGY

A major application of wavelets to technology has been in the area of data compression. The following list indicates the breadth of this application area. In each case the compression ratios indicated are what is roughly currently available, and are all products of Aware, Inc., of Cambridge, Mass., which is the leading commercial supplier of wavelet-based compression algorithms, in the form of software, chips, and plug-in boards for various application areas. Moreover, the compression ratio indicates compression to a version of the original signal which is indistinguishable from the original signal for the purposes at hand, and has been verified and tested by the industry experts in that given area. As one example, audio compression, listed at 8-1 compression ratio, has the property that the human ear cannot normally distinguish the compressed signal from the original, and the compression algorithm uses information about how the ear perceives sound and at what frequency scales.

- Audio compression - high fidelity at 8:1
- Still-image compression 20:1 (BW), 100:1 (Color)

- Seismic compression 20:1
- Radiology images 20:1
- Fingerprint images 25:1
- Video compression (color) 140:1

The basic idea in a compression algorithm in all of the above examples is to represent the digitized signal in terms of a wavelet expansion (the coefficients of this expansion will be the Discrete Wavelet Transform). Using a statistical analysis of the data type involved one carries out a systematic dropping of bits of these wavelet expansion coefficients at specific scales (this is the quantization process) to represent the same signal effectively with less bits, and an additional lossless compression is then applied to the result, which can then be either transmitted or archived. To recover the signal, one reverses the process with the exception of the quantization step, as those bits cannot be recovered. For further details about this compression process in the context of images, see, e.g., [12], and more information about specific technologies in all of the areas above is available, in particular, from Aware, Inc. in Cambridge, Mass.

One important feature of all of these algorithms is that one can download a compressed signal (or even an uncompressed signal represented in terms of its DWT), at any desired scale to obtain "snapshots" of the data, and download additional information later (or in the case of audio, to increase the fidelity at a later time). This technology is undergoing rapid development at the present time, and there is still much to be learned and understood in terms of modeling these compression ideas.

A second important area in which the DWT has played an important role is that of Assymetric Data Subscriber List (ADSL) technology. This is the basic copper wire twisted-pair communications link between American homes and their telephone companies. The spectral bands of this communication link are divided into three regions, the lowest being POTS ("Plain Old Telephone Service"), the second being a band for sending conventional digital data (linking computers for instance), and the high end of the band is reserved for digital video commmunication. The problem was that this was such a noisy channel that it was difficult to send video signals over this band in a meaningful manner. Very recently, Aware, Inc. announced a partnership with Analog Devices (a second Boston area company) to build transceivers which will implement such video communication in an effective manner, and this will be marketed to the telephone industry by an Alliance involving this partnership plus Westel, Newbridge, and AT&T, all of whom are involved in various aspects of the telecommunication industry. The technical report [8] which will appear soon in the proceedings of the International Communications Conference to be held in New Orleans in 1994 gives further information about this new advance in wavelet communications technology.

5 WAVELET-BASED NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

The wavelet represention of a sampled function of the form (7) allows one to use the scaling functions at a given scale (in this case at the arbitrary scale J corresponding to a mesh size of $h = 1/2^J$) as finite-element or Galerkin-type basis elements in a discrete approximation to some continuous problem (e.g., solving a partial differential equation numerically). In a number of recent papers these ideas have been carried out for various types of elliptic boundary value problems [11,10,9,2,4]. In addition one can use the multiscale representation of data as given in (8) to implement multigrid iterative schemes for solving such elliptic boundary problems where the solution by direct methods or by iterative methods at a single scale is prohibitive. In particular, one obtains an efficient multiscale algorithm for solving the model problem involving Laplace's equation for a domain with a very general boundary [3]. Moreover, a second model problem involving anisotropic coefficients in two dimensions with periodic boundary values admits a robust multigrid algorithm whose condition number is independent of the mesh size and of the anisotropy parameter [7]. In these multigrid applications of wavelets to numerical analysis the linear Mallat algorithms (transform from single scale to multiscale and conversely) play a major role. They allow one to map simply from one adjacent scale to another in a very effective manner, and that, along with the implicit orthogonality (and hence lack of redundancy), is one of the keys to their success in this applications (which is also true in their application to digital signal processing).

References

[1] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41:906–966, 1988.

[2] R. Glowinski, J. Periaux, M. Ravachol, T. W. Pan, R. O. Wells, Jr, and X. Zhou. Wavelet methods in computational fluid dynamics. In et al. M. Y. Hussaini, editor, *Algorithmic Trends in Computational Fluid Dynamics*, New York, pp. 259-276, 1993. Springer-Verlag.

[3] R. Glowinski, A. Rieder, R. O. Wells, and X. Zhou. A wavelet multigrid preconditioner for Dirichlet boundary value problems in general domains. Computational Mathematics Laboratory TR93-06, Rice University, 1993.

[4] Roland Glowinski, T. W. Pan, R. O. Wells, and Xiaodong Zhou. Wavelet and finite element solutions for the neumann problem using fictitious domains. Technical Report 92-01, Rice University, 1992. Computational Mathematics Laboratory.

[5] S. Mallat. Multiresolution approximation and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. Amer. Math. Soc.*, 315:69–87, 1989.

[6] S Pittner, J. Schneid, and C. W. Ueberhuber. *Wavelet Literature Survey.* Institute for Applied and Numerical Mathematics, Technical University Vienna, Wien, Austria, 1993.

[7] A. Rieder, R. O. Wells Jr., and X. Zhou. A wavelet approach to robust multilevel solvers for anisotropic elliptic problems. *Applied and Computational Harmonic Analysis*, 1994. To appear.

[8] Michael A. Tzannes, Marcos C. Tzannes, John Proakis, and Peter N. Heller. DMT systems, DWMT systems and digital filter banks. Conference Proceedins of ICC '94, New Orleans. IEEE, 1994. (Aware, Inc. Technical Report AD940308).

[9] R. O. Wells and Xiaodong Zhou. Representing the geometry of domains by wavelets with applications to partial differential equations. In J. Warren, editor, *Curves and Surfaces in Computer Graphics III*, volume 1834, pages 23–33. SPIE, 1992.

[10] R. O. Wells and Xiaodong Zhou. Wavelet solutions for the Dirichlet problem. Technical Report 92-02, Rice University, 1992. Computational Mathematics Laboratory.

[11] R. O. Wells and Xiaodong Zhou. Wavelet interpolation and approximate solutions of elliptic partial differential equations. In R. Wilson and E. A. Tanner, editors, *Noncompact Lie Groups.* Kluwer, 1994. To appear. Proceedings of NATO Advanced Research Workshop.

[12] W. Zettler, J. Huffman, and D. Linden. Application of compactly supported wavelets to image compression. In *Proc. SPIE: Image Processing Algorithms and Techniques*, volume 1244, pages 150–160. Intl. Soc. for Optical Engineering, 1990.