New Interpretation Of The Wigner Function

Jamil Daboul

Physics Department, Ben Gurion University of the Negev
84105 Beer Sheva, Israel
(E-mail: daboul@bguvms.bgu.ac.il)

Abstract

I define a two-sided or forward-backward propagator for the pseudo-diffusion equation of the "squeezed" Q function. This propagator leads to squeezing in one of the phase-space variables and anti-squeezing in the other. By noting that the Q function is related to the Wigner function by a special case of the above propagator, I am led to a new interpretation of the Wigner function.

1 Introduction

The Wigner representation of any operator A is defined by

$$W(A;p,q) \equiv \int_{-\infty}^{\infty} (q-a \mid A \mid q+a)\, e^{2iap} da = Tr\,(\,A\,\mathbf{W}(p,q)\,)\ , \tag{1}$$

where the rounded kets are eigenstates of the position operator, $\mathbf{Q}\,|x) = x\,|x)$, and $\mathbf{W}(p,q) \equiv \int_{-\infty}^{\infty} |\,q+a)(q-a\,|\,e^{2iap}da$ is a unitary and also a Hermitian operator, which can be interpreted as a displaced parity operator [2]. The Wigner representation yields functions of two variables, p and q, which may be looked upon as phase-space variables. These "Wigner functions" have interesting properties and are useful for various calculations [1]. The Wigner functions are often referred to as pseudo-probability functions, because they can take negative values, even when A is a positive operator, $A \geq 0$, such as the density operator ρ.

In contrast, the Husimi or Q representation [3] yield nonnegative functions for positive operators A: These functions are defined as follows

$$Q(A;p,q;\zeta) = \langle pq;\zeta \mid A \mid pq;\zeta\rangle = Tr\,(\,A\,\Pi(pq;\zeta)\,)\ , \quad \text{where} \quad \Pi(p,q;\lambda) \equiv \mid pq;\zeta\rangle\langle pq;\zeta\mid \tag{2}$$

are projection operators on the squeezed states $|pq;\zeta\rangle$, which are defined by [4]

$$|pq;\zeta\rangle = \mathbf{D}(p,q)\mathbf{S}(\zeta)|0\rangle\ , \quad \text{where} \quad \zeta \equiv ye^{i\varphi} \quad (-\infty < y < \infty) \tag{3}$$

and $|0\rangle$ is the ground state of a specific harmonic oscillator, $\mathbf{a}|0\rangle = 0$. (i.e. $\mathbf{a}$ is the annihilation operator with a definite frequency ω_0; Henceforth, we set $\hbar = m = \omega_0 = 1$, for simplicity.) In (3)

$$\mathbf{D}(p,q) = \exp[-i(q\mathbf{P} - p\mathbf{Q})]\ , \tag{4}$$

is the displacement operator which generates the coherent states when applied to $|0\rangle$, and

$$\mathbf{S}(\zeta) = \exp\left[\frac{1}{2}\left(\zeta\mathbf{a}^{\dagger 2} - \zeta^{*}\mathbf{a}^{2}\right)\right]\ , \quad \left(\mathbf{a} \equiv \frac{\mathbf{Q}+i\mathbf{P}}{\sqrt{2}}\right) \tag{5}$$

is the squeezing operator, where the squeeze parameter y vanishes in the coherent-state limit.

If A is a density matrix ρ, then its Q function $Q(\rho;p,q;\zeta)$ can naturally be interpreted as a probability distribution. To emphasize this fact, the Q functions were denoted by P in [5, 6], instead of Q here.

For simplicity, I shall from now on discuss only squeezings which are pure boosts, without rotation, i.e. with $\varphi \equiv 0$ in (3), and use the squeezing parameter $\lambda := e^{2y}$ instead of y.

The Q and the Wigner functions are related as follows [1, 6]:

$$Q(A;p,q;\lambda) = \iint \frac{dp'dq'}{\pi} \exp[-\lambda^{-1}(p-p')^2 - \lambda(q-q')^2]\, W(A;p',q'). \tag{6}$$

In this paper, I shall first recall in Sec.2 that the Q functions (2) satisfy the partial differential equation (7). This equation describes how the Q functions $Q(p,q;\lambda)$ get changed in phase space (p,q) as the squeezing parameter λ is increased. In Sec.3 I define a forward-backward propagator for this equation. Finally, in Sec.4 I show that the Gaussian factor in the integral (6) is equal to a special case of the above propagator. This fact will yield the new interpretation of the Wigner function.

2 The Pseudo-Diffusion Equation

In previous papers [5, 6], it was shown that the Q functions, and other quantities, obey the following partial differential equation

$$\heartsuit(p,q;\lambda)\; Q(A;p,q;\lambda) \equiv \left[\frac{\partial}{\partial\lambda} - \frac{1}{4}\left(\frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right)\right] Q(A;p,q;\lambda) = 0\ , \qquad \text{where} \quad \lambda := e^{2y}\ , \tag{7}$$

where y is the squeezing parameter, as defined in (3). Eq. (7) was called [5, 6] *pseudo-diffusion equation*, because (a) it resembles the diffusion equation in 2 dimensions [7], where the parameter λ plays the role of time, and (b) the coefficients of $\frac{\partial^2}{\partial p^2}$ and $\frac{\partial^2}{\partial q^2}$ in (7) have opposite signs. Therefore, this equation describes a diffusive process in the p variable and an infusive one in the q variable *for all* λ. In this way a thin distribution along the q-axis get continuously deformed into a thin distribution along the p-axis, as λ is increased from 0 to ∞.

3 Solutions by Separation of Variables

The pseudo-diffusion equation (7) was solved by two methods [6]: by Fourier transform and by separation of variables. I shall now recall the latter method: Writing the solution as a product of two functions, $Q(p,q;\lambda) = \theta(p,\lambda)\psi(q,\lambda)$, where θ depends only on p and λ, and ψ depends only on q and λ, we get

$$\begin{aligned} 0 = \frac{1}{Q}\heartsuit Q &\equiv \frac{1}{\theta\psi}\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\left[\frac{\partial^2}{\partial p^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial q^2}\right]\right)\theta\psi \\ &= \frac{1}{\theta}\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta(p;\lambda) - \frac{1}{\psi}\left(-\frac{\partial}{\partial\lambda} - \frac{1}{4\lambda^2}\frac{\partial^2}{\partial q^2}\right)\psi(q;\lambda)\ . \end{aligned} \tag{8}$$

Since the first term in (8) depends only on p and λ, while the second term in (8) depends only on q and λ, we conclude that each of them must be equal to a function of λ only, which we denote by $f(\lambda)$. In [6] the solutions for $f(\lambda) \neq 0$ were discussed. But for my purposes here, I shall only consider the case $f(\lambda) = 0$. For this case equation (8) yields the following two equations:

$$\left(\frac{\partial}{\partial\lambda} - \frac{1}{4}\frac{\partial^2}{\partial p^2}\right)\theta(p;\lambda) = 0 \tag{9}$$

$$\left(-\frac{\partial}{\partial\lambda} - \frac{1}{4\lambda^2}\frac{\partial^2}{\partial q^2}\right)\psi(q;\lambda) = \frac{1}{\lambda^2}\left(\frac{\partial}{\partial\lambda^{-1}} - \frac{1}{4}\frac{\partial^2}{\partial q^2}\right)\psi(q;\lambda) = 0\ , \tag{10}$$

where $\frac{\partial}{\partial\lambda} = -\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda^{-1}}$ was used in (10). We see that θ obeys a 1-dimensional diffusion equation in p, where $\frac{1}{4}\lambda$ plays the role of time. Similarly, ψ obeys a diffusion equation in q, but with $\frac{1}{4}\lambda^{-1}$ playing the role of time. The solutions of the diffusion equation are well known [7]. In particular, the propagators of Eqs. (9) and (10) are specific solutions, given by

$$G_1(p-p', \lambda-\mu) = \frac{1}{\sqrt{\pi\,(\lambda-\mu)}}\exp\left[-\frac{(p-p')^2}{\lambda-\mu}\right]\ , \qquad \text{for} \quad \lambda > \mu\ , \tag{11}$$

$$G_1(q-q', \lambda^{-1}-\sigma^{-1}) = \frac{1}{\sqrt{\pi\,(\lambda^{-1}-\sigma^{-1})}}\exp\left[-\frac{(q-q')^2}{\lambda^{-1}-\sigma^{-1}}\right]\ , \qquad \text{for} \quad \lambda < \sigma\ . \tag{12}$$

Clearly, the products of the above two propagators yield a different solution of the pseudo-diffusion equation (7) for every 4-tupel (p', q', μ, σ):

$$G(p-p', q-q'; \lambda, \mu, \sigma) \equiv G_1(p-p', \lambda-\mu)\, G_1(q-q', \lambda^{-1}-\sigma^{-1}) \qquad \text{for} \quad \mu < \lambda < \sigma \tag{13}$$

$$= \frac{1}{\pi\sqrt{(\lambda-\mu)(\lambda^{-1}-\sigma^{-1})}}\exp\left[-\frac{(p-p')^2}{\lambda-\mu} - \frac{(q-q')^2}{\lambda^{-1}-\sigma^{-1}}\right]\ . \tag{14}$$

I shall call these G functions *two-sided or forward-backward propagators* of the pseudo-diffusion equation (7), because they involve the two squeezing parameters, μ and σ, which are *on opposite sides of* λ. In particular, these G solutions have the proper limit when λ is approached from opposite directions:

$$\lim_{\mu\to\lambda-\epsilon\,,\ \sigma\to\lambda+\epsilon} G(p-p', q-q'; \lambda, \mu, \sigma) = \delta(p-p')\delta(q-q')\ . \tag{15}$$

Since the heart operator $\heartsuit$ is a linear, any superposition of the above 2-sided propagators will also be a solution of the pseudo-diffusion equation. In particular, if we fix the squeezing parameters μ and σ and integrate only over p' and q', we get solutions of the form

$$f(p,q;\lambda,\lambda) = \iint dp'dq'\ G(p-p', q-q'; \lambda, \mu, \sigma)\ f(p', q'; \mu, \sigma), \qquad \text{for} \quad \sigma > \lambda > \mu\ , \tag{16}$$

for any given function $f(p, q; \mu, \sigma)$, provided that the integrals (16) exist.

4 The New Interpretation of the Wigner Function

An extreme case of the 2-sided propagators (14) is obtained by choosing $\mu = 0$ and $\sigma = \infty$. These squeezing parameters correspond to the values $-\infty$ and $+\infty$ of the $y = \frac{1}{2}\ln\lambda$ variable, respectively. For this choice of μ and σ, λ is free to take any positive value $\infty > \lambda > 0$. Moreover, the square-root factors in the two propagators cancel out. For this case, Eq. (16) becomes

$$f(p,q;\lambda,\lambda) = \iint \frac{dp'dq'}{\pi}\exp[-\lambda^{-1}(p-p')^2 - \lambda(q-q')^2]\, f(p', q'; 0, \infty), \qquad \text{for} \quad \lambda > 0\ . \tag{17}$$

If we compare (17) with the well known relation (6) between the Q function and the Wigner function, we realize immediately that these two functions are simply related by the special 2-sided propagator $G(p-p', q-q'; \lambda, 0, \infty)$. Therefore, we are led in a natural way to the interpretation that *the Wigner function is a Q function, which is squeezed to $y = +\infty$ in the q variable and anti-squeezed to $y = -\infty$ in the p variable.*

Note that by applying the following relation

$$\int \frac{dp'}{\sqrt{\pi\lambda}} \exp[-\lambda^{-1}(p-p')^2]\, g(p') = \exp\left[\frac{\lambda}{4}\frac{d^2}{dp^2}\right] g(p)\ , \qquad \text{for} \quad \lambda > 0\ , \tag{18}$$

to (17), we obtain a formal solution $f(p,q;\lambda,\lambda)$ of the pseudo-diffusion equation (7), in terms of a differential operator applied to an arbitrary function $g(p,q) \equiv f(p,q;0,\infty)$ of p and q:

$$f(p,q;\lambda,\lambda) = \exp\left[\frac{1}{4}\left(\lambda\frac{\partial^2}{\partial p^2} + \frac{1}{\lambda}\frac{\partial^2}{\partial q^2}\right)\right] f(p,q;0,\infty)\ . \tag{19}$$

One can easily check, by simple differentiation with respect to λ, that this formal solution satisfies the pseudo-diffusion equation (7). In particular, if $g(p,q)$ is equal to the Wigner function of an operator A, then $f(p,q;\lambda,\lambda)$ is the corresponding Q function. This formal relatonship between these two functions was noted by Husimi [3].

As an application, we note that the relation (6) holds for every operator A, so that the corresponding two operators in Eqs. (1) and (2) are also related by the above special propagator:

$$\Pi(p,q;\lambda) = \iint \frac{dp'dq'}{\pi} \exp[-\lambda^{-1}(p-p')^2 - \lambda(q-q')^2]\, \mathbf{W}(p',q')\ . \tag{20}$$

5 Conclusions

A one-sided propagator, which we would get for example from (14) by choosing $\mu, \sigma < \lambda$, is not suitable for the pseudo-diffusion equation (7), because one of the Gaussian factors in (14) will blow up at infinity. By showing that a special 2-sided propagator takes the Wigner function into a Q function, I concluded that the Wigner function can be regarded as a Q function, which is squeezed backwards ($\mu = 0$) in the p variable and forwards ($\sigma = \infty$) in q variable.

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