HIGHER–ORDER SQUEEZING OF QUANTUM FIELD AND THE GENERALIZED UNCERTAINTY RELATIONS IN NON–DEGENERATE FOUR–WAVE MIXING

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Abstract

It is found that the field of the combined mode of the probe wave and the phase–conjugate wave in the process of non–degenerate four–wave mixing exhibits higher–order squeezing to all even orders. And the generalized uncertainty relations in this process are also presented.

With the development of techniques for making higher–order correlation measurement in quantum optics, the new concept of higher–order squeezing of the single–mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985[1,2]. Lately Xi-zeng Li and Ying Shan have calculated the higher–order squeezing in the process of degenerate four–wave mixing[3] and presented the higher–order uncertainty relations of the fields in single–mode squeezed states[4]. As a natural generalization of Hong and Mandel's work, we introduced the theory of higher–order squeezing of the quantum fields in two–mode squeezed states in 1993. In this paper we study for the first time the higher–order squeezing of the quantum field and the generalized uncertainty relations in non–degenerate four–wave mixing (NDFWM) by means of the above theory.

1 Definition of higher–order squeezing of two mode quantum fields

The real two mode output field $\hat{E}$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E} = \hat{E}_1 cos(\Omega t - \phi) + \hat{E}_2 sin(\Omega t - \phi), \tag{1}$$

$$[\hat{E}_1, \hat{E}_2] = 2iC_0. \tag{2}$$

Then the field is squeezed to the Nth–order in $\hat{E}_1 (N = 1, 2, 3, \cdots)$ if there exists a phase angle ϕ such that $< (\Delta\hat{E}_1)^N >$ is smaller than its value in a completely two–mode coherent state of the field, viz.,

$$< (\Delta\hat{E}_1)^N > \quad < \quad < (\Delta\hat{E}_1)^N >_{two-mode \ \ coh.s.} \tag{3}$$

This is the definition of higher–order squeezing of two mode quantum fields.

2 Scheme for generation of higher–order squeezing via NDFWM

The scheme is shown in the following figure:

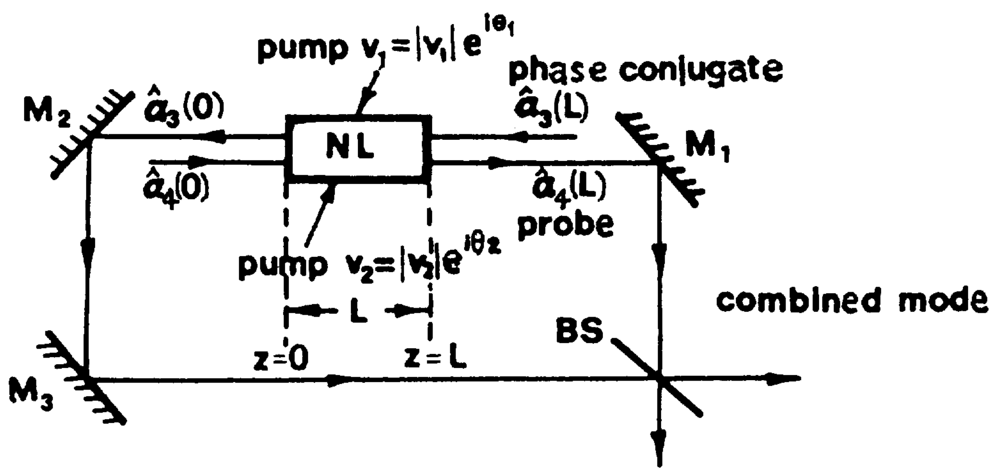

FIG. 1. Schematic for generation of higher–order squeezing via NDFWM. M_1, M_2, M_3 are mirrors, BS is the 50%–50% beam splitter

Where two strong, classical pump waves of complex amplitude ($v_1 = |v_1|e^{i\theta_1}$ and $v_2 = |v_2|e^{i\theta_2}$) with the same frequency Ω are incident on a nonlinear crystal possessing a third–order ($\chi^{(3)}$) nonlinearity. The length of the medium is L. $\hat{a}_4$ is the annihilation operator of the transmitted –probe wave with frequency $\omega_4, \hat{a}_3$ is the annihilation operator of the phase–conjugate wave with frequency ω_3, and

$$\Omega = \frac{\omega_3 + \omega_4}{2} \tag{4}$$

The effective Hamiltonian of this interaction system has the form of

$$\hat{H} = \hbar\omega_3\hat{a}_3^+\hat{a}_3 + \hbar\omega_4\hat{a}_4^+\hat{a}_4 + \hbar g_0(v_1 v_2 \hat{a}_3^+\hat{a}_4^+ e^{-2i\Omega t} + H.C) \tag{5}$$

where g_0 is the coupling const, t is the time propagation of light in NL crystal.

By solving the Heisenberg Equation of motion we get the output mode

$$\hat{a}_3(t) = [\mu\hat{a}_3(L) + \nu\hat{a}_4^+(0)]e^{-i\omega_3 t}, \quad (z = L - ct \quad \text{for} \quad \hat{a}_3) \tag{6}$$

$$\hat{a}_4(t) = [\mu\hat{a}_4(0) + \nu\hat{a}_3^+(L)]e^{-i\omega_4 t}, \quad (z = ct \quad \text{for} \quad \hat{a}_4) \tag{7}$$

where

$$\left.\begin{aligned} \mu &= sec|k|L, \\ \nu &= -ie^{i(\theta_1+\theta_2)}tan|k|L, \\ |k| &= \tfrac{g_0|v_1||v_2|}{c}. \end{aligned}\right\} \tag{8}$$

3 Combined mode and its quadrature components

It can be verified that the field of either $\hat{a}_3(0)$ or $\hat{a}_4(L)$ mode does not exhibit higher−order squeezing.

We consider the field of the combined mode of $\hat{a}_3(t)$ and $\hat{a}_4(t)$

$$\begin{aligned}\hat{E}(t) &= \sqrt{\frac{\omega_3}{2}}\hat{a}_3(t) - i\sqrt{\frac{\omega_4}{2}}\hat{a}_4(t) + (H.C) \\ &= \sqrt{\frac{\Omega}{2}}\lambda_3\hat{a}_3(t) - i\sqrt{\frac{\Omega}{2}}\lambda_4\hat{a}_4(t) + (H.C)\end{aligned} \tag{9}$$

where

$$\lambda_3 = \sqrt{\frac{\omega_3}{\Omega}}, \lambda_4 = \sqrt{\frac{\omega_4}{\Omega}} \tag{10}$$

and $-i$ denotes the phase delay. The units are chosen so that $\hbar = c = 1$.

$\hat{E}(t)$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E}(t) = \hat{E}_1 cos(\Omega t - \phi) + \hat{E}_2 sin(\Omega t - \phi), \tag{11}$$

where

$$\Omega = \frac{\omega_3 + \omega_4}{2}, \tag{12}$$

and ϕ is an arbitrary phase angle that may be chosen at will.

$\hat{E}_1$ can be expressed in term of initial modes $\hat{a}_3(L)$ and $\hat{a}_4(0)$,

$$\hat{E}_1 = g\hat{a}_3(L) + h\hat{a}_4(0) + g^*\hat{a}_3^+(L) + h^*\hat{a}_4^+(0), \tag{13}$$

where

$$g = \sqrt{\frac{\Omega}{2}}[\lambda_3\mu e^{-i\phi} + \lambda_4\nu^* e^{i(\phi+\pi/2)}]e^{i\epsilon t}, \tag{14}$$

$$h = \sqrt{\frac{\Omega}{2}}[\lambda_4\mu e^{-i(\phi+\pi/2)} + \lambda_3\nu^* e^{i\phi}]e^{-i\epsilon t}, \tag{15}$$

$$\epsilon = \Omega - \omega_3 = \omega_4 - \Omega, \tag{16}$$

ϵ is the modulation frequency.

Now we define

$$\hat{B} = g\hat{a}_3(L) + h\hat{a}_4(0), \tag{17}$$

$$\hat{B}^+ = g^*\hat{a}_3^+(L) + h^*\hat{a}_4^+(0), \tag{18}$$

then

$$\hat{E}_1 = \hat{B} + \hat{B}^+, \tag{19}$$

where $\hat{B}^+$ is the adjoint of $\hat{B}$.

4 Higher–order noise moment $<(\Delta\hat{E}_1)^N>$ and higher –order squeezing

By using the Campbell–Baker–Hausdorff formula, we get the Nth–order moment of $\Delta\hat{E}_1$,

$$\begin{aligned} <(\Delta\hat{E}_1)^N> &= <::(\Delta\hat{E}_1)^N::> + \frac{N^{(2)}}{1!}(\frac{1}{2}C_0) <::(\Delta\hat{E}_1)^{N-2}::> + \frac{N^{(4)}}{2!}(\frac{1}{2}C_0)^2 \\ & <::(\Delta\hat{E}_1)^{N-4}::> \\ & + \cdots + (N-1)!!C_0^{N/2}. \qquad \text{(N is even)} \end{aligned} \tag{20}$$

where

$$N^{(r)} = N(N-1)\cdots(N-r+1), \quad C_0 = \frac{1}{2i}[\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^+], \tag{21}$$

and :: :: denotes normal ordering with respect to $\hat{B}$ and $\hat{B}^+$.

We take the initial quantum state to be $|\alpha>_4 |0>_3$, which is a product of the coherent state $|\alpha>_4$ for $\hat{a}_4(0)$ mode and the vacuum state for $\hat{a}_3(L)$ mode. Since $|\alpha>_4 |0>_3$ is the eigenstate of $\hat{B}$, we get

$$\begin{aligned} <::(\Delta\hat{E}_1)^N::> &= <::(\Delta\hat{B} + \Delta\hat{B}^+)^N::> \\ &= \sum_{\gamma=0}^{N} \begin{bmatrix} N \\ \gamma \end{bmatrix} \ {}_3<0|_4<\alpha| :: (\Delta\hat{B}^+)^\gamma (\Delta\hat{B})^{N-\gamma} :: |\alpha>_4 |0>_3 = 0. \end{aligned} \tag{22}$$

Then from (20),

$$<(\Delta\hat{E}_1)^N> = (N-1)!!C_0^{N/2}, \tag{23}$$

$$\begin{aligned} C_0 &= [\hat{B}, \hat{B}^+] = |g|^2 + |h|^2, \\ &= \frac{\Omega}{2}\{(\lambda_3^2 + \lambda_4^2)(|\mu|^2 + |\nu|^2) + 2\lambda_3\lambda_4[\mu^*\nu^* e^{i(2\phi + \frac{\pi}{2})} + \mu\nu e^{-i(2\phi + \frac{\pi}{2})}]\}. \end{aligned} \tag{24}$$

where

$$\lambda_3^2 + \lambda_4^2 = 2, \qquad \lambda_3\lambda_4 = \sqrt{1 - \frac{\epsilon^2}{\Omega^2}}.$$

Substituting eqs. (8), (10), (24) into (23), we get the Nth–order moment of $\Delta\hat{E}_1$,

$$\begin{aligned} <(\Delta\hat{E}_1)^N> &= (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L \\ &- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L cos(2\phi - \theta_1 - \theta_2)]^{N/2}. \end{aligned} \tag{25}$$

If ϕ is chosen to satisfy

$$2\phi - \theta_1 - \theta_2 = 0, \quad \text{or} \quad cos(2\phi - \theta_1 - \theta_2) = 1,$$

then the above eq. (25) leads to the result

$$\begin{aligned} <(\Delta\hat{E}_1)^N> &= (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L \\ &- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2}} sec|k|L tan|k|L]^{N/2}. \end{aligned} \tag{26}$$

When $0 < |k|L < \pi$, the right-hand side is less than $(N-1)!!\Omega^{N/2}$, which is the corresponding Nth-order moment for two-mode coherent states. It follows that the field of the combined mode of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders.

The squeeze parameter q_N for measuring the degree of Nth-order squeezing is

$$q_N = \frac{<(\Delta\hat{E}_1)^N> - <(\Delta\hat{E}_1)^N>_{two-mode\ coh.s}}{<(\Delta\hat{E}_1)^N>_{two-mode\ coh.s}} \tag{27}$$

$$= [sec^2|k|L + tan^2|k|L - 2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}sec|k|Ltan|k|L]^{N/2} - 1. \tag{28}$$

We find that q_N is negative, and q_N increases with N. This gives out the conclusion that the degree of higher-order squeezing is greater than that of the second order.

5 Generalized uncertainty relations in NDFWM

$\hat{E}_2$ can be regarded as a special case of $\hat{E}_1$ if ϕ is replaced by $\phi + \pi/2$. Then if ϕ is chosen to satisfy $2\phi - \theta_1 - \theta_2 = 0$, from eq. (25) it follows that

$$<(\Delta\hat{E}_2)^N> = (N-1)!!\Omega^{N/2}[sec^2|k|L + tan^2|k|L + 2\sqrt{1-\frac{\epsilon^2}{\Omega^2}}sec|k|Ltan|k|L]^{N/2}. \tag{29}$$

when $0 < |k|L < \pi$, the right-hand side is greater than $(N-1)!!\Omega^{N/2}$.

From eqs. (26) and (29), we obtain

$$<(\Delta\hat{E}_1)^N> \cdot <(\Delta\hat{E}_2)^N> = [(N-1)!!]^2\Omega^N[1 + 4\frac{\epsilon^2}{\Omega^2}sec^2|k|Ltan^2|k|L]^{N/2}. \tag{30}$$

Eq. (30) shows that $<(\Delta\hat{E}_1)^N>$ and $<(\Delta\hat{E}_2)^N>$ can not be made arbitrarily small simultaneously. We call eq. (30) the generalized uncertainty relations in NDFWM, and the right-hand side is dependent on ϵ, Ω, N, and $|k|L$.

In the degenerate case $\omega_4 = \omega_2 = \Omega, \epsilon = 0$ from eqs. (26), (28) and (30) we obtain

$$<(\Delta\hat{E}_1)^N> = (N-1)!!\Omega^{N/2}[sec|k|L - tan|k|L]^N, \tag{31}$$

$$q_N = [sec|k|L - tan|k|L]^N - 1, \tag{32}$$

$$<(\Delta\hat{E}_1)^N> \cdot <(\Delta\hat{E}_2)^N> = [(N-1)!!]^2 \cdot \Omega^N. \tag{33}$$

When $N = 2$,

$$<(\Delta\hat{E}_1)^2> = \Omega[sec|k|L - tan|k|L]^2, \tag{34}$$

$$q_2 = [sec|k|L - tan|k|L]^2 - 1, \tag{35}$$

$$< (\Delta\hat{E}_1)^2 > \cdot < (\Delta\hat{E}_2)^2 > = \Omega^2 \quad (36)$$

These results are in agreement with the conclusions in the previous relevant references[3][5].

6 Acknowledgements

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