The Total Gaussian Class of Quasiprobabilities and its Relation to Squeezed-state Excitations

Alfred Wünsche
Arbeitsgruppe "Nichtklassische Strahlung" der Max-Planck-Gesellschaft
Rudower Chaussee 5, 12489 Berlin, Germany

Abstract

The class of quasiprobabilities obtainable from the Wigner quasiprobability by convolutions with the general class of Gaussian functions is investigated. It can be described by a three-dimensional, in general, complex vector parameter with the property of additivity when composing convolutions. The diagonal representation of this class of quasiprobabilities is connected with a generalization of the displaced Fock states in direction of squeezing. The subclass with real vector parameter is considered more in detail. It is related to the most important kinds of boson operator ordering. The properties of a specific set of discrete excitations of squeezed coherent states are given.

1 Introduction

The representation of density operators by quasiprobabilities forms one of the bridges between classical and quantum mechanics. Whereas the classical distribution function is uniquely defined and gives the probability density to find the system at the corresponding point of the phase space, a quantum-mechanical distribution function over the phase space is uniquely defined only in relation to a certain operator ordering and does not possess all properties of a true probability density, for example, positive definiteness or orthonormality of the involved states. The best compromise between classical and quantum mechanics is given by the Wigner quasiprobability $W(\alpha, \alpha^*)$ introduced by Wigner in 1932 [1] and corresponding to symmetrical (Weyl) ordering. However, other quasiprobabilities are in use and sometimes advantageous as the coherent-state quasiprobability $Q(\alpha, \alpha^*)$, the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$, or the one-parameter class of s-ordered quasiprobabilities ($-1 \leq s \leq +1$) which linearly interpolates between the coherent-state quasiprobability and the Glauber-Sudarshan quasiprobability with the Wigner quasiprobability in its center [2, 3, 4]. The quasiprobabilities are auxiliary functions in analogy to the classical distribution function and are appropriate for the convenient calculation of expectation values of operators being invariant quantities in quantum mechanics. Therefore, each of the quasiprobabilities must carry the complete information of the density operator and a reconstruction of the density operator from the quasiprobability must be possible. We consider here the general three-parameter class of quasiprobabilities obtainable by convolutions of the Wigner quasiprobability with the total class of normalized Gaussian functions of the phase-space variables and call this the total Gaussian class of quasiprobabilities. In particular, it contains the quasiprobabilities related to standard and antistandard ordering of the canonical operators and the linear interpolation between them.

2 The displacement structure of the quasiprobabilities

A strong and important restriction to the form of quasiprobabilities over a phase space with the topology of a plane results from the requirement that displacements of the whole system in the phase plane (Heisenberg-Weyl group) must lead to correspondingly displaced quasiprobabilities in analogy to classical mechanics. If the transition from the density operator ϱ to a normalized quasiprobability $F(\alpha, \alpha^*)$ is written by a transition operator $T(\alpha, \alpha^*)$ as follows

$$F(\alpha,\alpha^*) = \langle \varrho T(\alpha,\alpha^*) \rangle, \quad \int \frac{i}{2} d\alpha \wedge d\alpha^* \, F(\alpha,\alpha^*) = 1,$$
$$\frac{i}{2} d\alpha \wedge d\alpha^* = d\,\mathrm{Re}\,(\alpha) \wedge d\,\mathrm{Im}\,(\alpha), \quad \langle \ldots \rangle \equiv \mathrm{Trace}(\ldots), \tag{1}$$

then the requirement regarding displacements implies the following "displacement structure" of the transition operators

$$\begin{aligned} T(\alpha,\alpha^*) &= D(\alpha,\alpha^*) T(0,0) (D(\alpha,\alpha^*))^\dagger, \\ \int \frac{i}{2} d\alpha \wedge d\alpha^* \, T(\alpha,\alpha^*) &= I, \quad \langle T(\alpha,\alpha^*) \rangle = \langle T(0,0) \rangle = \frac{1}{\pi}, \end{aligned} \tag{2}$$

where the displacement operator $D(\alpha, \alpha^*)$ is defined by

$$D(\alpha,\alpha^*) \equiv \exp(\alpha a^\dagger - \alpha^* a), \quad [a, a^\dagger] = I, \tag{3}$$

with a and $a^\dagger$ as the boson annihilation and creation operator and with I as the unity operator. This means that the transition operators $T(\alpha, \alpha^*)$ provide a phase-space decomposition of the unity operator. The given trace of the transition operators is a consequence of the following identity which can be proved for arbitrary operators A [5, 6]

$$\int \frac{i}{2} d\alpha \wedge d\alpha^* \, D(\alpha,\alpha^*) A (D(\alpha,\alpha^*))^\dagger = \pi \langle A \rangle I. \tag{4}$$

The reconstruction of the density operator ϱ from the quasiprobability $F(\alpha, \alpha^*)$ can be made by an operator $\overline{T}(\alpha, \alpha^*)$ in the following way

$$\varrho = \pi \int \frac{i}{2} d\alpha \wedge d\alpha^* \, F(\alpha,\alpha^*) \overline{T}(\alpha,\alpha^*), \tag{5}$$

under the condition

$$\langle \overline{T}(\alpha,\alpha^*) T(\beta,\beta^*) \rangle = \frac{1}{\pi} \delta(\alpha - \beta, \alpha^* - \beta^*). \tag{6}$$

It can be proved that the operator $\overline{T}(\alpha, \alpha^*)$ possesses the same "displacement structure" as the operator $T(\alpha, \alpha^*)$ with all its consequences (phase-space decomposition of the unity operator, trace equal to $1/\pi$, see [6]).

3 The three-parameter Gaussian class of quasiprobabilities

The discussed restrictions from the displacement structure of the quasiprobabilities admit still a rich variety of possible quasiprobabilities. We consider here the three-parameter class of quasiprobabilities $F_{\boldsymbol{r}}(\alpha, \alpha^*)$ with the vector parameter $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ which can be obtained from the Wigner quasiprobability $W(\alpha, \alpha^*) \equiv F_{\boldsymbol{0}}(\alpha, \alpha^*)$, $(\boldsymbol{0} \equiv (0,0,0))$, by the following convolutions

$$\begin{aligned} F_{(r_1,r_2,r_3)}(\alpha, \alpha^*) &= g_{(r_1,r_2,r_3)}(\alpha, \alpha^*) * W(\alpha, \alpha^*) \\ &= \tilde{g}_{(r_1,r_2,r_3)}\left(\frac{2}{i}\frac{\partial}{\partial \alpha^*}, \frac{2}{i}\frac{\partial}{\partial \alpha}\right) W(\alpha, \alpha^*), \end{aligned} \tag{7}$$

with the normalized Gaussian functions g or their Fourier transforms $\tilde{g}$

$$\begin{aligned} g_{(r_1,r_2,r_3)}(\alpha, \alpha^*) &\equiv \frac{2}{\sqrt{\boldsymbol{r}^2}\pi} \exp\left\{ -\frac{1}{\boldsymbol{r}^2}\Big(r_1(\alpha^2 - \alpha^{*2}) + i r_2(\alpha^2 + \alpha^{*2}) + r_3 2\alpha\alpha^*\Big)\right\}, \\ \tilde{g}_{(r_1,r_2,r_3)}\left(\frac{2}{i}\frac{\partial}{\partial \alpha^*}, \frac{2}{i}\frac{\partial}{\partial \alpha}\right) &\equiv \exp\left\{\frac{1}{4}\left(r_1\left(\frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \alpha^{*2}}\right) - i r_2\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \alpha^{*2}}\right) + r_3 2\frac{\partial^2}{\partial \alpha \partial \alpha *}\right)\right\}, \\ \boldsymbol{r}^2 &\equiv r_1^2 + r_2^2 + r_3^2. \end{aligned} \tag{8}$$

This total Gaussian class of quasiprobabilities with, in general, complex vector parameters $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ contains the class of s-ordered quasiprobabilities as the special case $F_{(0,0,r_3)}(\alpha, \alpha^*)$ with real $r_3 = -s$. The subclass $F_{(r_1,0,0)}(\alpha, \alpha^*)$ with real r_1 and $-1 \leq r_1 \leq +1$ is related to the linear interpolation between standard and antistandard ordering of powers of the canonical operators Q and P that is considered more in detail in [6]. The connection between two arbitrary quasiprobabilities with the vector parameters $\boldsymbol{r}$ and $\boldsymbol{s}$ is given by

$$F_{\boldsymbol{r}}(\alpha, \alpha^*) = g_{\boldsymbol{r}-\boldsymbol{s}}(\alpha, \alpha^*) F_{\boldsymbol{s}}(\alpha, \alpha^*), \tag{9}$$

and the reconstruction of the density operator ϱ by

$$\varrho = \pi \int \frac{i}{2} d\alpha \wedge d\alpha^* \, F_{\boldsymbol{r}}(\alpha, \alpha^*) T_{-\boldsymbol{r}}(\alpha, \alpha^*). \tag{10}$$

An interesting subclass of the total Gaussian class of quasiprobabilities is given by the restriction to real vector parameters $\boldsymbol{r} \equiv (r_1, r_2, r_3))$ and by $\boldsymbol{r}^2 \leq 1$. The "diagonal representation" of this subclass leads to a generalization of the displaced Fock states in direction of a kind of displaced squeezed Fock states as we now will show.

4 Diagonal representation of the Gaussian class of quasiprobabilities with real vector parameters

From the Fock-state representation of the operator $T(0,0)$ in Eq.(2) in connection with Eq.(1) one obtains the following, in general, nondiagonal representation of the quasiprobabilities in displaced

Fock states $|\alpha, n\rangle$

$$\begin{aligned} F(\alpha,\alpha^*) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|T(0,0)|n\rangle\langle\alpha,n|\varrho|\alpha,m\rangle, \qquad \sum_{n=0}^{\infty}\langle n|T(0,0)|n\rangle = \frac{1}{\pi}, \\ |\alpha,n\rangle &\equiv D(\alpha,\alpha^*)|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger - \alpha^* I)^n|\alpha\rangle. \end{aligned} \tag{11}$$

The s-ordered class of quasiprobabilities is diagonal in the representation by the displaced Fock states according to ($s = -r_3$)

$$F_{(0,0,r_3)}(\alpha,\alpha^*) = \frac{2}{(1+r_3)\pi}\sum_{n=0}^{\infty}\left(-\frac{1-r_3}{1+r_3}\right)^n\langle\alpha,n|\varrho|\alpha,n\rangle. \tag{12}$$

The more general Gaussian class of quasiprobabilities with real vector parameters $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ can be diagonalized in the following way (proof is given in [6])

$$\begin{aligned} F_{(r_1,r_2,r_3)}(\alpha,\alpha^*) &= \frac{2}{(1+r)\pi}\sum_{n=0}^{\infty}\left(-\frac{1-r}{1+r}\right)^n\left\langle\alpha,n;\frac{r_1-ir_2}{r+r_3}\right|\varrho\left|\alpha,n;-\frac{r_1-ir_2}{r+r_3}\right\rangle, \\ r &\equiv \sqrt{r_1^2+r_2^2+r_3^2}, \quad 0 \le \frac{r_3}{r} \le 1, \end{aligned} \tag{13}$$

where we have introduced a set of discrete excitations of squeezed coherent states $|\alpha, n; \zeta\rangle$ with a complex squeezing parameter ζ in the nonunitary approach as follows

$$\begin{aligned} |\alpha,n;\zeta\rangle &\equiv D(\alpha,\alpha^*)\frac{1}{\sqrt{n!}}\left(\frac{a^\dagger-\zeta^* a}{\sqrt{1+\zeta\zeta^*}}\right)^n|0,0;\zeta\rangle, \\ |0,0;\zeta\rangle &\equiv (1+\zeta\zeta^*)^{\frac{1}{4}}\exp\left(-\frac{\zeta}{2}a^{\dagger 2}\right)|0\rangle = (1+\zeta\zeta^*)^{\frac{1}{4}}\sum_{m=0}^{\infty}\frac{(-1)^m\sqrt{(2m)!}}{2^m m!}\zeta^m|2m\rangle. \end{aligned} \tag{14}$$

The states $|\alpha, m; -\zeta\rangle$ and $|\alpha, n; \zeta\rangle$ with opposite squeezing parameters ζ are mutually orthonormalized and satisfy a completeness relation in the following way

$$\langle\alpha,m;-\zeta|\alpha,n;\zeta\rangle = \delta_{m,n}, \qquad \sum_{n=0}^{\infty}|\alpha,n;\zeta\rangle\langle\alpha,n;-\zeta| = I. \tag{15}$$

In case of vanishing squeezing parameter $\zeta = 0$ the states $|\alpha, n; \zeta\rangle$ become identical with the displaced Fock states $|\alpha, n\rangle$

$$|\alpha,n;0\rangle \equiv D(\alpha,\alpha^*)|n\rangle \equiv |\alpha,n\rangle. \tag{16}$$

Consider now the limiting case of maximal squeezing $|\zeta| = 1$ within the states $|\alpha, n; \zeta\rangle$. If one makes the transition to real variables q and p and to the canonical Hermitean operators Q and P according to

$$\alpha = \frac{q+ip}{\sqrt{2\hbar}}, \quad \alpha^* = \frac{q-ip}{\sqrt{2\hbar}}, \quad a = \frac{Q+iP}{\sqrt{2\hbar}}, \quad a^\dagger = \frac{Q-iP}{\sqrt{2\hbar}}, \tag{17}$$

then, in particular, one obtains

$$\begin{aligned}\left|\frac{q+ip}{\sqrt{2\hbar}},n;1\right\rangle &= D(q,p)\frac{(2\hbar\pi)^{\frac{1}{4}}}{\sqrt{n!}}\left(\frac{P}{i\sqrt{\hbar}}\right)^{n}|q=0\rangle \\ &= \exp\left(i\frac{pq}{2\hbar}\right)\frac{(2\hbar\pi)^{\frac{1}{4}}}{(i\sqrt{\hbar})^{n}\sqrt{n!}}(P-pI)^{n}|q\rangle, \\ \left|\frac{q+ip}{\sqrt{2\hbar}},n;-1\right\rangle &= D(q,p)\frac{(2\hbar\pi)^{\frac{1}{4}}}{\sqrt{n!}}\left(\frac{Q}{\sqrt{\hbar}}\right)^{n}|p=0\rangle \\ &= \exp\left(-i\frac{pq}{2\hbar}\right)\frac{(2\hbar\pi)^{\frac{1}{4}}}{(\sqrt{\hbar})^{n}\sqrt{n!}}(Q-qI)^{n}|p\rangle, \end{aligned} \tag{18}$$

where $D(q,p)$ denotes the displacement operator in the representation by the real variables q and p and $|q\rangle$ and $|p\rangle$ are the eigenstates of the operators Q and P, respectively, normalized in the usual way by means of the delta functions with the scalar product $\sqrt{2\hbar\pi}\langle q|p\rangle = \exp\left((ipq)/\hbar\right)$. The states in Eq.(18) represent discrete sets of excitations of the states $|q\rangle$ and $|p\rangle$ in analogy to the displaced Fock states $|\alpha,n\rangle$ as discrete sets of excitations of the coherent states $|\alpha\rangle$.

The states $|\alpha,n;\zeta\rangle$ with $|\zeta|>1$ are well defined by Eq.(14) but they are not normalizable in the usual sense or by means of the delta function. They are states of certain rigged Hilbert spaces since their scalar products with itself does not exit but it exists the scalar product with states from spaces of sufficiently well-behaved normalizable states that can be used for auxiliary purposes, for example, for the formulation of completeness relations on contours of the complex variable α. In this connection we introduce the following terminology of normalizability of states:

1. normalizable (scalar product of the state with itself exists meaning that they are states of the usual Hilbert space; case $|\zeta|<1$ in Eq.(14)),

2. weakly nonnormalizable (states can be considered as limiting cases of normalizable states or states of a certain rigged Hilbert space and can often be normalized with "neighbouring" states by means of the delta function; case $|\zeta|=1$ in Eq.(14)),

3. strongly nonnormalizable (states cannot be considered as limiting cases of normalizable states but they are states of more general rigged Hilbert spaces or spaces of linear functionals; case $|\zeta|>1$ in Eq.(14)).

If one admits strongly nonnormalizable states in Eq.(13) in a formal way, then one may omit the restriction to nonnegative values of r_3/r. In the case $r_3=0$ one has to do with weakly nonnormalizable states corresponding to $|\zeta|=1$ and both possible signs of the square root in $r=\sqrt{r_1^2+r_2^2}$ are admissible leading to two possible representations of equal rank.

5 The sphere of the Gaussian class of quasiprobabilities with real vector parameters

As the main class of quasiprobabilities, the Gaussian subclass of quasiprobabilities with real vector parameter $\boldsymbol{r}\equiv(r_1,r_2,r_3)$ and with $\boldsymbol{r}^2\leq 1$ forms the interior plus surface of a three-dimensional sphere with the Wigner quasiprobability $W(\alpha,\alpha^*)$ in its center, the coherent-state quasiprobability $Q(\alpha,\alpha^*)$ in the North pole, the Glauber-Sudarshan quasiprobability $P(\alpha,\alpha^*)$ in the South pole and the quasiprobabilities $F_{(\cos 2\varphi,\sin 2\varphi,0)}(\alpha,\alpha^*)$ corresponding to standard or antistandard ordering of the rotated canonical operators Q and P about an angle φ around the Equator (see fig.1 in [6]). Whereas at the surface of this sphere the quasiprobabilities are representable as the expectation values of transition operators of the dyadic form $1/\pi|\alpha,0;\zeta\rangle\langle\alpha,0;-\zeta|$ with squeezing parameters $\pm\zeta$ fixed for each diagonal through the center of the sphere (if we admit strongly nonnormalizable states; in the other case this is only true for the upper hemisphere), in the interior one has mixed states of $|\alpha,n;\zeta\rangle\langle\alpha,n;-\zeta|$, $(n=0,\ldots,\infty)$ as transition operators. This is in a certain analogy to the Poincaré sphere of pure and mixed polarization states where the pure polarization states are situated on the surface of this sphere (right-handed and left-handed circular polarization at the North and South pole and the different linear polarizations around the Equator in dependence on the direction of linear polarization, elliptical polarizations on general surface points) and the mixed polarizations in the interior of the sphere with the fully unpolarized state in the center.

6 Some representations of the states $|\beta,n;\zeta\rangle$

It is interesting to consider the properties of the states $|\beta,n;\zeta\rangle$ itself by the calculation of different representations and quasiprobabilities. These states comprise the squeezed coherent states as the special case $|\beta,0;\zeta\rangle$. We introduced these states in Eq.(14) in a nonnormalized form. First, a normalization factor can be calculated from the following scalar product (see [6])

$$\begin{aligned}\langle\beta,n;\zeta|\beta,n;\zeta\rangle &= \langle 0,n;\zeta|0,n;\zeta\rangle \\ &= \left(\frac{1+\zeta\zeta^*}{1-\zeta\zeta^*}\right)^{n+\frac{1}{2}}\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!^2(n-2k)!}\left(\frac{\sqrt{\zeta\zeta^*}}{1+\zeta\zeta^*}\right)^{2k}. \end{aligned} \tag{19}$$

The polynomials at the right-hand side of Eq.(19) do not belong, at least, to well-known polynomials with a fixed abbreviation.

Next, we calculate the Bargmann representation of the nonnormalized states $|\beta,n;\zeta\rangle$ with the following result of an analytic function of α^*

$$\begin{aligned} f(\alpha^*) &\equiv \langle 0|\exp(\alpha^* a)|\beta,n;\zeta\rangle \\ &= \frac{(1+\zeta\zeta^*)^{\frac{1}{4}}}{\sqrt{n!}}\left(\sqrt{\frac{\zeta^*}{2}}\right)^n H_n\left(\sqrt{\frac{1+\zeta\zeta^*}{2\zeta^*}}(\alpha^*-\beta^*)\right)\exp\left\{-\frac{\zeta}{2}(\alpha^*-\beta^*)^2+\alpha^*\beta-\frac{1}{2}\beta\beta^*\right\}, \end{aligned} \tag{20}$$

where $H_n(z)$ denotes the Hermite polynomials in the usual way. For the "position"representation one obtains

$$\begin{aligned} &\langle q|\beta,n;\zeta\rangle \\ &= \frac{1}{\sqrt{2^n n!}}\left(\sqrt{\frac{1+\zeta^*}{1-\zeta}}\right)^n H_n\left(\sqrt{\frac{1+\zeta\zeta^*}{(1-\zeta)(1+\zeta^*)\hbar}}\left(q-\sqrt{\frac{\hbar}{2}}(\beta+\beta^*)\right)\right) \end{aligned}$$

$$\left(\frac{1+\zeta\zeta^*}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{1-\zeta}}\exp\left\{-\frac{1+\zeta}{1-\zeta}\frac{1}{2\hbar}\left(q-\sqrt{\frac{\hbar}{2}}(\beta+\beta^*)\right)^2+\frac{(\beta-\beta^*)q}{\sqrt{2\hbar}}-\frac{\beta^2-\beta^{*2}}{4}\right\}, \tag{21}$$

and for the "momentum" representation

$$\begin{aligned}&\langle p|\beta,n;\zeta\rangle\\ =\ &\frac{(-i)^n}{\sqrt{2^n n!}}\left(\sqrt{\frac{1-\zeta^*}{1+\zeta}}\right)^n H_n\left(\sqrt{\frac{1+\zeta\zeta^*}{(1+\zeta)(1-\zeta^*)\hbar}}\left(p+i\sqrt{\frac{\hbar}{2}}(\beta-\beta^*)\right)\right)\\ &\left(\frac{1+\zeta\zeta^*}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{1+\zeta}}\exp\left\{-\frac{1-\zeta}{1+\zeta}\frac{1}{2\hbar}\left(p+i\sqrt{\frac{\hbar}{2}}(\beta-\beta^*)\right)^2-i\frac{p(\beta+\beta^*)}{\sqrt{2\hbar}}+\frac{\beta^2-\beta^{*2}}{4}\right\}.\end{aligned} \tag{22}$$

From Eqs.(20) and (19) one finds the coherent-state quasiprobability $Q(\alpha,\alpha^*)$ for the normalized states $|\beta,n;\zeta\rangle_{norm}$. We give it only for the states $|0,n;\zeta\rangle_{norm}$ because the transition to the states $|\beta,n;\zeta\rangle_{norm}$ can be simply made by the substitutions $\alpha\rightarrow\alpha-\beta$ and $\alpha^*\rightarrow\alpha^*-\beta^*$. The result for $|0,n;\zeta\rangle_{norm}$ is

$$\begin{aligned}Q(\alpha,\alpha^*)&\equiv\frac{1}{\pi}\frac{\langle\alpha|0,n;\zeta\rangle\langle 0,n;\zeta|\alpha\rangle}{\langle 0,n;\zeta|0,n;\zeta\rangle}\\ &=\frac{\frac{1}{n!}\left(\frac{\sqrt{\zeta\zeta^*}}{2}\frac{1-\zeta\zeta^*}{1+\zeta\zeta^*}\right)^n}{\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!^2(n-2k)!}\left(\frac{\sqrt{\zeta\zeta^*}}{1+\zeta\zeta^*}\right)^{2k}}H_n\left(\sqrt{\frac{1+\zeta\zeta^*}{2\zeta}}\alpha\right)H_n\left(\sqrt{\frac{1+\zeta\zeta^*}{2\zeta^*}}\alpha^*\right)\\ &\quad\frac{\sqrt{1-\zeta\zeta^*}}{\pi}\exp\left\{-\left(\alpha\alpha^*+\frac{\zeta^*}{2}\alpha^2+\frac{\zeta}{2}\alpha^{*2}\right)\right\}.\end{aligned} \tag{23}$$

By convolution of $Q(\alpha,\alpha^*)$ with $2/\pi\exp(2\alpha\alpha^*)$ one obtains from Eq.(23) the Wigner quasiprobability for the normalized states $|0,n;\zeta\rangle_{norm}$ with the result

$$\begin{aligned}W(\alpha,\alpha^*)&=\frac{(-1)^n}{\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!^2(n-2k)!}\left(\frac{\sqrt{\zeta\zeta^*}}{1+\zeta\zeta^*}\right)^{2k}}\\ &\quad\sum_{j=0}^{n}\frac{(-1)^j n!}{j!^2(n-j)!}\left(\frac{\sqrt{\zeta\zeta^*}}{1+\zeta\zeta^*}\right)^j H_j\left(\sqrt{\frac{1+\zeta\zeta^*}{1-\zeta\zeta^*}}\frac{\alpha+\zeta\alpha^*}{\sqrt{\zeta}}\right)H_j\left(\sqrt{\frac{1+\zeta\zeta^*}{1-\zeta\zeta^*}}\frac{\alpha^*+\zeta^*\alpha}{\sqrt{\zeta^*}}\right)\\ &\quad\frac{2}{\pi}\exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)}{1-\zeta\zeta^*}\right\}.\end{aligned} \tag{24}$$

The transition from $|0,n;\zeta\rangle_{norm}$ to $|\beta,n;\zeta\rangle_{norm}$ can be made again in Eq.(24) by the simple substitutions $\alpha\rightarrow\alpha-\beta$ and $\alpha^*\rightarrow\alpha^*-\beta^*$. In figs.(1-6) we represent the Wigner quasiprobability in its real representation $W(q,p)$ with the normalization $\int dq\wedge dpW(q,p)=1$ for the first 6 states

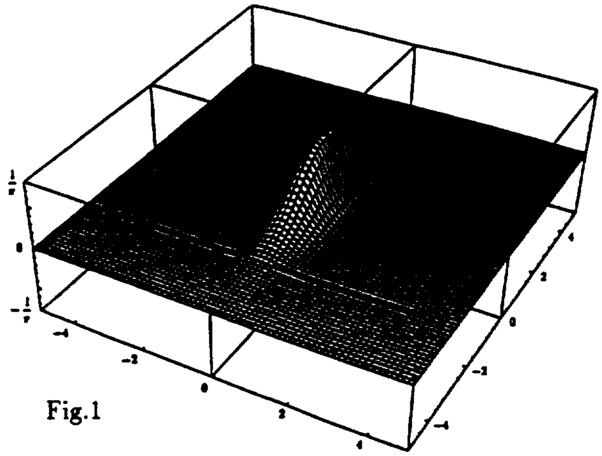

Fig.1

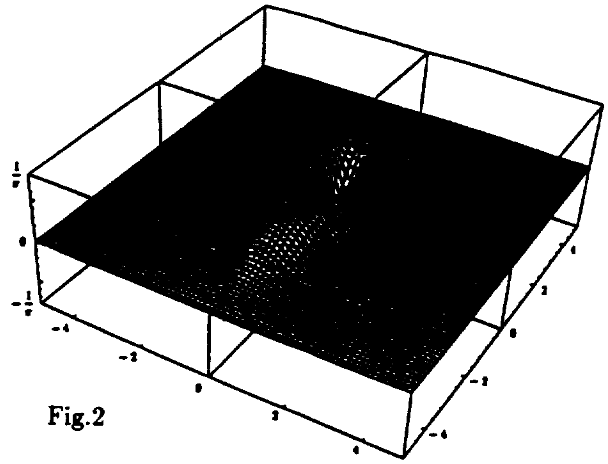

Fig.2

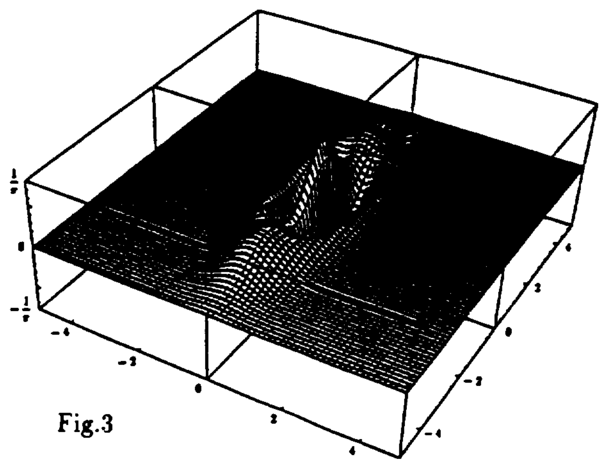

Fig.3

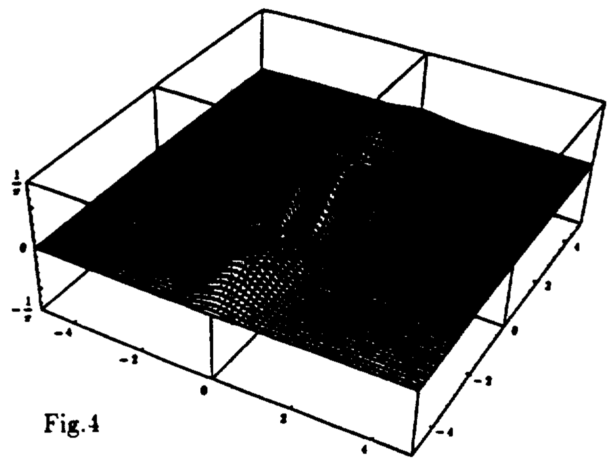

Fig.4

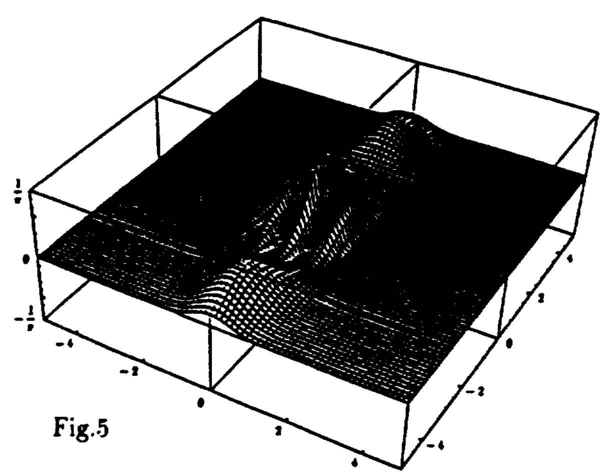

Fig.5

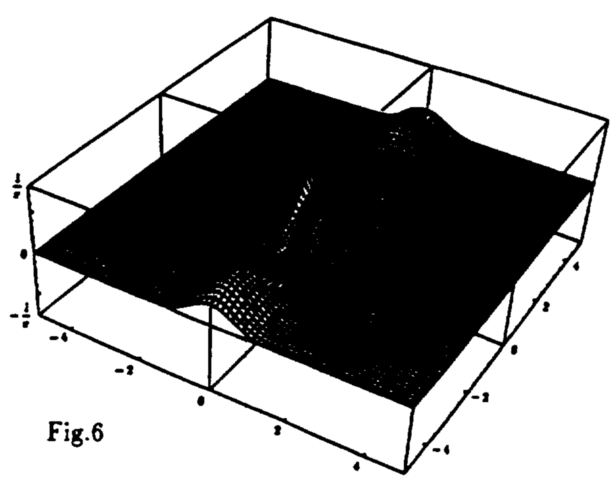

Fig.6

Fig.1-6:

Wigner quasiprobability $W(q,p)$ to states $|0,n;\zeta\rangle_{norm}$ for $n = 0, 1, \ldots, 5$, $\zeta = +0.5$ and $\hbar = 1$.

$|0,n;\zeta\rangle_{norm}$, i.e. for $n = 0,1,\ldots,5$, with the squeezing parameter $\zeta = +0.5$ and with $\hbar = 1$. For $\zeta = -0.5$ one obtains the same pictures only rotated about an angle $\pi/2$.

Let us give here additionally the explicit expressions of three partial classes of quasiprobabilities from the total Gaussian class for the normalized squeezed vacuum states $|0,0;\zeta\rangle_{norm}$

$$
\begin{aligned}
F_{(r_1,0,0)}(\alpha,\alpha^*) &= \frac{2}{\pi}\sqrt{\frac{1-\zeta\zeta^*}{(1+r_1^2)(1-\zeta\zeta^*)-2r_1(\zeta-\zeta^*)}} \\
&\quad \exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)+r_1(1-\zeta\zeta^*)(\alpha^2-\alpha^{*2})}{(1+r_1^2)(1-\zeta\zeta^*)-2r_1(\zeta-\zeta^*)}\right\}, \\
F_{(0,r_2,0)}(\alpha,\alpha^*) &= \frac{2}{\pi}\sqrt{\frac{1-\zeta\zeta^*}{(1+r_2^2)(1-\zeta\zeta^*)-i2r_2(\zeta+\zeta^*)}} \\
&\quad \exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)+ir_2(1-\zeta\zeta^*)(\alpha^2+\alpha^{*2})}{(1+r_2^2)(1-\zeta\zeta^*)-i2r_2(\zeta+\zeta^*)}\right\}, \\
F_{(0,0,r_3)}(\alpha,\alpha^*) &= \frac{2}{\pi}\sqrt{\frac{1-\zeta\zeta^*}{(1+r_3^2)(1-\zeta\zeta^*)+2r_3(1+\zeta\zeta^*)}} \\
&\quad \exp\left\{-\frac{2(\alpha+\zeta\alpha^*)(\alpha^*+\zeta^*\alpha)+2r_3(1-\zeta\zeta^*)\alpha\alpha^*}{(1+r_3^2)(1-\zeta\zeta^*)+2r_3(1+\zeta\zeta^*)}\right\}.
\end{aligned}
\tag{25}
$$

The modulus of the complex squeezing parameter ζ determines the amount of squeezing whereas the phase of the squeezing parameter ζ determines the position of the squeezing axes. In particular, the squeezing axes are parallel to the coordinate axes for real $\zeta = \zeta^*$. In this case the class of quasiprobabilities $F_{(r_1,0,0)}(\alpha,\alpha^*)$ simplifies. The squeezing axes are diagonal to the coordinate axes for imaginary $\zeta = -\zeta^*$ and then the class of quasiprobabilities $F_{(0,r_2,0)}(\alpha,\alpha^*)$ simplifies. The usually considered class of quasiprobabilities $F_{(0,0,r_3)}(\alpha,\alpha^*)$ contains the interesting value of the parameter r_3 for which the denominator in the exponential function vanishes and the quasiprobability becomes a singular function. This point depends on the modulus of the squeezing parameter and is given by

$$
r_3^{sing} = -\frac{1-|\zeta|}{1+|\zeta|}, \quad \left(r_3^{'\,sing} = -\frac{1+|\zeta|}{1-|\zeta|}\right),
\tag{26}
$$

where the second solution given in brackets seems to be not of interest. For parameters r_3 less or equal this singularity point the corresponding quasiprobabilities can be only considered as generalized functions. Recall that the quasiprobabilities for squeezed coherent states $|\beta,0;\zeta\rangle$ can be obtained again from the quasiprobabilities for $|0,0;\zeta\rangle$ by the mentioned argument displacements.

Last, we found for the number representation of the states $|0,n;\zeta\rangle_{norm}$

$$
\begin{aligned}
|0,n;\zeta\rangle_{norm} &= \frac{\left(\sqrt{1-\zeta\zeta^*}\right)^{n+\frac{1}{2}}}{\sqrt{\sum_{k=0}^{[\frac{n}{2}]}\frac{n!}{k!^2(n-2k)!}\left(\frac{\sqrt{\zeta\zeta^*}}{1+\zeta\zeta^*}\right)^{2k}}} \sum_{j=-[\frac{n}{2}]}^{\infty} \frac{(-1)^j}{2^j}\sqrt{\frac{(n+2j)!}{n!}}\zeta^j \\
&\quad \left(\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{l!(l+j)!(n-2l)!}\left(\frac{\zeta\zeta^*}{4(1+\zeta\zeta^*)}\right)^l\right)|n+2j\rangle.
\end{aligned}
\tag{27}
$$

It contains only even or odd number states in dependence on n as an even or odd number. For large modulus of the squeezing parameter ζ the resulting number distribution becomes relatively broad and uniform over even or odd numbers. The transition from the states $|0,n;\zeta\rangle_{norm}$ to the displaced states $|\beta,n;\zeta\rangle_{norm}$ is here more complicated as in the case of the quasiprobabilities. Generally, if an arbitrary state $|\psi\rangle$ has the number representation

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{28}$$

then the displaced state $D(\beta,\beta^*)|\psi\rangle$ has the number representation

$$\begin{aligned} D(\beta,\beta^*)|\psi\rangle &= \exp\left(-\frac{\beta\beta^*}{2}\right)\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{c_n}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\beta^{m-j}(-\beta^*)^{n-j}\right)|m\rangle \\ &= \exp\left(-\frac{\beta\beta^*}{2}\right)\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\sqrt{\frac{n!}{m!}}\beta^{m-n}L_n^{m-n}(\beta\beta^*)c_n\right)|m\rangle \\ &= \exp\left(-\frac{\beta\beta^*}{2}\right)\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\sqrt{\frac{m!}{n!}}(-\beta^*)^{n-m}L_m^{n-m}(\beta\beta^*)c_n\right)|m\rangle, \end{aligned} \tag{29}$$

where $L_n^{\nu}(z)$ denotes the Laguerre polynomials in the usual way. This is a kind of discrete convolution of the primary number representation.

7 Conclusion

We investigated the total Gaussian class of quasiprobabilities and its diagonal representation in case of real vector parameters. Another interesting special case is given for real r_3 and imaginary r_1 and r_2. It seems that this case may be treated in analogy to the usual s-parametrized class of quasiprobabilities by transition to new boson operators via a Bogolyubov transformation. Some points and proofs are given more in detail in [6] but some are new in the present paper, in particular, all formulae of section 6 for the states $|\beta,n;\zeta\rangle$ are given here for the first time.

References

[1] R.J. Glauber, Phys. Rev. 131, 2766 (1963).

[2] K.E. Cahill and R.J. Glauber, Phys. Rev. 177, 1857 and 1882 (1969).

[3] G.S. Agarwal and E. Wolf, Phys. Rev. D2, 2161 (1970).

[4] M. Hillary, R.F. O'Connell, M.O. Scully and E.P. Wigner, Physics Report 106, 121 (1984).

[5] R.F. Bishop and A. Vourdas, Phys. Rev. A50, 4488 (1994).

[6] A. Wünsche, submitted to Phys. Rev. A (July 1995); this paper contains a more complete list of references to the subject than given here.