Generalization of the time-energy uncertainty relation of Anandan-Aharonov Type

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Abstract

A new type of time-energy uncertainty relation was proposed recently by Anandan and Aharonov. Their formula to estimate the lower bound of time-integral of the energy-fluctuation in a quantum state is generalized to the one involving a set of quantum states. This is achieved by obtaining an explicit formula for the distance between two finitely separated points in the Grassman manifold.

I. Introduction

We first review briefiy the conventional time-energy uncertainty relation in quantum mechanics. Let A be an ovservable without explicit time-dependence and $|\psi(t)\rangle$ be a normalized quantum state vector obeying the Schrödinger equation with a hermitian Hamiltonian H. If we define ΔA and τ_A by

$$\Delta A = \sqrt{\langle\psi(t)|A^2|\psi(t)\rangle - \langle\psi(t)|A|\psi(t)\rangle^2} \ , \tag{1}$$

$$\tau_A = \left|\frac{d}{dt}\langle\psi(t)|A|\psi(t)\rangle\right|^{-1} \Delta A \ , \tag{2}$$

and take the equation

$$\frac{d}{dt}\langle\psi(t)|A|\psi(t)\rangle = \frac{1}{i\hbar}\langle\psi(t)|\ [A,H]\ |\psi(t)\rangle \tag{3}$$

into account, we are led to the uncertainty relation [1]

$$\tau_A \Delta H \geq \frac{\hbar}{2} \ . \tag{4}$$

The quantity τ_A is interpreted as the time necessary for the distribution of $\langle\psi(t)|A|\psi(t)\rangle$ to be recognized to have clearly changed its shape.

In contrast with the result given above, Anandan and Aharonov [2] have recently succeeded in obtaining quite an interesting inequality. They consider the case that the $|\psi(t)\rangle$ develops in time obeying

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \ , \tag{5}$$

$$\langle\psi(t)|\psi(t)\rangle = 1 \ , \tag{6}$$

where $H(t)$ is an operator which is hermitian and might be time-dependent. They conclude that

$$\int_{t_1}^{t_2} \Delta\mathcal{E}(t)dt \geq \hbar\mathrm{Arccos}(|\langle\psi(t_1)|\psi(t_2)\rangle|) \ , \tag{7}$$

where $\Delta\mathcal{E}(t)$ is given by

$$\Delta\mathcal{E}(t) = \sqrt{\langle\psi(t)|H(t)^2|\psi(t)\rangle - \langle\psi(t)|H(t)|\psi(t)\rangle^2} \ . \tag{8}$$

The inequality (7), which we refer to as the Anandan-Aharonov time-energy uncertainty relation, has been derived through a geometrical investigation of the set of normalized quantum state vectors. The r.h.s. of (7) can be regarded as the distance between two points in a complex projective space.

Here, we seek the generalized version of (7). We consider a set of N orthonormal vectors $\{|\psi_i(t)\rangle : i = 1, 2, \ldots, N\}$ satisfying

$$\langle\psi_i(t)|\psi_j(t)\rangle = \delta_{ij} \ , \quad i, j = 1, 2, \ldots, N, \tag{9}$$

each of which obeying the Schrödinger equation (5). We define $N \times N$ matrices $A(t_1, t_2)$ and $K(t_1, t_2)$ by

$$A(t_1, t_2) = (a_{ij}(t_1, t_2)) \ , \quad a_{ij}(t_1, t_2) = \langle\psi_i(t_1)|\psi_j(t_2)\rangle \ , \tag{10}$$

$$K(t_1, t_2) = A^\dagger(t_1, t_2) A(t_1, t_2) \tag{11}$$

and $\kappa_i(t_1, t_2), i = 1, 2, \ldots, N$, to be the eigenvalues of $K(t_1, t_2)$. Defining the generalization of (8) by

$$\Delta\mathcal{E}_N(t) = \sqrt{\sum_{i=1}^{N} \langle\psi_i(t)|H(t)^2|\psi_i(t)\rangle - \sum_{i,j=1}^{N} |\langle\psi_i(t)|H(t)|\psi_j(t)\rangle|^2} \ , \tag{12}$$

we find that $\Delta\mathcal{E}_N(t)$ satisfies

$$\int_{t_1}^{t_2} \Delta\mathcal{E}_N(t) dt \geq \hbar \sqrt{\sum_{i=1}^{N} \left\{ \mathrm{Arccos} \sqrt{\kappa_i(t_1, t_2)} \right\}^2} \ . \tag{13}$$

The inequality (13) can be written in an operator form as

$$\begin{aligned} &\int_{t_1}^{t_2} \sqrt{\mathrm{Tr}(P(t)[H(t), [H(t), P(t)]])} \mathrm{dt} \\ &\qquad \geq \sqrt{2}\hbar \sqrt{\mathrm{Tr}(\{\mathrm{Arccos}\sqrt{P(t_1)P(t_2)}\}^2)} \ , \end{aligned} \tag{14}$$

where $P(t)$ is defined by

$$P(t) = \sum_{i=1}^{N} |\psi_i(t)\rangle\langle\psi_i(t)|, \tag{15}$$

and Tr denotes the trace in the Hilbert space. The result (13) is obtained through a geometrical investigation of the Grassmann manifold G_N mentioned below.

II. Distance formula for the Grassmann manifold

Given a Hilbert space h,we consider vectors $|\psi_i\rangle, i = 1,2,\ldots,N$,belonging to h and satisfying $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. We call the set

$$\Psi = (|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle) \tag{16}$$

an N-frame of h and the set

$$[\Psi] = \{\Psi u : u \in U(N)\} \tag{17}$$

an N-plane of h,where Ψu is defined by

$$\Psi u = (\sum_{i=1}^{N} |\psi_i\rangle u_{i1}, \sum_{j=1}^{N} |\psi_j\rangle u_{j2}, \ldots, \sum_{k=1}^{N} |\psi_k\rangle u_{kN}). \tag{18}$$

It is clear that the $[\Psi]$ and the projection operator $P = \sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i|$ are invariant under the replacement $\Psi \longrightarrow \Psi u$. We denote the set of all the Ψ's of h by S_N. Then the set G_N defined by

$$G_N = \{[\Psi] : \Psi \in S_N\} \tag{19}$$

is known to constitute a manifold of complex dimension $N(\dim \mathrm{h} - N)$ and is called the Grassmann manifold.

To an N-frame $\Psi(t) = (|\psi_1(t)\rangle, |\psi_2(t)\rangle, \ldots, |\psi_N(t)\rangle) \in S_N, 0 \le t \le 1$, there correspond an N-plane $[\Psi(t)] \in G_N$ and a projection operator $P(t) = \sum_{i=1}^{N} |\psi_i(t)\rangle\langle\psi_i(t)|$. Since the eigenvalues of $P(1)$ are equal to those of $P(0)$ including multiplicities,there exists a unitary operator W such that

$$P(1) = W^{\dagger} P(0) W, \quad W = e^{iY}, \quad Y^{\dagger} = Y. \tag{20}$$

We define the distance $d([\Psi(0)], [\Psi(1)])$ between two points $[\Psi(0)]$ and $[\Psi(1)]$ of the Grassmann manifold G_N by

$$d([\Psi(0)], [\Psi(1)]) = \underset{Y \in \Sigma}{\mathrm{Min}} \|Y\|, \tag{21}$$

where Σ is the set of hermitian operators specified by $P(0)$ and $P(1)$ in the following way:

$$\Sigma = \{Y : Y = Y(P(0), P(1)) = -Y(P(1), P(0)) = Y^{\dagger}, e^{-iY} P(0) e^{iY} = P(1)\}. \tag{22}$$

After some manipulations,we find that the distance is given by the formula

$$d([\Psi(0)],[\Psi(1)]) = \sqrt{2\sum_{i=1}^{N}(\mathrm{Arccos}\sqrt{\kappa_i})^2}, \tag{23}$$

where κ_i is defined below (11) and satisfies $0 \leq \kappa_i \leq 1$.

We also find that the above defined distance in G_N satisfies the property of distance:

$$d([\Psi],[\Phi]) = d([\Phi],[\Psi]) \geq 0, \tag{24}$$

$$d([\Psi],[\Phi]) = 0 \iff [\Psi] = [\Phi], \tag{25}$$

$$d([\Psi],[\Phi]) \leq d([\Psi],[\Xi]) + d([\Xi],[\Phi]), \tag{26}$$

for any $[\Psi],[\Phi],[\Xi] \in G_N$.

III. Time-energy uncertainty relation

The projection operator $P(t)$ is defined by (15) and $|\psi_i(t)\rangle, i = 1,2,\ldots,N$, develops in time obeying (5). We then have

$$\begin{aligned} P(t+dt) = P(t) &+ \frac{dt}{i\hbar}[H(t),P(t)] \\ &+ \frac{(dt)^2}{2(i\hbar)^2}\left\{i\hbar[\frac{dH(t)}{dt},P(t)] + [H(t),[H(t),P(t)]]\right\} + \cdots . \end{aligned} \tag{27}$$

When $[\Psi(0)]$ and $[\Psi(1)]$ are close to each other, $\kappa_i, i = 1,2,\ldots,N$, are nearly equal to 1. Noticing that $(\mathrm{Arccos}\sqrt{\kappa})^2 \approx 1-\kappa$ for $\kappa \approx 1$, we see

$$d([\Psi(t)],[\Psi(t+dt)]) \approx \sqrt{2\sum_{i=1}^{N}(1-\kappa_i(t))}, \tag{28}$$

where $\kappa_i(t)$'s are obtained from $P(t)$ and $P(t+dt)$ by similar procedures to those of previous sections. Since, in the above case, we have $\mathrm{Tr}P(t) = N$ and

$$\mathrm{Tr}(P(t)P(t+dt)) = \sum_{i=1}^{N}\kappa_i(t), \tag{29}$$

(28) can be rewritten as

$$d([\Psi(t)],[\Psi(t+dt)]) = \sqrt{2\mathrm{Tr}(P(t)\{P(t)-P(t+dt)\})}. \tag{30}$$

Now we have

$$
\begin{aligned}
d([\Psi(t)],[\Psi(t+dt)]) &= \frac{|dt|}{\hbar}\sqrt{\mathrm{Tr}(P(t)[H(t),[H(t),P(t)]])} \\
&= \frac{|dt|}{\hbar}\sqrt{\mathrm{Tr}([P(t),H(t)][H(t),P(t)])} \\
&= \left\|\frac{dP(t)}{dt}\right\| |dt|. \\
&= \|dP(t)\|.
\end{aligned}
\tag{31}
$$

It can be easily seen that the r.h.s. of (31) is proportional to $\Delta\mathcal{E}_N(t)$ defined by (12). Now we are led to

$$
d([\Psi(t)],[\Psi(t+dt)]) = \frac{\sqrt{2}}{\hbar}\Delta\mathcal{E}_N(t)|dt|. \tag{32}
$$

For finitely separated $[\Psi(t_1)]$ and $[\Psi(t_2)]$ in G_N, the triangle inequality (26) implies

$$
\int_{t_1}^{t_2}\Delta\mathcal{E}_N(t)dt \geq \frac{\hbar}{\sqrt{2}}d([\Psi(t_1)],[\Psi(t_2)]), t_2 \geq t_1. \tag{33}
$$

The formula (23) then leads us to (13) or (14). For details , see [3].

References

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